

Mode Treatment Effect

Neng-Chieh Chang*

Abstract

Mean, median, and mode are three essential measures of the centrality of probability distributions. In program evaluation, the average treatment effect (mean) and the quantile treatment effect (median) have been intensively studied in the past decades. The mode treatment effect, however, has long been neglected in program evaluation. This paper fills the gap by discussing both the estimation and inference of the mode treatment effect. I propose both traditional kernel and machine learning methods to estimate the mode treatment effect. I also derive the asymptotic properties of the proposed estimators and find that both estimators follow the asymptotic normality but with the rate of convergence slower than the regular rate \sqrt{N} , which is different from the rates of the classical average and quantile treatment effect estimators.

1 Introduction

The effects of policies on the distribution of outcomes have long been of central interest in many areas of empirical economics. A policy maker might be interested in the difference of the distribution of outcome under treatment and the distribution of outcome in the absence of treatment. The empirical studies of distributional effects include but not are not limited to [Freeman \(1980\)](#), [Card \(1996\)](#), [DiNardo, Fortin, & Lemieux \(1995\)](#), and [Bitler, Gelbach, & Hoynes \(2006\)](#). Most researches use the difference of the averages or quantiles of the treated and untreated distribution, known as average treatment effect and quantile treatment effect, as a summary for the effect of treatment on distribution. The mode of a distribution, which is also an important summary statistics of data, has long been ignored in the literature. This paper fills up the gap by studying the mode treatment effect: the difference of the modes of the treated and untreated distribution. Compared to the average and the quantile treatment effect, the mode treatment effect has two advantages: (1) mode captures the most probable value of the distribution under treatment and in the absence of treatment. It provides a better summary of centrality than average and quantile when the distributions are highly skewed; (2) mode is robust to heavy-tailed distributions where outliers don't follow the same behavior as the majority of a sample. In economic studies, it is especially often to confront a skewed and heavy-tailed distribution when the outcome of interest is income or wage.

*Department of Economics, University of California Los Angeles, 315 Portola Plaza, Los Angeles, CA 90095, USA. email: nengchiehchang@g.ucla.edu

This paper discusses the estimation and inference of the mode treatment effect under the Strong Ignorability assumption (Rosenbaum & Rubin, 1983), which states that conditional on a vector of control variables the treatment is randomly assigned. The first estimator I propose is the kernel estimator. I estimate the density function of the outcome distribution using the kernel method and define the maximum of the estimated density function as the estimator of the mode. While the kernel estimator is a straightforward estimator, it requires to estimate the conditional density function in the process, and the estimation of the conditional density function may be difficult in practice when there exist more than two or three control variables, due to the curse of dimension. The kernel estimator is appropriate if there are less than three control variables. In practice, however, researchers may want to include as many control variables as possible in order to make their identification robust. In this circumstance, the curse of dimension may lead to inaccurate estimation and misleading inference.

To address this problem, I propose the ML estimator. The key feature of the proposed ML estimator is that it translates the estimation of the conditional density function into the estimation of conditional expectation, which we can apply a rich set of ML methods, such as Lasso, random forests, neural nets, and etc, to estimate. This feature provides researchers with the flexibility to apply ML methods to estimate the density function of the outcome distribution. By the virtue of ML methods, the proposed ML estimator can handle the situation when there exist many control variables, even the number of control variables is comparable to or more than the sample size. However, it is well-known that the regularization bias embedded in ML methods may lead to the bias of the final estimator and misleading inference (Chernozhukov et al., 2018). To solve this problem, I further derive the Neyman-orthogonal scores (Chernozhukov et al., 2018) for each estimation which requires the first-step estimation of the conditional expectation. These Neyman-orthogonal scores, to my best knowledge, are new results. The proposed ML estimator is built on the newly derived Neyman-orthogonal score, and hence, it is robust to the regularization bias of the first-step ML estimation.

I derive the asymptotic properties for both the proposed kernel and ML estimators. I show that both estimators are consistent and asymptotically normal with the rate of convergence $\sqrt{Nh^3}$, where N is the sample size and h is the bandwidth of the chosen kernel, which is slower than the traditional rate of convergence \sqrt{N} in the estimation of mean and quantile. In fact, this rate of convergence complies the intuition. While the estimators of mean and quantile are the weighted average of all the available observations, only a small portion of observations near the mode provides the information to the estimator of the mode. This explains the slower rate of convergence for the proposed estimators.

This paper contributes to the program evaluation literature which includes the studies of average treatment effect: Rosenbaum & Rubin (1983), Heckman & Robb (1985), Heckman, Ichimura, & Todd (1997), Hahn (1998), and Hirano, Imbens, & Ridder (2003); the studies of quantile treatment effect: Abadie, Angrist, & Imbens (2002), Chernozhukov & Hansen (2005), and Firpo (2007); the studies of mode estimation and mode regression: Parzen (1962), Eddy et al. (1980), Lee (1989), Yao & Li (2014), and Chen, Genovese, Tibshirani, Wasserman, et al. (2016); as well as the causal inference of ML methods: Belloni et al. (2012), Belloni et al. (2014), Chernozhukov et al. (2015), Belloni et al. (2017), Chernozhukov et al. (2018), and Athey et al. (2019). This paper is also closely related to the robustness of average treat-

ment effect estimation discussed in (Robins & Rotnitzky, 1995) and the general discussion in (Chernozhukov, Escanciano, Ichimura, & Newey, 2016). The asymptotic properties of the robust estimators discussed in these papers remain unaffected if only one of the first-step estimation with classical nonparametric method is inconsistent.

Plan of the paper. Section 2 sets up the notation and framework for the discussion of the mode treatment effect. Section 3 discusses the kernel method and derives the asymptotic properties. Section 4 presents the ML estimator for density estimation and the corresponding Neyman-orthogonal score. I combine the Neyman-orthogonal score with the cross-fitting algorithm to propose the ML estimator of the mode treatment effect, and derive its asymptotic properties. Section 5 concludes this paper.

2 Notation and Framework

Let Y be a continuous outcome variable of interest, D the binary treatment indicator, and X $d \times 1$ vector of control variables. Denote by Y_1 an individual's potential outcome when $D = 1$ and Y_0 if $D = 0$. Let $f_{Y_1}(y)$ and $f_{Y_0}(y)$ be the marginal probability density function (p.d.f.) of Y_1 and Y_0 , respectively. The modes of Y_1 and Y_0 are the values that appear with the highest probability. That is,

$$\theta_1^* \equiv \arg \max_{y \in \mathcal{Y}_1} f_{Y_1}(y) \text{ and } \theta_0^* \equiv \arg \max_{y \in \mathcal{Y}_0} f_{Y_0}(y),$$

where \mathcal{Y}_1 and \mathcal{Y}_0 are the supports of Y_1 and Y_0 . Here I assume that θ_1^* and θ_0^* are unique, meaning that both Y_1 and Y_0 are unimodal. I also assume that the modes θ_1^* and θ_0^* are in the interior of the common supports of Y_1 and Y_0 . These conditions are formally stated in the following assumption:

Assumption 1 (Uni-mode)

- For all $\epsilon > 0$,

$$\sup_{y: |y - \theta_1^*| > \epsilon} f_{Y_1}(y) < f_{Y_1}(\theta_1^*) \text{ for } y \in \mathcal{Y}_1,$$

and

$$\sup_{y: |y - \theta_0^*| > \epsilon} f_{Y_0}(y) < f_{Y_0}(\theta_0^*) \text{ for } y \in \mathcal{Y}_0.$$

- $\theta_1^*, \theta_0^* \in \text{Int}(\mathcal{Y}_1 \cap \mathcal{Y}_0)$.

Assumption 1 has been widely adopted in many studies (Parzen, 1962; Eddy et al., 1980; Lee, 1989; Yao & Li, 2014). Under Assumption 1, the mode treatment effect is uniquely defined as $\Delta^* \equiv \theta_1^* - \theta_0^*$. The following states the strong ignorability assumption (Rosenbaum & Rubin, 1983):

Assumption 2 (strong ignorability)

- $(Y_0, Y_1) \perp D \mid X$

- $0 < P(D = 1 | X) < 1$

The first part of Assumption 2 assumes that potential outcomes are independent of treatment after conditioning on the observable covariates X . The second part states that for all values of X , both treatment status occur with a positive probability. Under the strong ignorability condition, both f_{Y_1} and f_{Y_0} can be identified from the observable variables (Y, D, X) since

$$f_{Y|D=1,X}(y | x) = f_{Y_1|D=1,X}(y | x) = f_{Y_1|X}(y | x),$$

and thus

$$f_{Y_1}(y) = E \left[f_{Y_1|X}(y | X) \right] = E \left[f_{Y|D=1,X}(y | X) \right]. \quad (2.1)$$

Similarly, we have

$$f_{Y_0}(y) = E \left[f_{Y|D=0,X}(y | X) \right]. \quad (2.2)$$

Equation (2.1) and (2.2) shows the identification result of the density function f_{Y_1} and f_{Y_0} . Then it is straightforward to identify their modes θ_1^* and θ_0^* :

$$\theta_1^* = \arg \max_{y \in \mathcal{Y}_1} E \left[f_{Y|D=1,X}(y | X) \right] \quad \text{and} \quad \theta_0^* = \arg \max_{y \in \mathcal{Y}_0} E \left[f_{Y|D=0,X}(y | X) \right]. \quad (2.3)$$

If both $f_{Y|D=1,X}(y | X)$ and $f_{Y|D=0,X}(y | X)$ are differentiable with respect to y , we can further identify the modes using the first-order conditions under Assumption 1:

$$E \left[f_{Y|D=1,X}^{(1)}(\theta_1^* | X) \right] = 0 \quad \text{and} \quad E \left[f_{Y|D=0,X}^{(1)}(\theta_0^* | X) \right] = 0, \quad (2.4)$$

where $m^{(s)}(y, x) \equiv \partial^s m(y, x) / \partial y^s$ denotes the partial derivatives with respect to y .

Equation (2.1)-(2.4) provide us a direct way to estimate the modes θ_1^* and θ_0^* . Intuitively, we estimate the density functions $f_{Y_1}(y)$ and $f_{Y_0}(y)$ in the first step and use the maximizers of the estimated density functions as the estimators of the modes. Section 3 and 4 presents the kernel and ML estimation method, respectively.

3 The Kernel Estimation

In this section, I propose kernel estimators for θ_1^* , θ_0^* , and the mode treatment effect $\Delta^* = \theta_1^* - \theta_0^*$. Let $K(\cdot)$ be a kernel function with bandwidth h . Define the estimators of the density functions $f_{Y_1}(y)$ and $f_{Y_0}(y)$ as,

$$\hat{f}_{Y_1}(y) = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|D=1,X}(y | X_i),$$

$$\hat{f}_{Y_0}(y) = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|D=0,X}(y | X_i)$$

with the kernel estimators

$$\hat{f}_{Y|D=1,X}(y|x) = \frac{\sum_{j=1}^n D_j K_h(y - Y_j) K_h(x - X_j)}{\sum_{j=1}^n D_j K_h(x - X_j)},$$

$$\hat{f}_{Y|D=0,X}(y|x) = \frac{\sum_{j=1}^n (1 - D_j) K_h(y - Y_j) K_h(x - X_j)}{\sum_{j=1}^n (1 - D_j) K_h(x - X_j)}$$

where $K_h(y - Y_j) = h^{-1}K\left(\frac{y - Y_j}{h}\right)$ and

$$K_h(x - X_j) = h^{-d}K\left(\frac{x_1 - X_{j1}}{h}\right) \times \dots \times K\left(\frac{x_d - X_{jd}}{h}\right).$$

Then it is straightforward to define the estimators of the modes θ_1^* and θ_0^* :

$$\hat{\theta}_1 \equiv \arg \max_y \hat{f}_{Y_1}(y),$$

$$\hat{\theta}_0 \equiv \arg \max_y \hat{f}_{Y_0}(y).$$

The estimator of the mode treatment effect Δ^* is $\hat{\Delta} \equiv \hat{\theta}_1 - \hat{\theta}_0$. Through out the paper, I impose the following conditions on the kernel $K(\cdot)$:

Assumption 3:

- $|K(u)| \leq \bar{K} < \infty$.
- $\int K(u) du = 1, \int uK(u) du = 0, \int u^2K(u) du < \infty$.
- $K(u)$ is differentiable.

The first part of Assumption 3 requires that $K(u)$ is bounded. Although the second part implies that $K(u)$ is a first-order kernel, the arguments in this paper can be easily extended to higher-order kernels. We assume the first-order kernel here just for simplicity. The third part imposes enough smoothness on $K(u)$.

Theorem 1 (Consistency) *Suppose Assumption 1-3 hold. Assume that the density functions $f_{Y|D=1,X}(y|x)$ and $f_{Y|D=0,X}(y|x)$ are (i) continuous in y , (ii) bounded by some function $d(x)$ with $E[d(X)] < \infty$ for all $y \in \mathcal{Y}$, and (iii) $y \in \mathcal{Y}$ and $x \in \mathcal{X}$ with compact \mathcal{Y} and \mathcal{X} . We also assume that the density functions $f_{X|D=1}(x)$ and $f_{X|D=0}(x)$ are bounded away from zero. If $n \rightarrow \infty$, $h \rightarrow 0$, and $\ln n (nh^{d+1})^{-1} \rightarrow 0$, then we have $\hat{\theta}_1 \xrightarrow{P} \theta_1^*$ and $\hat{\theta}_0 \xrightarrow{P} \theta_0^*$.*

Theorem 2 (Asymptotic Normality) *Suppose that the assumptions of Theorem 1 hold. Assume that $f_{Y|X,D=1}^{(2)}(y|x)$ and $f_{Y|X,D=0}^{(2)}(y|x)$ are continuous at $y = \theta_1^*$ and $y = \theta_0^*$ for*

all x , respectively. If $n \rightarrow \infty$, $h \rightarrow 0$, $\sqrt{nh^3} (\ln n) (nh^{d+3})^{-1} \rightarrow 0$, $(\ln n) (nh^{d+5})^{-1} \rightarrow 0$, and $\sqrt{nh^3} h^2 \rightarrow 0$, then

$$\sqrt{nh^3} (\hat{\theta}_1 - \theta_1^*) \xrightarrow{d} N(0, M_1^{-1} V_1 M_1^{-1})$$

$$\sqrt{nh^3} (\hat{\theta}_0 - \theta_0^*) \xrightarrow{d} N(0, M_0^{-1} V_0 M_0^{-1})$$

where

$$M_1 \equiv E \left[f_{Y|X,D=1}^{(2)}(\theta_1^* | X) \right],$$

$$M_0 \equiv E \left[f_{Y|X,D=0}^{(2)}(\theta_0^* | X) \right],$$

$$V_1 = \kappa_0^{(1)} E \left[\frac{f_{Y|X,D=1}(\theta_1^* | X)}{P(D=1 | X)} \right],$$

$$V_0 = \kappa_0^{(1)} E \left[\frac{f_{Y|X,D=0}(\theta_0^* | X)}{P(D=0 | X)} \right],$$

and $\kappa_0^{(1)} = \int K^{(1)}(u)^2 du$. Further, we have

$$\sqrt{nh^3} (\hat{\Delta} - \Delta^*) \xrightarrow{d} N(0, M_1 V_1 M_1 + M_0 V_0 M_0).$$

Theorem 1 and 2 show that the asymptotic properties of the estimator of the mode treatment effect. We can see that the proposed estimators follows the asymptotic normality but with the rate of convergence slower than the regular rate \sqrt{N} . The intuition is that, unlike the estimation of the average and the quantile treatment effect, the estimation of modes only uses a small portion of total observations which are around the modes. The usage rate of observations determines that the rate of convergence is slower than the regular rate \sqrt{N} .

To estimate the asymptotic variances, we define $\pi_0(X) \equiv P(D=1 | X)$ to be the propensity score. The consistent variance estimators are

$$\hat{M}_1 = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i),$$

$$\hat{M}_0 = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=0}^{(2)}(\hat{\theta}_0 | X_i),$$

$$\hat{V}_1 = \kappa_0^{(1)} \frac{1}{n} \sum_{i=1}^n \frac{\hat{f}_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\hat{\pi}(X_i)},$$

$$\hat{V}_0 = \kappa_0^{(1)} \frac{1}{n} \sum_{i=1}^n \frac{\hat{f}_{Y|X,D=0}(\hat{\theta}_0 | X_i)}{\hat{\pi}(X_i)}.$$

Theorem 3 (Variance Estimation) *Suppose that the assumptions in Theorem 2 hold. Let $\hat{\pi}(x)$ be an uniformly consistent estimator for $\pi_0(x)$. If $n \rightarrow \infty$, $h \rightarrow 0$, and $\ln n (nh^{d+5}) \rightarrow 0$, then $\hat{M}_1 \xrightarrow{p} M_1$, $\hat{M}_0 \xrightarrow{p} M_0$, $\hat{V}_1 \xrightarrow{p} V_1$, $\hat{V}_0 \xrightarrow{p} V_0$. Thus we have $\hat{M}_1^{-1} \hat{V}_1 \hat{M}_1^{-1} \xrightarrow{p} M_1^{-1} V_1 M_1^{-1}$ and $\hat{M}_0^{-1} \hat{V}_0 \hat{M}_0^{-1} \xrightarrow{p} M_0^{-1} V_0 M_0^{-1}$.*

4 The Machine Learning Estimation

In this section, I propose the ML estimator of the mode treatment effect. The ML estimator can accommodate a large number of control variables, potentially more than the sample size. This flexibility will enable researcher to include as many control variables they consider important to make their identification assumptions more plausible. The key to implement ML methods is to replace the estimation of the conditional density function with the estimation of the conditional expectation. To begin with, the estimation of the conditional density function in the traditional kernel estimation is

$$\hat{f}_{Y|D=1,X}(y|x) = \frac{\sum_{j=1}^n D_j K_h(y - Y_j) K_h(x - X_j)}{\sum_{j=1}^n D_j K_h(x - X_j)}.$$

Notice that we can divide both the numerator and the denominator by $\sum_{j=1}^n K_h(x - X_j)$ to obtain

$$\hat{f}_{Y|D=1,X}(y|x) = \frac{\sum_{j=1}^n D_j K_h(y - Y_j) K_h(x - X_j) / \sum_{j=1}^n K_h(x - X_j)}{\sum_{j=1}^n D_j K_h(x - X_j) / \sum_{j=1}^n K_h(x - X_j)}.$$

The numerator is an kernel estimator of $E[DK_h(y - Y) | X]$ and the denominator is an kernel estimator of the propensity score $E[D | X] = \pi(X)$. Hence, $\hat{f}_{Y|D=1,X}(y|x)$ is an estimator of $E[DK_h(y - Y) | X] / \pi(X)$. Then the marginal density estimator

$$\hat{f}_{Y_1}(y) = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|D=1,X}(y | X_i)$$

defined in the previous section can be interpreted as an estimator of

$$E \left[\frac{E[DK_h(y - Y) | X]}{\pi(X)} \right] = E \left[\frac{DK_h(y - Y)}{\pi(X)} \right].$$

Therefore, we can use the machine learning estimator of $E \left[\frac{DK_h(y - Y)}{\pi(X)} \right]$ as an estimator for $f_{Y_1}(y)$. We have successfully translate the estimation of the conditional density function into the estimation of the conditional expectation, which is the propensity score $\pi(X)$.

Here we pursue a little bit further to construct the Neyman-orthogonal score ([Chernozhukov et al., 2018](#)) for the robustness of the first-step estimation:

$$m_1(Z, y, \eta_{10}) = \frac{DK_h(y - Y)}{\pi_0(X)} - \frac{D - \pi_0(X)}{\pi_0(X)} E[K_h(y - Y) | X, D = 1], \quad (4.1)$$

where $Z = (Y, D, X)$ and $\eta_0 = (\pi_0, g_{10})$ with $g_{10}(X) \equiv E[K_h(y - Y) | X, D = 1]$. Similarly, the Neyman-orthogonal score for $f_{Y_0}(y)$ is

$$m_2(Z, y, \eta_{20}) = \frac{(1 - D)K_h(y - Y)}{1 - \pi_0(X)} - \frac{\pi_0 - D(X)}{1 - \pi_0(X)} E[K_h(y - Y) | X, D = 0], \quad (4.2)$$

where $\eta_{20} = (\pi_0, g_{20})$ with $g_{20}(X) \equiv E[K_h(y - Y) | X, D = 0]$. Equation (4.1) and (4.2), to my best knowledge, should be the new results for density estimation. The Neyman orthogonality will make the estimation of the density functions more robust to the first-step estimation. Now I combine (4.1) and (4.2) with the cross-fitting algorithm ([Chernozhukov et al., 2018](#)) to propose the new estimator:

Definition.

1. Take a K -fold random partition $(I_k)_{k=1}^K$ of $[N] = \{1, \dots, N\}$ such that the size of each I_k is $n = N/K$. For each $k \in [K] = \{1, \dots, K\}$, define the auxiliary sample $I_k^c \equiv \{1, \dots, N\}$.
2. For each $k \in [K]$, use the auxiliary sample I_k^c to construct machine learning estimators

$$\hat{\pi}_k(x), \hat{g}_{1k}(x), \text{ and } \hat{g}_{2k}(x)$$

of $\pi_0(x)$, $g_{10}(x)$, and $g_{20}(x)$.

3. Construct the estimator of $f_{Y_1}(y)$ and $f_{Y_0}(y)$:

$$\hat{f}_{Y_1}(y) = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k} [m_1(Z, y, \hat{\eta}_{1k})] \quad \text{and} \quad \hat{f}_{Y_0}(y) = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k} [m_2(Z, y, \hat{\eta}_{2k})]$$

where $\mathbb{E}_{n,k} [m(Z)] = n^{-1} \sum_{i \in I_k} m(Z_i)$.

4. Construct the estimator for θ_1^* and θ_0^*

$$\hat{\theta}_1 = \arg \max_y \hat{f}_{Y_1}(y) \quad \text{and} \quad \hat{\theta}_0 = \arg \max_y \hat{f}_{Y_0}(y).$$

5. Construct the estimator for the mode treatment effect $\hat{\Delta} = \hat{\theta}_1 - \hat{\theta}_0$.

Theorem 4.1. *Suppose that with probability $1 - o(1)$, $\|\hat{\eta}_{1k} - \eta_{10}\|_{P,2} \leq \varepsilon_N$, $\|\hat{\pi}_k - 1/2\|_{P,\infty} \leq 1/2 - \kappa$, and $\|\hat{\pi}_k - \pi_0\|_{P,2}^2 + \|\hat{\pi}_k - \pi_0\|_{P,2} \times \|\hat{g}_{1k} - g_{10}\|_{P,2} \leq (\varepsilon_N)^2$. If $\varepsilon_N = o((Nh^3)^{-1/4})$ and $Nh^7 \rightarrow 0$, then we have*

$$\begin{aligned} \sqrt{nh^3} (\hat{\theta}_1 - \theta_1^*) &\xrightarrow{d} N(0, M_1^{-1} V_1 M_1^{-1}), \\ \sqrt{nh^3} (\hat{\theta}_0 - \theta_0^*) &\xrightarrow{d} N(0, M_0^{-1} V_0 M_0^{-1}). \end{aligned}$$

As for the variance estimation, recall that the kernel estimator of M_1 in the previous section is $\hat{M}_1 = N^{-1} \sum_{i=1}^N \hat{f}_{Y|D=1,X}^{(2)}(\hat{\theta}_1 | x)$, where

$$\hat{f}_{Y|D=1,X}^{(2)}(y|x) = \frac{\sum_{j=1}^n D_j K_h(y - Y_j)^{(2)} K_h(x - X_j)}{\sum_{j=1}^n D_j K_h(x - X_j)}.$$

Notice that we can divide both the numerator and the denominator by $\sum_{j=1}^n K_h(x - X_j)$ to obtain

$$\hat{f}_{Y|D=1,X}^{(2)}(y|x) = \frac{\sum_{j=1}^n D_j K_h(y - Y_j)^{(2)} K_h(x - X_j) / \sum_{j=1}^n K_h(x - X_j)}{\sum_{j=1}^n D_j K_h(x - X_j) / \sum_{j=1}^n K_h(x - X_j)}.$$

Observe that the numerator is an kernel estimator of $E[DK_h(y - Y) | X]$ and the denominator is an kernel estimator of the propensity score $E[D | X] = \pi(X)$. Hence, $\hat{f}_{Y|D=1,X}^{(2)}(y|x)$ is an estimator of $E[DK_h^{(2)}(y - Y) | X] / \pi(X)$. Hence, we can use the machine learning estimator of $E\left[\frac{DK_h^{(2)}(y-Y)}{\pi(X)}\right]$ as an estimator for M_1 . We can also construct a DML estimator using the Neyman-orthogonal functional form

$$\frac{DK_h^{(2)}(y - Y)}{\pi(X)} - \frac{D - \pi_0(X)}{\pi_0(X)} E\left[K_h^{(2)}(y - Y) | X, D = 1\right]$$

In step 1, we use machine learning methods to estimate $\pi_0(X)$ and $E[K_h^{(2)}(\hat{\theta}_1 - Y) | X, D = 1]$ using auxiliary sample I_k^c . In Step 2, we construct the DML estimator of M_1 :

$$\hat{M}_1 = \frac{1}{K} \sum_{k=1}^K \sum_{i \in I_k} \frac{D_i K_h^{(2)}(\hat{\theta}_1 - Y_i)}{\hat{\pi}(X_i)} - \frac{D_i - \hat{\pi}_0(X_i)}{\hat{\pi}_0(X_i)} \hat{E}\left[K_h^{(2)}(\hat{\theta}_1 - Y) | X_i, D = 1\right].$$

By the general DML theory (Chernozhukov et al., 2018), \hat{M}_1 is a consistent estimator of M_1 . Similarly, we can construct the DML estimators for V_1 , M_0 , and V_0 using the following table:

	Original Form	Equivalent Form
M_1	$E\left[f_{Y X,D=1}^{(2)}(\theta_1^* X)\right]$	$E\left[\frac{DK_h^{(2)}(\theta_1^* - Y)}{\pi(X)} - \frac{D - \pi_0(X)}{\pi_0(X)} E\left[K_h^{(2)}(y - Y) X, D = 1\right]\right]$
V_1	$\kappa_0^{(1)} E\left[\frac{f_{Y X,D=1}(\theta_1^* X)}{P(D=1 X)}\right]$	$E\left[\frac{DK_h(\theta_1^* - Y)}{\pi(X)^2} - 2\frac{D - \pi_0(X)}{\pi_0^2(X)} E\left[K_h(y - Y) X, D = 1\right]\right]$
M_0	$E\left[f_{Y X,D=0}^{(2)}(\theta_1^* X)\right]$	$E\left[\frac{(1-D)K_h^{(2)}(\theta_1^* - Y)}{1 - \pi(X)} - \frac{\pi_0(X) - D}{1 - \pi_0(X)} E\left[K_h^{(2)}(y - Y) X, D = 0\right]\right]$
V_0	$\kappa_0^{(1)} E\left[\frac{f_{Y X,D=0}(\theta_1^* X)}{P(D=0 X)}\right]$	$E\left[\frac{(1-D)K_h(\theta_1^* - Y)}{(1 - \pi(X))^2} - 2\frac{\pi_0(X) - D}{(1 - \pi_0(X))^2} E\left[K_h(y - Y) X, D = 0\right]\right]$

5 Conclusion

This paper studies the estimation and inference of the mode treatment effect, which has been ignored in the treatment effect literature compared to the estimation of the average and the

quantile treatment effect estimation. I propose both kernel and ML estimators to accommodate a variety of data sets faced by researchers. I also derive the asymptotic properties of the proposed estimators. I show that both estimators are consistent and asymptotically normal with the rate of convergence $\sqrt{Nh^3}$.

References

- Abadie, A., Angrist, J., & Imbens, G. (2002). Instrumental variables estimates of the effect of subsidized training on the quantiles of trainee earnings. *Econometrica*, 70(1), 91–117.
- Athey, S., Tibshirani, J., Wager, S., et al. (2019). Generalized random forests. *The Annals of Statistics*, 47(2), 1148–1178.
- Belloni, A., Chen, D., Chernozhukov, V., & Hansen, C. (2012). Sparse models and methods for optimal instruments with an application to eminent domain. *Econometrica*, 80(6), 2369–2429.
- Belloni, A., Chernozhukov, V., Fernández-Val, I., & Hansen, C. (2017). Program evaluation and causal inference with high-dimensional data. *Econometrica*, 85(1), 233–298.
- Belloni, A., Chernozhukov, V., & Hansen, C. (2014). Inference on treatment effects after selection among high-dimensional controls†. *The Review of Economic Studies*, 81(2), 608–650.
- Bitler, M. P., Gelbach, J. B., & Hoynes, H. W. (2006). What mean impacts miss: Distributional effects of welfare reform experiments. *American Economic Review*, 96(4), 988–1012.
- Card, D. (1996). The effect of unions on the structure of wages: A longitudinal analysis. *Econometrica: Journal of the Econometric Society*, 957–979.
- Chen, Y.-C., Genovese, C. R., Tibshirani, R. J., Wasserman, L., et al. (2016). Nonparametric modal regression. *The Annals of Statistics*, 44(2), 489–514.
- Chernozhukov, V., Chetverikov, D., Demirer, M., Duflo, E., Hansen, C., Newey, W., & Robins, J. (2018). Double/debiased machine learning for treatment and structural parameters. *The Econometrics Journal*, 21(1), C1–C68.
- Chernozhukov, V., Escanciano, J. C., Ichimura, H., & Newey, W. K. (2016). Locally robust semiparametric estimation. *arXiv preprint arXiv:1608.00033*.
- Chernozhukov, V., & Hansen, C. (2005). An iv model of quantile treatment effects. *Econometrica*, 73(1), 245–261.
- Chernozhukov, V., Hansen, C., & Spindler, M. (2015). Valid post-selection and post-regularization inference: An elementary, general approach. *Annu. Rev. Econ.*, 7(1), 649–688.

- DiNardo, J., Fortin, N. M., & Lemieux, T. (1995). *Labor market institutions and the distribution of wages, 1973-1992: A semiparametric approach* (Tech. Rep.). National bureau of economic research.
- Eddy, W. F., et al. (1980). Optimum kernel estimators of the mode. *The Annals of Statistics*, 8(4), 870–882.
- Firpo, S. (2007). Efficient semiparametric estimation of quantile treatment effects. *Econometrica*, 75(1), 259–276.
- Freeman, R. B. (1980). Unionism and the dispersion of wages. *ILR Review*, 34(1), 3–23.
- Hahn, J. (1998). On the role of the propensity score in efficient semiparametric estimation of average treatment effects. *Econometrica*, 315–331.
- Hansen, B. E. (2008). Uniform convergence rates for kernel estimation with dependent data. *Econometric Theory*, 24(3), 726–748.
- Heckman, J., Ichimura, H., & Todd, P. E. (1997). Matching as an econometric evaluation estimator: Evidence from evaluating a job training programme. *The review of economic studies*, 64(4), 605–654.
- Heckman, J., & Robb, R. (1985). Alternative methods for evaluating the impact of interventions: An overview. *Journal of econometrics*, 30(1-2), 239–267.
- Hirano, K., Imbens, G. W., & Ridder, G. (2003). Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica*, 71(4), 1161–1189.
- Lee, M.-J. (1989). Mode regression. *Journal of Econometrics*, 42(3), 337–349.
- Newey, W. K., & McFadden, D. (1994). Large sample estimation and hypothesis testing. *Handbook of econometrics*, 4, 2111–2245.
- Parzen, E. (1962). On estimation of a probability density function and mode. *The annals of mathematical statistics*, 33(3), 1065–1076.
- Robins, J. M., & Rotnitzky, A. (1995). Semiparametric efficiency in multivariate regression models with missing data. *Journal of the American Statistical Association*, 90(429), 122–129.
- Rosenbaum, P. R., & Rubin, D. B. (1983). The central role of the propensity score in observational studies for causal effects. *Biometrika*, 70(1), 41–55.
- Tauchén, G. (1985). Diagnostic testing and evaluation of maximum likelihood models. *Journal of Econometrics*, 30(1-2), 415–443.
- Van der Vaart, A. W. (2000). *Asymptotic statistics* (Vol. 3). Cambridge university press.
- Yao, W., & Li, L. (2014). A new regression model: modal linear regression. *Scandinavian Journal of Statistics*, 41(3), 656–671.

6 Appendix

Proof of Theorem 1: We only present the proof of the first claim, $\hat{\theta}_1 \xrightarrow{p} \theta_1^*$, since the second claim follows from the same arguments. The proof proceeds in two steps. In Step 1, we show the uniform law of large number holds

$$\sup_y |\hat{f}_{Y_1}(y) - f_{Y_1}(y)| = o_p(1).$$

In Step 2, we establish the consistency $\hat{\theta}_1 \xrightarrow{p} \theta_1^*$ using the same argument of Theorem 5.7 in [Van der Vaart \(2000\)](#).

Step 1. Notice that we have the decomposition

$$\begin{aligned} \hat{f}_{Y_1}(y) - f_{Y_1}(y) &= \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|D=1,X}(y | X_i) - E \left[f_{Y|D=1,X}(y | X) \right] \\ &= \underbrace{\frac{1}{n} \sum_{i=1}^n \left(\hat{f}_{Y|D=1,X}(y | X_i) - f_{Y|D=1,X}(y | X_i) \right)}_{A(y)} \\ &\quad + \underbrace{\frac{1}{n} \sum_{i=1}^n f_{Y|D=1,X}(y | X_i) - E \left[f_{Y|D=1,X}(y | X) \right]}_{B(y)}. \end{aligned}$$

Hence,

$$\sup_y |\hat{f}_{Y_1}(y) - f_{Y_1}(y)| \leq \sup_y |A(y)| + \sup_y |B(y)|$$

By Theorem 6 in [Hansen \(2008\)](#) (uniform rates of convergence of kernel estimators), the first term $\sup_y |A(y)|$ is bounded by

$$\begin{aligned} \sup_y |A(y)| &\leq \sup_y \frac{1}{n} \sum_{i=1}^n \left| \hat{f}_{Y|D=1,X}(y | X_i) - f_{Y|D=1,X}(y | X_i) \right| \\ &\leq \sup_y \sup_x \left| \hat{f}_{Y|D=1,X}(y | x) - f_{Y|D=1,X}(y | x) \right| \\ &\leq \sup_{x,y} \left| \hat{f}_{Y|D=1,X}(y | x) - f_{Y|D=1,X}(y | x) \right| \\ &= O_p \left(\sqrt{\frac{\ln n}{nh^{d+1}}} + h^2 \right) \\ &= o_p(1). \end{aligned}$$

On the other hand, by Lemma 1 of [Tauchen \(1985\)](#) (uniform law of large numbers), we have

$$\sup_{y \in \mathcal{Y}} |B(y)| = \sup_{y \in \mathcal{Y}} \left| \frac{1}{n} \sum_{i=1}^n f_{Y|D=1,X}(y | X_i) - E \left[f_{Y|D=1,X}(y | X) \right] \right| \xrightarrow{p} 0.$$

Combining the results of $\sup_y |A(y)|$ and $\sup_y |B(y)|$ gives

$$\sup_y |\hat{f}_{Y_1}(y) - f_{Y_1}(y)| = o_p(1).$$

Step 2. The definition of $\hat{\theta}_1$ implies that $\hat{f}_{Y_1}(\hat{\theta}_1) \geq \hat{f}_{Y_1}(\theta_1^*)$. Therefore, we have

$$\begin{aligned} f_{Y_1}(\theta_1^*) - f_{Y_1}(\hat{\theta}_1) &= f_{Y_1}(\theta_1^*) - \hat{f}_{Y_1}(\theta_1^*) + \hat{f}_{Y_1}(\theta_1^*) - f_{Y_1}(\hat{\theta}_1) \\ &\leq f_{Y_1}(\theta_1^*) - \hat{f}_{Y_1}(\theta_1^*) + \hat{f}_{Y_1}(\hat{\theta}_1) - f_{Y_1}(\hat{\theta}_1) \\ &\leq 2 \sup_y |\hat{f}_{Y_1}(y) - f_{Y_1}(y)|. \end{aligned}$$

By Step 1, we have that for any $\delta > 0$,

$$P\left(f_{Y_1}(\theta_1^*) - f_{Y_1}(\hat{\theta}_1) > \delta\right) \leq P\left(\sup_y |\hat{f}_{Y_1}(y) - f_{Y_1}(y)| > \delta/2\right) \rightarrow 0.$$

Further, Assumption 1 implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{y: |y - \theta_1^*| > \varepsilon} f_{Y_1}(y) < f_{Y_1}(\theta_1^*) - \delta.$$

Then the following inequality holds

$$\begin{aligned} P\left(|\hat{\theta}_1 - \theta_1^*| > \varepsilon\right) &\leq P\left(f_{Y_1}(\hat{\theta}_1) < f_{Y_1}(\theta_1^*) - \delta\right) \\ &\leq P\left(f_{Y_1}(\theta_1^*) - f_{Y_1}(\hat{\theta}_1) > \delta\right) \rightarrow 0. \end{aligned}$$

Thus, we prove the consistency $\hat{\theta}_1 \xrightarrow{P} \theta_1^*$.

Proof of Theorem 2: Here we focus on the result for $\hat{\theta}_1$ only. Notice that the first-order condition for $\hat{\theta}_1$ gives

$$\begin{aligned} 0 &= \hat{f}_{Y_1}^{(1)}(\hat{\theta}_1) = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(1)}(\hat{\theta}_1 | X_i) \\ &= \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) + \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(2)}(\tilde{\theta}_1 | X_i) (\hat{\theta}_1 - \theta_1^*), \end{aligned}$$

where $\tilde{\theta}_1 \in (\hat{\theta}_1, \theta_1^*)$. Then we have

$$\sqrt{nh^3} (\hat{\theta}_1 - \theta_1^*) = - \left[\frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(2)}(\tilde{\theta}_1 | X_i) \right]^{-1} \left(\frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) \right).$$

The proof proceeds in six steps. In Step 1, we show that the first term of r.h.s converges to $M_1 = E \left[f_{Y|X,D=1}^{(2)}(\theta_1^* | X) \right]$ in probability. In Step 2-5, we show the asymptotic normality of the second term. Then, by Slutsky's theorem, we can show the asymptotic normality for $\hat{\theta}_1$. In Step 6, we show the asymptotic normality for $\hat{\Delta}$.

For convenience, we define $\gamma_{10}(x) \equiv f_{Y,X|D=1}^{(1)}(\theta_1^*, x)$, $\gamma_{20}(x) \equiv f_{X|D=1}(x)$, and

$$\begin{aligned}\hat{\gamma}_1(x) &\equiv \frac{1}{n} \sum_{j=1}^n \frac{D_j K_h^{(1)}(\theta_1^* - Y_j) K_h(x - X_j)}{P(D=1)} \\ \hat{\gamma}_2(x) &\equiv \frac{1}{n} \sum_{j=1}^n \frac{D_j K_h(x - X_j)}{P(D=1)}.\end{aligned}$$

In these notations, we can express $\hat{f}_{Y|X,D=1}^{(1)}(\theta_1^* | x)$ and $f_{Y|X,D=1}^{(1)}(\theta_1^* | x)$ as $\hat{\gamma}_1(x) / \hat{\gamma}_2(x)$ and $\gamma_{10}(x) / \gamma_{20}(x)$, respectively. Also, let $\gamma_0 = (\gamma_{10}, \gamma_{20})'$ and $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2)'$.

Step 1. In this step, we show that $n^{-1} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) \xrightarrow{p} E \left[f_{Y|X,D=1}^{(2)}(\theta_1^* | X) \right]$.

Notice that

$$\frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) = \frac{1}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(2)}(\theta_1^* | X_i) + A_1 + A_2$$

where

$$A_1 = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) - f_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i)$$

and

$$A_2 = \frac{1}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) - f_{Y|X,D=1}^{(2)}(\theta_1^* | X_i).$$

Since $\frac{1}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(2)}(\theta_1^* | X_i) \xrightarrow{p} E \left[f_{Y|X,D=1}^{(2)}(\theta_1^* | X) \right]$ by the law of large numbers, we only have to show that $A_1 = o_p(1)$ and $A_2 = o_p(1)$. Note that

$$\begin{aligned}|A_1| &\leq \frac{1}{n} \sum_{i=1}^n \left| \hat{f}_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) - f_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) \right| \\ &\leq \sup_{y,x} \left| \hat{f}_{Y|X,D=1}^{(2)}(y | x) - f_{Y|X,D=1}^{(2)}(y | x) \right| \\ &= O_p \left(\sqrt{\frac{\ln n}{nh^{d+5}}} + h^2 \right) \\ &= o_p(1),\end{aligned}$$

where the first equality follows from the uniform rates of convergence of kernel estimators (Hansen, 2008). For A_2 , we use the argument in Lemma 4.3 of Newey & McFadden (1994). By consistency of $\hat{\theta}_1$, and thus $\tilde{\theta}_1$, there is $\delta_n \rightarrow 0$ such that $\|\tilde{\theta}_1 - \theta_1^*\| \leq \delta_n$ with probability approaching to one. Define

$$\Delta_n(X_i) = \sup_{\|y - \theta_1^*\| \leq \delta_n} \left\| f_{Y|X, D=1}^{(2)}(y | X_i) - f_{Y|X, D=1}^{(2)}(\theta_1^* | X_i) \right\|.$$

By the continuity of $f_{Y|X, D=1}^{(2)}(y | X_i)$ at θ_1^* , $\Delta_n(X_i) \xrightarrow{P} 0$. Hence, by the dominated convergence theorem, we have $E[\Delta_n(X_i)] \rightarrow 0$. Then, by Markov's inequality,

$$P\left(\frac{1}{n} \sum_{i=1}^n \Delta_n(X_i) > \epsilon\right) \leq E[\Delta_n(X_i)] / \epsilon \rightarrow 0.$$

Therefore, we have

$$|A_2| \leq \frac{1}{n} \sum_{i=1}^n \Delta_n(X_i) + o_p(1) = o_p(1).$$

Step 2. In this step, we show

$$\frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) = \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n f_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) + \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n G(Z_i, \hat{\gamma} - \gamma_0) + o_p(1),$$

where $G(z, \gamma) = \gamma_{20}(x)^{-1} \left[1, -\frac{\gamma_{10}(x)}{\gamma_{20}(x)}\right] \gamma(x)$ and $z = (y, x, d)$ denotes data observation. To do this, it suffices to show

$$\frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \left[\hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) - f_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) - G(Z_i, \hat{\gamma} - \gamma_0) \right] = o_p(1).$$

Using the notation of γ , we have

$$\hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | x) - f_{Y|X, D=1}^{(1)}(\theta_1^* | x) = \frac{\hat{\gamma}_1(x)}{\hat{\gamma}_2(x)} - \frac{\gamma_{10}(x)}{\gamma_{20}(x)}.$$

The following argument follows from [Newey & McFadden \(1994\)](#). Consider the algebra relation $\tilde{a}/\tilde{b} - a/b = b^{-1} \left[1 - \tilde{b}^{-1}(\tilde{b} - b)\right] \left[\tilde{a} - a - (a/b)(\tilde{b} - b)\right]$. The linear part of the r.h.s is $b^{-1} \left[\tilde{a} - a - (a/b)(\tilde{b} - b)\right]$, and the remaining term is of higher order. By letting $a = \gamma_{10}$, $\tilde{a} = \hat{\gamma}_1$, $b = \gamma_{20}$, and $\tilde{b} = \hat{\gamma}_2$, this linear term corresponds to the linear functional $G(Z_i, \hat{\gamma} - \gamma_0)$. The remaining higher-order term will satisfy

$$\begin{aligned} & \left| \frac{\gamma_1(x)}{\gamma_2(x)} - \frac{\gamma_{10}(x)}{\gamma_{20}(x)} - G(z, \gamma - \gamma_0) \right| \\ & \leq |\gamma_2(x)|^{-1} \gamma_{20}(x)^{-1} \left[1 + \frac{\gamma_{10}(x)}{\gamma_{20}(x)}\right] \left[(\gamma_1(x) - \gamma_{10}(x))^2 + (\gamma_2(x) - \gamma_{20}(x))^2 \right] \\ & \leq C \sup_{x \in \mathcal{X}} \|\gamma(x) - \gamma_0(x)\|^2 \end{aligned}$$

for some constant C if γ_2 and γ_{20} are bounded away from zero. Hence Lemma 1 holds if $\sqrt{nh^3} \sup_{x \in \mathcal{X}} \|\hat{\gamma}(x) - \gamma_0(x)\|^2 \xrightarrow{p} 0$. By the uniform rates of convergence of kernel estimators (Hansen, 2008), we have

$$\begin{aligned} \sup_{x \in \mathcal{X}} \|\hat{\gamma}(x) - \gamma_0(x)\|^2 &= \sup_{x \in \mathcal{X}} \left((\hat{\gamma}_1(x) - \gamma_{10}(x))^2 + (\hat{\gamma}_2(x) - \gamma_{20}(x))^2 \right) \\ &\leq \sup_{x \in \mathcal{X}} (\hat{\gamma}_1(x) - \gamma_{10}(x))^2 + \sup_{x \in \mathcal{X}} (\hat{\gamma}_2(x) - \gamma_{20}(x))^2 \\ &= O_p \left[(\ln n) (nh^{d+3})^{-1} + h^4 \right] + O_p \left[(\ln n) (nh^d)^{-1} + h^4 \right] \\ &= O_p \left[(\ln n) (nh^{d+3})^{-1} + h^4 \right]. \end{aligned}$$

The rates of h and n imply that $\sqrt{nh^3} \sup_{x \in \mathcal{X}} \|\hat{\gamma}(x) - \gamma_0(x)\|^2 \xrightarrow{p} 0$.

Step 3. In this step, we show

$$\begin{aligned} \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) &= \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n f_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) + \sqrt{nh^3} \int G(z, \hat{\gamma} - \gamma_0) dF_0(z) \\ &\quad + o_p(1), \end{aligned}$$

where F_0 is the c.d.f. of z . To do this, it suffices to show that

$$\sqrt{nh^3} \left\{ \frac{1}{n} \sum_{i=1}^n G(Z_i, \hat{\gamma} - \gamma_0) - \int G(z, \hat{\gamma} - \gamma_0) dF_0(z) \right\} = o_p(1).$$

Let $\bar{\gamma} \equiv E[\hat{\gamma}]$ and by the linearity of $G(z, \gamma)$, we have the decomposition

$$G(z, \hat{\gamma} - \gamma_0) = G(z, \hat{\gamma} - \bar{\gamma}) + G(z, \bar{\gamma} - \gamma_0).$$

Therefore we just need to show that

$$\sqrt{nh^3} \left\{ \frac{1}{n} \sum_{i=1}^n G(Z_i, \hat{\gamma} - \bar{\gamma}) - \int G(z, \hat{\gamma} - \bar{\gamma}) dF_0(z) \right\} = o_p(1)$$

and

$$\sqrt{nh^3} \left\{ \frac{1}{n} \sum_{i=1}^n G(Z_i, \bar{\gamma} - \gamma_0) - \int G(z, \bar{\gamma} - \gamma_0) dF_0(z) \right\} = o_p(1).$$

The second condition holds by the central limit theorem since

$$\sqrt{nh^3} \left\{ \frac{1}{n} \sum_{i=1}^n G(Z_i, \bar{\gamma} - \gamma_0) - \int G(z, \bar{\gamma} - \gamma_0) dF_0(z) \right\} = \sqrt{nh^3} O_p(n^{-1/2}) = o_p(1).$$

It remains to show the first condition. We follow the arguments in [Newey & McFadden \(1994\)](#). Define $q_j \equiv \left(\frac{D_j K_h^{(1)}(\theta_1^* - Y_j)}{P(D=1)}, \frac{D_j}{P(D=1)} \right)'$, we can rewrite

$$\hat{\gamma}(x) = \begin{bmatrix} \hat{\gamma}_1(x) \\ \hat{\gamma}_2(x) \end{bmatrix} = \frac{1}{n} \sum_{j=1}^n q_j K_h(x - X_j).$$

We also define

$$\begin{aligned} m(Z_i, Z_j) &= G \left[Z_i, q_j K_h(\cdot - X_j) \right] \\ m_1(z) &= \int m(z, \tilde{z}) dF_0(\tilde{z}) = G(z, \bar{\gamma}) \\ m_2(z) &= \int m(\tilde{z}, z) dF_0(\tilde{z}) = \int G[\tilde{z}, q K_h(\cdot - X)] dF_0(\tilde{z}). \end{aligned}$$

Then the l.h.s. of the first condition equals

$$\begin{aligned} & \sqrt{nh^3} \left\{ \frac{1}{n} \sum_{i=1}^n G(Z_i, \hat{\gamma} - \bar{\gamma}) - \int G(z, \hat{\gamma} - \bar{\gamma}) dF_0(z) \right\} \\ &= \sqrt{nh^3} \left\{ \frac{1}{n} \sum_{i=1}^n G(z, \hat{\gamma}) - \frac{1}{n} \sum_{i=1}^n G(z, \bar{\gamma}) - \int G(z, \hat{\gamma}) dF_0(z) + \int G(z, \bar{\gamma}) dF_0(z) \right\} \\ &= \sqrt{nh^3} \left\{ \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n m(Z_i, Z_j) - \frac{1}{n} \sum_{i=1}^n m_1(Z_i) - \frac{1}{n} \sum_{i=1}^n m_2(Z_i) + E[m_1(z)] \right\} \\ &= \sqrt{nh^3} \times O_p \left\{ E[|m(Z_1, Z_1)|] / n + \left(E[|m(Z_1, Z_2)|^2] \right)^{1/2} / n \right\}, \end{aligned}$$

where the last equality follows from Lemma 8.4 of [Newey & McFadden \(1994\)](#). The last term converges to zero in probability if we can control the convergence rates of $E[|m(Z_1, Z_1)|]$ and $E[|m(Z_1, Z_2)|^2]$. Notice that we have $|G(z, \gamma)| \leq b(z) \|\gamma\|_2$ with

$$b(z) = \left\| f_{X|D=1}(x)^{-1} \left[1, -f_{Y|X, D=1}^{(1)}(\theta_1^*, x) \right] \right\|_2$$

where $\|\cdot\|_2$ denotes the ℓ_2 norm. Then $E \left[\left| G(z, q K_h(\cdot - x)) \right| \right] \leq b(z) h^{-d} \|q\|_2$ by the boundedness of $K(u)$. By that $f_{X|D=1}(x)$ is bounded away from zero and $f_{Y|X, D=1}(\theta_1^*, x)$ is

bounded from above, we have that $E [b(z)^2] \leq \infty$. Therefore, we have

$$\begin{aligned}
& \sqrt{nh^3} \times O_p \left\{ E [|m(Z_1, Z_1)|] / n + \left(E [|m(Z_1, Z_2)|^2] \right)^{1/2} / n \right\} \\
&= \sqrt{nh^3} \times O_p \left\{ E [\|q\|_2 b(Z_1)] / n + \left(E [\|q\|_2^2 b(Z_1)^2] \right)^{1/2} (nh^d)^{-1} \right\} \\
&= \sqrt{nh^3} \times O_p (n^{-1}h^{-d-2}) \\
&= o_p(1)
\end{aligned}$$

by the assumptions on n and h . The additional h^{-2} in the rates of convergence follows from that q contains $K_h^{(1)}(u) = h^{-2}K^{(1)}(u/h)$ with bounded $K^{(1)}(u)$.

Step 4. In this step, we show that

$$\frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \hat{f}_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) = \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n f_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) + \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n v(X_i) q_i + o_p(1),$$

where $v(X_i) = \frac{P(D=1)}{P(D=1|X_i)} \left[1, -\frac{\gamma_{10}(X_i)}{\gamma_{20}(X_i)} \right]$ and $q_i = \left(\frac{D_i K_h^{(1)}(\theta_1^* - Y_i)}{P(D=1)}, \frac{D_i}{P(D=1)} \right)'$. To do this, it suffices to show that

$$\sqrt{nh^3} \int G(z, \hat{\gamma} - \gamma_0) dF_0(z) - \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n v(X_i) q_i = o_p(1)$$

Notice that

$$\begin{aligned}
\int G(z, \gamma) dF_0(z) &= \int \gamma_{20}(x)^{-1} \left[1, -\frac{\gamma_{10}(x)}{\gamma_{20}(x)} \right] \gamma(x) f_X(x) dx \\
&= \int f_{X|D=1}(x)^{-1} \left[1, -\frac{\gamma_{10}(x)}{\gamma_{20}(x)} \right] \gamma(x) f_X(x) dx \\
&= \int \frac{P(D=1)}{P(D=1|X=x)} \left[1, -\frac{\gamma_{10}(x)}{\gamma_{20}(x)} \right] \gamma(x) dx \\
&= \int v(x) \gamma(x) dx,
\end{aligned}$$

where $f_X(x)$ is the density function of X and $v(x) = \frac{P(D=1)}{P(D=1|X=x)} \left[1, -\frac{\gamma_{10}(x)}{\gamma_{20}(x)} \right]$. Also, we have

$$\begin{aligned}
v(x) \gamma_0(x) &= \frac{P(D=1)}{P(D=1|X=x)} \left[1, -\frac{\gamma_{10}(x)}{\gamma_{20}(x)} \right] \begin{bmatrix} \gamma_{10}(x) \\ \gamma_{20}(x) \end{bmatrix} \\
&= \frac{P(D=1)}{P(D=1|X=x)} (\gamma_{10}(x) - \gamma_{10}(x)) \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int G(z, \hat{\gamma} - \gamma_0) dF_0(z) &= \int v(x) \hat{\gamma}(x) dx - \int v(x) \gamma_0(x) dx = \int v(x) \hat{\gamma}(x) dx \\
&= \frac{1}{n} \sum_{i=1}^n \int v(x) q_i K_h(x - X_i) dx \\
&= \frac{1}{n} \sum_{i=1}^n v(X_i) q_i + \left(\frac{1}{n} \sum_{i=1}^n \int v(x) q_i K_h(x - X_i) dx - \frac{1}{n} \sum_{i=1}^n v(X_i) q_i \right) \\
&= \frac{1}{n} \sum_{i=1}^n v(X_i) q_i + \frac{1}{n} \sum_{i=1}^n \left[\int v(x) K_h(x - X_i) dx - v(X_i) \right] q_i.
\end{aligned}$$

By Chebyshev's inequality, sufficient conditions for $\sqrt{nh^3}$ times the second term in the last line converging to zero in probability are that

$$\sqrt{nh^3} E \left[\left(\int v(x) K_h(x - X_i) dx - v(X_i) \right) q_i \right] \rightarrow 0$$

and

$$E \left[\|q_i\|^2 \left\| \int v(x) K_h(x - X_i) dx - v(X_i) \right\|^2 \right] \rightarrow 0.$$

The expectation in the first condition is the difference of $E \left[\left(\int v(x) K_h(x - X_i) dx \right) q_i \right]$ and $E \left[v(X_i) q_i \right]$. We begin with the second term $E \left[v(X_i) q_i \right]$. Notice that

$$\begin{aligned}
E \left[v(X_i) q_i \right] &= E \left[v(X_i) E \left[q_i \mid X_i \right] \right] \\
&= E \left[v(X_i) E \left[\left(\frac{D_i K_h^{(1)}(\theta_1^* - Y_i)}{\frac{P(D=1)}{P(D_i)}} \right) \mid X_i \right] \right] \\
&= E \left[v(X_i) \frac{P(D=1 \mid X_i)}{P(D=1)} E \left[\left(K_h^{(1)} \left(\frac{\theta_1^* - Y_i}{1} \right) \right) \mid X_i, D=1 \right] \right]
\end{aligned}$$

by the law of iterated expectations. The inner conditional expectation in the last line satisfies

$$\begin{aligned}
E \left[K_h^{(1)} (\theta_1^* - Y_i) \mid X_i, D = 1 \right] &= \frac{1}{h^2} E \left[K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \mid X_i, D = 1 \right] \\
&= \frac{1}{h^2} \int K^{(1)} \left(\frac{\theta_1^* - y}{h} \right) f_{Y|X, D=1} (y \mid X_i) dy \\
&= \frac{1}{h} \int K \left(\frac{\theta_1^* - y}{h} \right) f_{Y|X, D=1}^{(1)} (y \mid X_i) dy \\
&= \int K(u) f_{Y|X, D=1}^{(1)} (\theta_1^* + hu \mid X_i) du \\
&= \int K(u) f_{Y|X, D=1}^{(1)} (\theta_1^* \mid X_i) du \\
&\quad + \int huK(u) f_{Y|X, D=1}^{(1)} (\theta_1^* \mid X_i) du \\
&\quad + \int \frac{h^2 u^2}{2} K(u) f_{Y|X, D=1}^{(3)} (\tilde{\theta}_1 \mid X_i) du \\
&= f_{Y|X, D=1}^{(1)} (\theta_1^* \mid X_i) + \frac{h^2}{2} \kappa_2 f_{Y|X, D=1}^{(3)} (\tilde{\theta}_1 \mid X_i)
\end{aligned}$$

with $\tilde{\theta}_1 \in (\theta_1^*, \theta_1^* + hu)$ and $\kappa_2 = \int u^2 K(u) du$. The third equality follows from integration by parts and the fourth from change of variables. Hence,

$$\begin{aligned}
E [v(X_i) q_i] &= E \left[v(X_i) \frac{P(D=1 \mid X_i)}{P(D=1)} \left(f_{Y|X, D=1}^{(1)} \left(\frac{\theta_1^*}{1} \mid X_i \right) \right) \right] \\
&\quad + \frac{h^2}{2} \kappa_2 E \left[v(X_i) \frac{P(D=1 \mid X_i)}{P(D=1)} \left(f_{Y|X, D=1}^{(3)} \left(\frac{\tilde{\theta}_1}{0} \mid X_i \right) \right) \right] \\
&= E \left[v(X_i) \frac{P(D=1 \mid X_i)}{P(D=1)} \left(f_{Y|X, D=1}^{(1)} \left(\frac{\theta_1^*}{1} \mid X_i \right) \right) \right] + O(h^2) \\
&= \int v(x) \frac{P(D=1 \mid X_i = x)}{P(D=1)} \left(f_{Y|X, D=1}^{(1)} \left(\frac{\theta_1^*}{1} \mid X_i = x \right) \right) f_X(x) dx + O(h^2) \\
&= \int v(x) \left(f_{Y|X, D=1}^{(1)} \left(\frac{\theta_1^*}{1} \mid X_i = x \right) \right) f_{X|D=1}(x) dx + O(h^2) \\
&= \int v(x) \left(\frac{f_{Y, X|D=1}^{(1)} (\theta_1^* \mid X_i = x)}{f_{X|D=1}(x)} \right) dx + O(h^2) \\
&= \int v(x) \gamma_0(x) dx + O(h^2).
\end{aligned}$$

Using the same arguments, we can also show that

$$\begin{aligned} E \left[\left(\int v(x) K_h(x - X_i) dx \right) q_i \right] &= E \left[\left(\int v(X_i + hu) K(u) du \right) q_i \right] \\ &= \int \left(\int v(x + hu) K(u) du \right) \gamma_0(x) dx + O(h^2) \end{aligned}$$

Then the first condition equals

$$\begin{aligned} &\sqrt{nh^3} \left\| E \left[\left(\int v(x) K_h(x - X_i) dx - v(X_i) \right) q_i \right] \right\| \\ &= \sqrt{nh^3} \left\| \int \left(\int v(x + hu) K(u) du \right) \gamma_0(x) dx - \int v(x) \gamma_0(x) dx + O(h^2) \right\|. \end{aligned}$$

Following the argument in Theorem 8.11 of [Newey & McFadden \(1994\)](#), the last line satisfies

$$\begin{aligned} &\sqrt{nh^3} \left\| \int \int v(x) K(u) \gamma_0(x - hu) dudx - \int v(x) \gamma_0(x) dx + O(h^2) \right\| \\ &= \sqrt{nh^3} \left\| \int v(x) \left\{ \int [\gamma_0(x - hu) - \gamma_0(x)] du \right\} dx + O(h^2) \right\| \\ &\leq \sqrt{nh^3} \int \|v(x)\| \left\| \int [\gamma_0(x - hu) - \gamma_0(x)] du \right\| dx + O(\sqrt{nh^3}h^2) \\ &\leq \sqrt{nh^3}Ch^2 \int \|v(x)\| dx + O(\sqrt{nh^3}h^2) \\ &= O(\sqrt{nh^3}h^2). \end{aligned}$$

Therefore the first condition holds if $\sqrt{nh^3}h^2 \rightarrow 0$.

Recall that the second condition we would like to show is

$$E \left[\|q_i\|^2 \left\| \int v(x) K_h(x - X_i) dx - v(X_i) \right\|^2 \right] \rightarrow 0.$$

By Cauchy Schwartz inequality, it suffices to show that

$$E \left[\left\| \int v(x) K_h(x - X_i) dx - v(X_i) \right\|^4 \right] \rightarrow 0.$$

By the continuity of $v(x)$, $v(x + hu) \rightarrow v(x)$ for all x and u as $h \rightarrow 0$. By the dominated convergence theorem, $\int v(x) K_h(x - x_i) dx = \int v(x + hu) K(u) du \rightarrow \int v(x) K(u) du = v(x)$ for all x . Therefore we have

$$E \left[\left\| \int v(x) K_h(x - X_i) dx - v(X_i) \right\|^4 \right] = E \left[\left\| \int v(X_i + hu) K(u) du - v(X_i) \right\|^4 \right] \rightarrow 0.$$

Step 5. By Step 4 and the definition of $v(X_i)$ and q_i , we have

$$\begin{aligned}
\frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) &= \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) + \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n v(X_i) q_i + o_p(1) \\
&= \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) + o_p(1) \\
&+ \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} \left[K_h^{(1)}(\theta_1^* - Y_i) - f_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) \right] \\
&= \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) \left[1 - \frac{D_i}{P(D=1 | X_i)} \right] \\
&+ \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} K_h^{(1)}(\theta_1^* - Y_i) + o_p(1).
\end{aligned}$$

Since we have $E \left[f_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) \left[1 - \frac{D_i}{P(D=1 | X_i)} \right] \right] = 0$ by the law of iterated expectations, the central limit theorem holds for the first term of r.h.s. Hence,

$$\begin{aligned}
\frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) &= O_p(\sqrt{nh^3} n^{-1/2}) + \frac{\sqrt{nh^3}}{nh^2} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right) \\
&+ o_p(1) \\
&= O_p(\sqrt{h^3}) + \frac{1}{\sqrt{nh}} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right) + o_p(1) \\
&= \frac{1}{\sqrt{nh}} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right) + o_p(1).
\end{aligned}$$

In this step, we show that

$$\frac{1}{\sqrt{nh}} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right) \xrightarrow{d} N(0, V),$$

where $V = \kappa_0^{(1)} E \left[\frac{f_{Y|X,D=1}(\theta_1^* | X)}{P(D=1 | X)} \right]$ and $\kappa_0^{(1)} = \int K^{(1)}(u)^2 du$.

For convenience, we define $\hat{g}(\theta_1^*) \equiv (nh^2)^{-1} \sum_{i=1}^n \frac{D_i}{P(D=1 | X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right)$. Then it is equivalent to show that

$$\sqrt{nh^3} (\hat{g}(\theta_1^*) - 0) \xrightarrow{d} N(0, V).$$

To use central limit theorem, we have to calculate $E [\hat{g}(\theta_1^*)]$ and $Var(\hat{g}(\theta_1^*))$.

$$\begin{aligned} E[\hat{g}(\theta_1^*)] &= \frac{1}{h^2} E \left[\frac{D_i}{P(D=1 | X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \right] \\ &= \frac{1}{h^2} E \left[\frac{1}{P(D=1 | X_i)} E \left[D_i K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \mid X_i \right] \right] \\ &= \frac{1}{h^2} E \left[E \left[K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \mid X_i, D=1 \right] \right]. \end{aligned}$$

Since $h^{-2} E \left[K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \mid X_i, D=1 \right] = f_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) + \frac{h^2}{2} \kappa_2 f_{Y|X, D=1}^{(3)}(\tilde{\theta}_1 | X_i)$ from the calculation in Step 4, then

$$E[\hat{g}(\theta_1^*)] = E \left[f_{Y|X, D=1}^{(1)}(\theta_1^* | X_i) \right] + O(h^2) = 0 + O(h^2).$$

For the variance,

$$\begin{aligned} Var(\hat{g}(\theta_1^*)) &= \frac{1}{nh^4} Var \left(\frac{D_i}{P(D=1 | X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \right) \\ &= \frac{1}{nh^4} E \left[\left(\frac{D_i}{P(D=1 | X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \right)^2 \right] \\ &\quad - \frac{1}{nh^4} \left(E \left[\frac{D_i}{P(D=1 | X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \right] \right)^2 \\ &= \frac{1}{nh^4} E \left[\left(\frac{D_i}{P(D=1 | X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \right)^2 \right] + \frac{1}{nh^4} O(h^4) \\ &= \frac{1}{nh^4} E \left[\frac{1}{P(D=1 | X_i)^2} E \left[D_i^2 K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right)^2 \mid X \right] \right] + \frac{1}{nh^4} O(h^4) \\ &= \frac{1}{nh^4} E \left[\frac{1}{P(D=1 | X_i)} E \left[K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right)^2 \mid X, D=1 \right] \right] + \frac{1}{nh^4} O(h^4). \end{aligned}$$

The inner expectation in the last line equals

$$\begin{aligned}
E \left[K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right)^2 \mid X, D = 1 \right] &= \int K^{(1)} \left(\frac{\theta_1^* - y}{h} \right)^2 f_{Y|X, D=1}(y \mid X) dy \\
&= h \int K^{(1)}(u)^2 f_{Y|X, D=1}(\theta_1^* + hu \mid X) du \\
&= h f_{Y|X, D=1}(\theta_1^* \mid X) \int K^{(1)}(u)^2 du \\
&\quad + h^2 f_{Y|X, D=1}^{(1)}(\tilde{\theta}_1 \mid X) \int u K^{(1)}(u)^2 du,
\end{aligned}$$

where $\tilde{\theta}_1 \in (\theta_1^*, \theta_1^* + hu)$. Define $\kappa_0^{(1)} = \int K^{(1)}(u)^2 du$ and $\kappa_1^{(1)} = \int u K^{(1)}(u)^2 du$, the variance equals

$$\begin{aligned}
\text{Var}(\hat{g}(\theta_1^*)) &= \frac{1}{nh^4} E \left[\frac{h}{P(D=1 \mid X_i)} \kappa_0^{(1)} f_{Y|X, D=1}(\theta_1^* \mid X) \right] \\
&\quad + \frac{1}{nh^4} E \left[\frac{h^2}{P(D=1 \mid X_i)} \kappa_1^{(1)} f_{Y|X, D=1}^{(1)}(\theta_1^* \mid X) \right] + \frac{1}{nh^4} O(h^4) \\
&= \frac{1}{nh^3} \left(\kappa_0^{(1)} E \left[\frac{1}{P(D=1 \mid X_i)} f_{Y|X, D=1}(\theta_1^* \mid X) \right] + O(h) + O(h^3) \right) \\
&= \frac{1}{nh^3} (V + O(h) + O(h^3)).
\end{aligned}$$

Then we are ready to apply the central limit theorem.

Let

$$Z_{n,i} \equiv (nh)^{-1/2} \left(\frac{D_i}{P(D=1 \mid X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) - E \left[\frac{D_i}{P(D=1 \mid X_i)} K^{(1)} \left(\frac{\theta_1^* - Y_i}{h} \right) \right] \right),$$

then $E[Z_{n,i}] = 0$ and $\text{Var}(Z_{n,i}) = h^3 \text{Var}(\hat{g}(\theta_1^*)) = n^{-1}V + o(n^{-1})$. Then

$$\begin{aligned}
\sqrt{nh^3}(\hat{g}(\theta_1^*) - 0) &= \sqrt{nh^3}(\hat{g}(\theta_1^*) - E[\hat{g}(\theta_1^*)]) + \sqrt{nh^3}(E[\hat{g}(\theta_1^*)] - 0) \\
&= \sqrt{nh^3}(\hat{g}(\theta_1^*) - E[\hat{g}(\theta_1^*)]) + \sqrt{nh^3}O(h^2) \\
&= \sum_{i=1}^n Z_{n,i} + \sqrt{nh^3}O(h^2) \\
&\xrightarrow{d} N(0, V)
\end{aligned}$$

by Liapunov CLT and $\sqrt{nh^3}h^2 \rightarrow 0$.

Step 6. In this step, we show that

$$\sqrt{nh^3} \begin{bmatrix} \hat{\theta}_1 - \theta_1^* \\ \hat{\theta}_0 - \theta_0^* \end{bmatrix} \xrightarrow{d} N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} M_1 V_1 M_1 & 0 \\ 0 & M_0 V_0 M_0 \end{bmatrix} \right)$$

and thus, by the delta method, we have

$$\sqrt{nh^3} \left(\hat{\Delta} - \Delta^* \right) \xrightarrow{d} N \left(0, M_1 V_1 M_1 + M_0 V_0 M_0 \right).$$

To show the joint distribution we adopt vector notations. The first-order conditions of $\hat{\theta}_1$ and $\hat{\theta}_0$ give

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{f}_{Y_1}(\hat{\theta}_1) \\ \hat{f}_{Y_0}(\hat{\theta}_0) \end{bmatrix} = \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} \hat{f}_{Y|X,D=1}^{(1)}(\hat{\theta}_1 | X_i) \\ \hat{f}_{Y|X,D=0}^{(1)}(\hat{\theta}_0 | X_i) \end{bmatrix} = \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} \hat{f}_{Y|X,D=1}^{(1)}(\theta_1^* | X_i) \\ \hat{f}_{Y|X,D=0}^{(1)}(\theta_0^* | X_i) \end{bmatrix} + J_n \begin{bmatrix} \hat{\theta}_1 - \theta_1^* \\ \hat{\theta}_0 - \theta_0^* \end{bmatrix}$$

where

$$\begin{aligned} J_n &= \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} \frac{\partial \hat{f}_{Y|X,D=1}^{(1)}(\tilde{\theta}_1 | X_i)}{\partial \theta_1} & \frac{\partial \hat{f}_{Y|X,D=1}^{(1)}(\tilde{\theta}_1 | X_i)}{\partial \theta_0} \\ \frac{\partial \hat{f}_{Y|X,D=0}^{(1)}(\tilde{\theta}_0 | X_i)}{\partial \theta_1} & \frac{\partial \hat{f}_{Y|X,D=0}^{(1)}(\tilde{\theta}_0 | X_i)}{\partial \theta_0} \end{bmatrix} \\ &= \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} \hat{f}_{Y|X,D=1}^{(2)}(\tilde{\theta}_1 | X_i) & 0 \\ 0 & \hat{f}_{Y|X,D=0}^{(2)}(\tilde{\theta}_0 | X_i) \end{bmatrix}. \end{aligned}$$

Hence we have

$$\begin{aligned} \sqrt{nh^3} \begin{bmatrix} \hat{\theta}_1 - \theta_1^* \\ \hat{\theta}_0 - \theta_0^* \end{bmatrix} &= (J_n^{-1}) \frac{\sqrt{nh^3}}{n} \sum_{i=1}^n \begin{bmatrix} \hat{f}_{Y|X,D=1}^{(1)}(\hat{\theta}_1 | X_i) \\ \hat{f}_{Y|X,D=0}^{(1)}(\hat{\theta}_0 | X_i) \end{bmatrix} \\ &= (J_n^{-1}) \frac{1}{\sqrt{nh}} \sum_{i=1}^n \begin{bmatrix} \frac{D_i}{P(D=1|X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right) \\ \frac{1-D_i}{P(D=0|X_i)} K^{(1)}\left(\frac{\theta_0^* - Y_i}{h}\right) \end{bmatrix} + \begin{bmatrix} o_p(1) \\ o_p(1) \end{bmatrix} \end{aligned}$$

where the last equality follows from the Step 5 in the proof of Theorem 1. Since

$$J_n \xrightarrow{p} \begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix}$$

and

$$\frac{1}{\sqrt{nh}} \sum_{i=1}^n \begin{bmatrix} \frac{D_i}{P(D=1|X_i)} K^{(1)}\left(\frac{\theta_1^* - Y_i}{h}\right) \\ \frac{1-D_i}{P(D=0|X_i)} K^{(1)}\left(\frac{\theta_0^* - Y_i}{h}\right) \end{bmatrix} \xrightarrow{d} N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} V_1 & 0 \\ 0 & V_0 \end{bmatrix} \right),$$

then by Slutsky's theorem we have

$$\sqrt{nh^3} \begin{bmatrix} \hat{\theta}_1 - \theta_1^* \\ \hat{\theta}_0 - \theta_0^* \end{bmatrix} \xrightarrow{d} N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} M_1 V_1 M_1 & 0 \\ 0 & M_0 V_0 M_0 \end{bmatrix} \right).$$

Proof of Theorem 3. It is enough to show the results of \hat{M}_1 and \hat{V}_1 . We first show that $\hat{M}_1 \xrightarrow{p} M_1$. By adding and subtracting additional terms, we have

$$\hat{M}_1 = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i) = \frac{1}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(2)}(\theta_1^* | X_i) + A_1 + A_2$$

where

$$A_1 = \frac{1}{n} \sum_{i=1}^n \hat{f}_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i) - f_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i)$$

and

$$A_2 = \frac{1}{n} \sum_{i=1}^n f_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i) - f_{Y|X,D=1}^{(2)}(\theta_1^* | X_i).$$

If we can show that $A_1 = o_p(1)$ and $A_2 = o_p(1)$, then $\hat{M} \xrightarrow{p} M$ by law of large numbers. Note that

$$\begin{aligned} |A_1| &\leq \frac{1}{n} \sum_{i=1}^n \left| \hat{f}_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i) - f_{Y|X,D=1}^{(2)}(\hat{\theta}_1 | X_i) \right| \\ &\leq \sup_{y,x} \left| \hat{f}_{Y|X,D=1}^{(2)}(y | x) - f_{Y|X,D=1}^{(2)}(y | x) \right| \\ &= O_p \left(\sqrt{\frac{\ln n}{nh^{d+5}}} + h^2 \right) \\ &= o_p(1), \end{aligned}$$

where the first equality follows from the uniform rates of convergence of kernel estimators (Hansen, 2008). For A_2 , we use the argument in Lemma 4.3 of Newey & McFadden (1994). By consistency of $\hat{\theta}_1$ there is $\delta_n \rightarrow 0$ such that $\|\hat{\theta}_1 - \theta_1^*\| \leq \delta_n$ with probability approaching to one. Define $\Delta_n(Z_i) = \sup_{\|y - \theta_1^*\| \leq \delta_n} \left| f_{Y|X,D=1}^{(2)}(y | X_i) - f_{Y|X,D=1}^{(2)}(\theta_1^* | X_i) \right|$. By the continuity of $f_{Y|X,D=1}^{(2)}(y | X_i)$ at θ_1^* , $\Delta_n(Z_i) \xrightarrow{p} 0$. By the dominated convergence theorem, we have $E[\Delta_n(Z_i)] \rightarrow 0$. By Markov inequality, $P(n^{-1} \sum_{i=1}^n \Delta_n(Z_i) > \epsilon) \leq E[\Delta_n(Z_i)] / \epsilon \rightarrow 0$. Therefore, we have

$$|A_2| \leq \frac{1}{n} \sum_{i=1}^n \Delta_n(Z_i) = o_p(1).$$

Next we show that $\hat{V}_1 \xrightarrow{p} V_1$. We can rewrite

$$\hat{V}_1/\kappa_0^{(1)} = \frac{1}{n} \sum_{i=1}^n \frac{\hat{f}_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\hat{\pi}(X_i)} = \frac{1}{n} \sum_{i=1}^n \frac{f_{Y|X,D=1}(\theta_1^* | X_i)}{\pi(X_i)} + B_1 + B_2$$

with

$$B_1 = \frac{1}{n} \sum_{i=1}^n \frac{\hat{f}_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\hat{\pi}(X_i)} - \frac{f_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\pi(X_i)}$$

and

$$B_2 = \frac{1}{n} \sum_{i=1}^n \frac{f_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\pi(X_i)} - \frac{f_{Y|X,D=1}(\theta_1^* | X_i)}{\pi(X_i)}.$$

It remains to show that $B_1 = o_p(1)$ and $B_2 = o_p(1)$. The result of B_2 follows from the same arguments as in the proof of A_2 if $f_{Y|X,D=1}(y | X_i)$ is continuous at θ_1^* . Thus, we only focus on B_1 . For convenience, define $f(y | x) = f_{Y|X,D=1}(y | x)$. For π bounded away from zero, we have

$$\begin{aligned} \frac{\hat{f}(y | x)}{\hat{\pi}(x)} - \frac{f(y | x)}{\pi(x)} &= \frac{\pi(x) \hat{f}(y | x) - \hat{\pi}(x) f(y | x)}{\hat{\pi}(x) \pi(x)} \\ &= \frac{\pi(x) \hat{f}(y | x) - \pi(x) f(y | x) + \pi(x) f(y | x) - \hat{\pi}(x) f(y | x)}{\hat{\pi}(x) \pi(x)} \\ &= \frac{\hat{f}(y | x) - f(y | x)}{\hat{\pi}(x)} + \frac{f(y | x)}{\hat{\pi}(x) \pi(x)} (\hat{\pi}(x) - \pi(x)) \\ &\leq C \left(\left(\hat{f}(y | x) - f(y | x) \right) + (\hat{\pi}(x) - \pi(x)) \right) \end{aligned}$$

for some $C > 0$. By the uniform rates of convergence of kernel estimators ([Hansen, 2008](#)), we have

$$\begin{aligned} |B_1| &\leq \frac{1}{n} \sum_{i=1}^n \left| \frac{\hat{f}_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\hat{\pi}(X_i)} - \frac{f_{Y|X,D=1}(\hat{\theta}_1 | X_i)}{\pi(X_i)} \right| \\ &\leq C \sup_{y,x} \left(\left| \hat{f}(y | x) - f(y | x) \right| + |\hat{\pi}(x) - \pi(x)| \right) \\ &= O_p \left(\sqrt{\frac{\ln n}{nh^{d+1}}} + h^2 \right) + \sup_x |\hat{\pi}(x) - \pi(x)| \\ &= o_p(1) \end{aligned}$$

by the rates of n and h and the uniform convergence of $\hat{\pi}(x)$.

Proof of Theorem 4: Suppose that

$$\hat{f}_{Y_1}(y) = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1(Z, y, \hat{\eta}_{1k})]$$

is differentiable with respect to y . Define

$$\hat{f}_{Y_1}^{(1)}(y) \equiv \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(1)}(Z, y, \hat{\eta}_{1k})]$$

where $m_1^{(1)}(Z, y, \hat{\eta}_{1k}) \equiv \partial m_1(Z, y, \hat{\eta}_{1k})/\partial y$.

By the definition of $\hat{\theta}_1$, we have

$$\begin{aligned} 0 &= \hat{f}_{Y_1}^{(1)}(\hat{\theta}_1) = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(1)}(Z, \hat{\theta}_1, \hat{\eta}_{1k})] \\ &= \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})] + \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(2)}(Z, \tilde{\theta}_1, \hat{\eta}_{1k})](\hat{\theta}_1 - \theta_1^*) \end{aligned}$$

and

$$\sqrt{Nh^3}(\hat{\theta}_1 - \theta_1^*) = - \left[\frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(2)}(Z, \tilde{\theta}_1, \hat{\eta}_{1k})] \right]^{-1} \left(\frac{\sqrt{Nh^3}}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})] \right).$$

In Step 1 and 2 below, we will show that

$$\frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(2)}(Z, \tilde{\theta}_1, \hat{\eta}_{1k})] \xrightarrow{p} M_1$$

and

$$\frac{\sqrt{Nh^3}}{K} \sum_{k=1}^K \mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})] \xrightarrow{d} N(0, V_1),$$

respectively. Hence, we can obtain the final result

$$\sqrt{Nh^3}(\hat{\theta}_1 - \theta_1^*) \xrightarrow{d} N(0, M_1^{-1}V_1M_1^{-1}).$$

Step 1. Since K is a fixed integer, which is independent of N , it suffices to show that for each $k \in [K]$,

$$\mathbb{E}_{n,k}[m_1^{(2)}(Z, \tilde{\theta}_1, \hat{\eta}_{1k})] \xrightarrow{p} M_1.$$

Then we can show this convergence using the same argument in Step 1 in the proof of Theorem 2.

Step 2. Since K is a fixed integer, which is independent of N , it is enough to consider the convergence of $\mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})]$. Notice that

$$\mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})] = \frac{1}{n} \sum_{i \in I_k} m_1^{(1)}(Z, \theta_1^*, \eta_{10}) + R_{2k}$$

where

$$R_{2,k} = \mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})] - \frac{1}{n} \sum_{i \in I_k} m_1^{(1)}(Z, \theta_1^*, \eta_{10}).$$

Then by triangular inequality,

$$\|R_{2,k}\| \leq \frac{I_{1,k} + I_{2,k}}{\sqrt{n}},$$

where

$$I_{1,k} \equiv \left\| \mathbb{G}_{n,k} \left[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k}) \right] - \mathbb{G}_{n,k} \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right] \right\|,$$

$$I_{2,k} \equiv \sqrt{n} \left\| E_P \left[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k}) \mid (W_i)_{i \in I_k^c} \right] - E_P \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right] \right\|.$$

Two auxiliary results will be used to bound $I_{1,k}$ and $I_{2,k}$:

$$\sup_{\eta_1 \in \mathcal{T}_N} \left(E \left[\left\| m_1^{(1)}(Z, \theta_1^*, \eta_1) - m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right\|^2 \right] \right)^{1/2} \leq \varepsilon_N, \quad (\text{A.1})$$

$$\sup_{r \in (0,1), \eta_1 \in \mathcal{T}_N} \left\| \partial_r^2 E \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10} + r(\eta_1 - \eta_{10})) \right] \right\| \leq (\varepsilon_N)^2, \quad (\text{A.2})$$

where \mathcal{T}_N is the set of all $\eta_1 = (\pi_0, g_{10})$ consisting of square-integrable functions π_0 and g_{10} such that

$$\begin{aligned} & \|\eta_1 - \eta_{10}\|_{P,2} \leq \varepsilon_N, \\ & \|\pi - 1/2\|_{P,\infty} \leq 1/2 - \kappa, \\ & \|\pi - \pi_0\|_{P,2}^2 + \|\pi - \pi_0\|_{P,2} \times \|g_1 - g_{10}\|_{P,2} \leq (\varepsilon_N)^2. \end{aligned}$$

Then by assumption, we have $\hat{\eta}_{1k} \in \mathcal{T}_N$ with probability $1 - o(1)$.

To bound $I_{1,k}$, note that conditional on $(W_i)_{i \in I_k^c}$ the estimator $\hat{\eta}_{1k}$ is nonstochastic. Under the event that $\hat{\eta}_{1k} \in \mathcal{T}_N$, we have

$$\begin{aligned} E_P \left[I_{1,k}^2 \mid (W_i)_{i \in I_k^c} \right] &= E_P \left[\left\| m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k}) - m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right\|^2 \mid (W_i)_{i \in I_k^c} \right] \\ &\leq \sup_{\eta_1 \in \mathcal{T}_N} E_P \left[\left\| m_1^{(1)}(Z, \theta_1^*, \eta_1) - m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right\|^2 \mid (W_i)_{i \in I_k^c} \right] \\ &= \sup_{\eta_1 \in \mathcal{T}_N} E_P \left[\left\| m_1^{(1)}(Z, \theta_1^*, \eta_1) - m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right\|^2 \right] \\ &= (\varepsilon_N)^2 \end{aligned}$$

by (A.1). Hence, $I_{1,k} = O_P(\varepsilon_N)$. To bound $I_{2,k}$, define the following function

$$f_k(r) = E_P \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10} + r(\hat{\eta}_{1k} - \eta_{10})) \mid (W_i)_{i \in I_k^c} \right] - E \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right]$$

for $r \in [0, 1)$. By Taylor series expansion, we have

$$f_k(1) = f_k(0) + f'_k(0) + f''_k(\tilde{r})/2, \text{ for some } \tilde{r} \in (0, 1).$$

Note that $f_k(0) = E \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \mid (W_i)_{i \in I_k^c} \right] = E \left[m_1^{(1)}(Z, \theta_1^*, \eta_{10}) \right] = O(h^2)$ by the calculation in Step 4 in the proof of Theorem 2. Further, on the event $\hat{\eta}_{1k} \in \mathcal{T}_N$,

$$\| f'_k(0) \| = \| \partial_{\eta_1} E[m_1^{(1)}(Z, \theta_1^*, \eta_{10})] [\hat{\eta}_{1k} - \eta_{10}] \| = 0$$

by the orthogonality. Also, on the event $\hat{\eta}_{1k} \in \mathcal{T}_N$,

$$\| f''_k(\tilde{r}) \| \leq \sup_{r \in (0,1)} \| f''_k(r) \| \leq (\varepsilon_N)^2$$

by (A.2). Thus,

$$I_{2,k} = \sqrt{n} \| f_k(1) \| = O_P \left(\sqrt{n} (\varepsilon_N)^2 + \sqrt{n} h^2 \right).$$

Together with the result on $I_{1,k}$, we have

$$\begin{aligned} \| R_{2,k} \| &\leq \frac{I_{1,k} + I_{2,k}}{\sqrt{n}} \\ &= O_P \left(n^{-1/2} \varepsilon_N + (\varepsilon_N)^2 + h^2 \right) \end{aligned}$$

Hence,

$$\sqrt{Nh^3} \| R_{2,k} \| = O_P(\sqrt{h^3} \varepsilon_N + \sqrt{Nh^3} \varepsilon_N^2 + \sqrt{Nh^3} h^2) = o_P(1)$$

by the assumptions on the rate of convergence that $\varepsilon_N = o((Nh^3)^{-1/4})$ and $Nh^7 \rightarrow 0$. Therefore,

$$\mathbb{E}_{n,k}[m_1^{(1)}(Z, \theta_1^*, \hat{\eta}_{1k})] = \frac{1}{n} \sum_{i \in I_k} m_1^{(1)}(Z, \theta_1^*, \eta_{10}) + o_P(1) \xrightarrow{d} N(0, V_1).$$