On pure complex spectrum for truncations of random orthogonal matrices and Kac polynomials.

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Abstract

Let $O(2n+\ell)$ be the group of orthogonal matrices of size $(2n+\ell)\times(2n+\ell)$ equipped with the probability distribution given by normalized Haar measure. We study the probability

$$p_{2n}^{(\ell)} = \mathbb{P}\left[M_{2n} \text{ has no real eigenvalues}\right],$$

where M_{2n} is the $2n\times 2n$ left top minor of a $(2n+\ell)\times(2n+\ell)$ orthogonal matrix. We prove that this probability is given in terms of a determinant identity minus a weighted Hankel matrix of size $n\times n$ that depends on the truncation parameter ℓ. For $\ell=1$ the matrix coincides with the Hilbert matrix and we prove

$$p_{2n}^{(1)} \sim n^{-3/8}, \text{ when } n\to\infty.$$

We also discuss connections of the above to the persistence probability for random Kac polynomials.

1 Introduction

In this paper we consider truncations of random orthogonal matrices distributed according to Haar measure. We are mostly interested in the set of real eigenvalues of these matrices and in the so-called persistence probability. This is the probability of the truncated random orthogonal matrix having no real eigenvalue. However, before we go into details regarding the results, we first want to give further motivation for considering this model in terms of random Kac polynomials. Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of statistically independent copies of a random variable ξ with zero mean and unit variance. Then we define the random polynomial

$$K_N(z) = \sum_{k=0}^{N} a_k z^k \tag{1.1}$$

having random real-valued coefficients and random roots. Their probability distribution and quantitative properties are a centrepiece of the theory of random polynomials dating back to

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the 18^{th} century and attracting lots of attention since then. It was shown in [52], under mild conditions on the probability distribution of ξ, that the normalized counting measure of the zeros converges to the uniform distribution on the unit circle when $N \to \infty$. Restricting ourselves to real roots only, the first major problem is determining their number

$$\mathcal{N}_{\mathbb{R}}(N) = \#\left\{x \in \mathbb{R} : K_N(x) = 0\right\},$$

and its asymptotic behaviour when $N \to \infty$. The problem of calculating $\mathbb{E}\left[\mathcal{N}_{\mathbb{R}}(N)\right]$ has a long history. The first significant results were obtained in a series of papers by Littlewood & Offord [39, 40, 41] and later improved by Kac in [33] where the author derives in the case of $a_i \overset{d}{=} \mathcal{N}(0,1)$

$$\mathbb{E}\left[\mathcal{N}_{\mathbb{R}}(N)\right] = \left(\frac{2}{\pi} + o(1)\right)\log N, \quad N \to \infty.$$

Later on this was shown to be universal for a wide class of probability distributions for ξ by Kac [34], Erdös & Offord [20], Ibragimov & Maslova [30, 31, 32, 29], Tao & Vu [54], see especially [54, Sec. 1.2] for further references.

Further studies of real zeros of Kac polynomials led researchers to the calculation of correlation functions. In [8] Bleher and Di found an explicit formula for all correlation functions between zeros which is given in terms of a multidimensional integral of Gaussian type. Unfortunately, their answer does not allow further progress in the study of the distribution of the zeros and did not show any special structure of the correlation functions. However, P. Forrester found in [23], generalizing methods of [27], a Pfaffian structure in the seemingly different model of eigenvalues of truncated random orthogonal matrices when $N \to \infty$. He argued that Blaschke products of the eigenvalues of rank-one truncated random orthogonal matrices converge to an infinite series with $\mathcal{N}(0,1)$-distributed coefficients. Here rank-one refers to truncations by one column and one row. Comparing this to (1.1), it is natural to call this series

$$K_\infty(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k \in \mathbb{R} \text{ are i.i.d. } N(0,1), \tag{1.2}$$

Kac series. This manifests the connection of roots of random polynomials and eigenvalues of truncated random orthogonal matrices for large N. The model of truncated random orthogonal matrices was first studied in [38] where it was shown that the eigenvalues form a Pfaffian Point Process (PPP), meaning that the correlation functions can be expressed in terms of Pfaffians Pf of the form

$$\rho_k^{(N)}(z_1, z_2, \ldots, z_k) = \text{Pf}\left\{\mathcal{K}_{2\times 2}^{(N)}(z_p, z_q)\right\}_{p,q=1}^{k},$$

$$\mathcal{K}_{2\times 2}^{(N)}(z,w) = \begin{pmatrix} K_{11}^{(N)}(z,w) & K_{12}^{(N)}(z,w) \\ -K_{12}^{(N)}(w,z) & K_{22}^{(N)}(z,w) \end{pmatrix},$$

with skew-symmetric functions $K_{11}(z,w) = -K_{11}(w,z)$ and $K_{22}(z,w) = -K_{22}(w,z)$. For the definition and basic properties of Pfaffians we refer the reader to Appendix B. Later Matsumoto and Shirai in [43], without using any relation to random matrices, proved that the roots of the random Kac series (1.2) form a PPP as well, with corresponding kernel being just the pointwise limit of the one obtained in [23]. This was another strong evidence that there is a hidden Pfaffian structure behind random roots of Kac polynomials. This Pfaffian structure was recently explained in [46] in terms of Gaussian Stationary Process with $\operatorname{sech}(t/2)$ correlation. This Gaussian process was introduced to the area in [13] when studying the so-called persistence probability for Kac polynomials. The persistence probability of Kac polynomials is defined by

$$p_N := \mathbb{P}\left[\mathcal{N}_{\mathbb{R}}(N) = 0\right] = 2\mathbb{P}\left[K_N(x) > 0, \forall x \in \mathbb{R}\right].^1$$

Obviously, this question makes no sense for odd values of N, and therefore from now and on we put $N = 2n$ for some integer n. One can also study more complicated probabilities of having some prescribed number of real roots as well, i.e.

$$p_{N,k} := \mathbb{P}\left[\mathcal{N}_{\mathbb{R}}(N) = k\right]. \tag{1.3}$$

First results on the persistence probability were obtained in [40], where it is proved that $p_{2n} = O(1/\log n)$. Only 60 years later power-like decay

$$p_{2n} \sim n^{-4\theta_{\text{Kac}}}, \text{when } n \to \infty, \tag{1.4}$$

for some unknown θ_{Kac}, was proven in [13] reducing the problem to the study of persistence probabilities for Gaussian Stationary Processes (GSP) Y_t with correlation function

$$R(t) = \langle Y_0 Y_t \rangle = \operatorname{sech}(t/2),$$

where $\langle \cdot \rangle$ denotes the expectation value. The authors showed that

$$\mathbb{P}\left[Y_t \geqslant 0, \forall\, 0 \leqslant t \leqslant T\right] \sim e^{-\theta_{\text{Kac}} T},$$

with θ_{Kac} being the same as in (1.4). Despite being explicitly defined, the constant θ_{Kac} remains unknown and the best known results are $\theta_{\text{Kac}} \in (1/8, 1/4]$ (theoretically, [55]) and $\theta_{\text{Kac}} \approx 0.1875 \pm 0.01$ (numerically, [13],[49]). Over time this constant became very popular as it had appeared in many applications such as persistence of integrated Brownian motion [2], [56], no flipping probabilities in Ising spin model [16], persistence probabilities for solutions of diffusion and heat equations with random initial data [49, 42, 48, 14]. However no single model was rigorously solved and the constant remained unknown. Gaussian stationary processes and the calculation of the corresponding persistence constant using Pfaffian structure is the main content of the upcoming paper [47].

In the paper, however, using the connection suggested in [23] and explained before, we analyse the model of truncated orthogonal matrices and the corresponding "persistence" probability. Let $\ell \in \mathbb{N}$ and $O(N+\ell)$ be the group of orthogonal $(N+\ell) \times (N+\ell)$ matrices equipped with the probability distribution given by normalized Haar measure. Decomposing $O \in O(N+\ell)$ according to

$$O = \begin{pmatrix} M_N & B_{N\times\ell} \\ C_{\ell\times N} & D_\ell \end{pmatrix}, \tag{1.5}$$

the main result of the paper can be formulated as follows.

Theorem 1.1. *Let $\{M_{2n}\}$ be the ensemble of the $2n\times 2n$ top left minor of the orthogonal matrices of size $(2n+\ell) \times (2n+\ell)$ chosen uniformly (with respect to Haar measure) at random. Then the "persistence" probability*

$$p_{2n}^{(\ell)} := \mathbb{P}\left[M_{2n} \text{ has no real eigenvalues }\right] = \det\left(I_n - \mathcal{D}_n^{(\ell)} H_n^{(\ell)} \mathcal{D}_n^{(\ell)}\right), \tag{1.6}$$

is a determinant given in terms of the $n \times n$ Hankel matrix

$$\left(H_n^{(\ell)}\right)_{p,q} = \frac{B(p+q+1/2, \ell)}{2^{\ell-1}\Gamma^2\left(\frac{\ell}{2}\right)}, \quad p, q = \overline{0, n-1},$$

[1] usually the persistence probability is defined as half of what we use here

where B stands for the Beta-function, and the $n \times n$ diagonal matrix $\mathcal{D}^{(\ell)}$ with diagonal elements

$$\left(\mathcal{D}_n^{(\ell)}\right)_{p,p} = \sqrt{\frac{\Gamma(2p+\ell)}{\Gamma(2p+1)}}, \quad p = \overline{0, n-1}. \tag{1.7}$$

Remark 1.1. *For $\ell = 1$ the above expression further simplifies to*

$$p_{2n}^{(1)} = \det\left(I_n - H_n^{(1)}\right), \text{ where } \left(H_n^{(1)}\right)_{p,q} = \frac{1}{\pi(p+q+1/2)}, \tag{1.8}$$

$p, q = \overline{0, n-1}$. From now on we use the superscript $^{(\ell)}$ only when $\ell \neq 1$, otherwise we drop it.

In fact our method allows us to find not only the probability of having no real eigenvalues, but also the moment generating function of the number of real eigenvalues.

Proposition 1.2. *Let $\mathcal{N}_n^{(\ell)}(\mathbb{R})$ denote a number of real eigenvalues of the random matrix M_{2n} taken from the ensemble defined above. Then for any $s \in \mathbb{C}$*

$$\left\langle e^{s\mathcal{N}_n^{(\ell)}(\mathbb{R})}\right\rangle_{M_{2n}} = \det\left(I_n - \left(1 - e^{2s}\right)\mathcal{D}_n^{(\ell)} H_n^{(\ell)} \mathcal{D}_n^{(\ell)}\right), \tag{1.9}$$

where $\langle \cdot \rangle_{M_{2n}}$ denotes the expectation value with respect to the ensemble.

A similar result for the moment generating function of the number of real eigenvalues for products of truncated orthogonal matrices was recently also obtained in [24]. The result was obtained in the regime of large truncations $\ell \geqslant N$ and is expressed in terms of Meijer G-functions, which makes its asymptotic analysis not feasible by our methods. We are interested in the application of our result to the study of random Kac polynomials, and therefore we stay in the so-called universality class of weak non-orthogonality (see review [26] and references therein).

Proposition 1.3. *Identity (1.9) gives access to the probability of all eigenvalues being real. The result reads*

$$p_{2n,2n}^{(\ell)} = \frac{G\left(n+\frac{\ell}{2}\right) G\left(n+\frac{\ell+1}{2}\right) G(n+\ell)\, G\left(n+\ell-\frac{1}{2}\right)}{\Gamma^{2n}\left(\frac{\ell}{2}\right) G\left(\frac{\ell}{2}\right) G\left(\frac{\ell+1}{2}\right) G(\ell)\, G\left(2n+\ell-\frac{1}{2}\right)}, \tag{1.10}$$

where G is the Barnes G-function and $p_{2n,2n}^{(\ell)}$ is defined similar to (1.3). For $n \to \infty$ and ℓ either growing with n or being fixed the probability of pure real spectrum has the asymptotic expansion

$$\lim_{n\to\infty} \frac{\log p_{2n,2n}^{(\ell)}}{n^2} = \begin{cases} -2\log 2, & \ell/n \to 0, \\ \phi(\alpha), & \ell/n = \alpha \in (0,\infty), \\ -\log 2, & \ell/n \to \infty. \end{cases} \tag{1.11}$$

where

$$\phi(\alpha) = -\log 2 - \alpha\left(1 + \frac{3}{4}\alpha\right)\log\alpha - \frac{\alpha}{2} + (1+\alpha)^2 \log(1+\alpha) - \left(1 + \frac{\alpha}{2}\right)^2 \log(2+\alpha),$$

with $\phi(0) = -2\log 2$ and $\phi(\infty) = -\log 2$.

Formula (1.10) was previously obtained in [25, Cor. 2] by a different method. Our main result here is the asymptotics (1.11) which clearly gives an interpolation between the weak non-orthogonality class (small ℓ) and the Real Ginibre Ensemble ($\ell \gg n$) in terms of the probability $p_{2n,2n}^{(\ell)}$. The corresponding result for the Real Ginibre ensemble can be found in [19, Cor. 7.1].

The second main result of this paper is the asymptotic analysis of the probability (1.8).

Theorem 1.4. *The asymptotics*

$$\log \det \left(I_n - H_n\right) = -2\theta \log n + o\left(\log n\right) \tag{1.12}$$

holds as $n \to \infty$ with

$$\theta = -\frac{1}{2\pi} \int_0^\infty \log\left(1 - \operatorname{sech}(\pi u)\right) du = \frac{3}{16}. \tag{1.13}$$

In particular this implies that

$$\lim_{n\to\infty} \frac{\log p_{2n}}{\log n} = -\frac{3}{8}. \tag{1.14}$$

Remarks 1.2. (i) *The constant 3/8 was recently also encountered in [10] in the context of persistence probability of so-called Peron polynomials. This is a very intriguing coincidence because Kac polynomials are defined through random coefficients, while the probability distribution on the space of Perron polynomials is defined in a far more complicated way.*

(ii) *The relation of truncated orthogonal matrices to Kac polynomials in the case $\ell = 1$ explained earlier on strongly indicates that the decay exponent $4\theta_{Kac}$ in* (1.4) *for the persistence probability of Kac polynomials is given by*

$$\theta_{Kac} = \frac{3}{16}. \tag{1.15}$$

as well and therefore $4\theta_{Kac} = 12/16 = 3/4$. This will be content of the upcoming paper [47]. Also note that introducing the coefficient in (1.12) *as "2θ" is justified by the following observation: All eigenvalues of the random matrix M_{2n} are positioned inside the unit disk and can model random roots of a polynomial* (1.1) *only within this domain. However, random roots of Kac polynomial lying outside of the unit disk in the large N limit are independent and equally distributed (up to a transformation $z \to 1/z$) with those inside. Therefore, the persistence constant for Kac polynomials is expected to be twice as big as the one for truncated orthogonal matrices.*

(iii) *Related asymptotics of the form* (1.12) *for* $\det\left(I - \alpha H_n\right)$ *with $|\alpha| < 1$ were proven in [21]. It is shown there that*

$$\det\left(I - \alpha H_n\right) = -\frac{1}{2\pi^2}\big(\arcsin^2(\alpha) + \pi \arcsin(\alpha)\big) \log n + o(\log n).$$

The proof works for a larger class of Hankel matrices but does not generalize to $\alpha = 1$ which we need in our case. This result yields that the moment generating function of the number of real eigenvalues can be written as

$$\left\langle e^{s\mathcal{N}_n}\right\rangle = \left(\frac{1}{8} - \frac{2}{\pi^2}\left[\arccos\frac{e^s}{\sqrt{2}}\right]^2\right) \log n + o\left(\log n\right), \quad s \in \left(-\infty, \frac{\log 2}{2}\right].$$

To conclude, we would like to mention that the same expression previously appeared in [16, Eq. (7)] when studying the problem of the number of persistent spins in the Ising model on the half-line.

The matrix H_n is a variation of the Hilbert matrix, known since the end of the 19^{th} century. Its spectral properties are well studied but, to the best of our knowledge, the known results do not allow us to compute the determinant we are interested in. The determinant can also be considered in the context of so-called Toeplitz $\pm$ Hankel determinants. These determinants are of independent interest where we refer to the review [12] and to [11, 3, 4, 5, 6] for recent progress in that area. The results in the mentioned papers solely deal with Toeplitz + Hankel determinants of particular forms, more precisely of Toeplitz and Hankel matrices having either the same symbol or symbols which differ by just a factor $e^{\pm it}$. Our result corresponds to symbols

$$\sigma^T\left(e^{it}\right) \equiv 1, \quad \sigma^H\left(e^{it}\right) = ie^{-it/2},$$

where the last one has a jump discontinuity at $t = 0$.

The paper is organised as follows. In Section 2 we prove Theorem 1.1 and derive (1.9) - (1.11). The proof of Theorem 1.4 is the main content of Section 3. The proofs of a number of auxilliary results is deferred to Sections 4–6. Finally, in Section 7 we discuss some open problems and conjectures in the scope of our interests.

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2 Ensemble of truncated orthogonal matrices

In this section we give the details for the derivation of (1.6). First we find the joint probability distribution function for the eigenvalues of M_{2n}. It was previously calculated with a mistake in the coefficients in [38] and then corrected by a different method in [44] in the case of "large" truncations. For rank-one truncations ($\ell = 1$) this was also calculated in [37, Thm. 6.4, Rmk. 2]. We follow the strategy of [38], to obtain the result for small truncations, and fill some holes and correct some minor mistakes in their proof. In the second part we note that the distribution fits in the framework of Point Processes Associated to Weights developed in [51, 9]. This enables us to calculate the corresponding averages in terms of a family of skew-orthogonal polynomials found in [23].

Proof of Theorem 1.1. We start by considering the orthogonal group of size $N + \ell$ with the probability distribution defined by the measure

$$\mathrm{d}\,\mu\,(O) = \frac{1}{v_{N+\ell}}\delta\left(O^T O - I_{N+\ell}\right)\mathrm{d}\,O, \tag{2.1}$$

where $\mathrm{d}\,O = \prod_{i,j=1}^{N+\ell} O_{i,j}$ is the flat Lebesgue measure on $\mathbb{R}^{(N+\ell)^2}$ and

$$v_{N+\ell} = \int_{\mathbb{R}^{(N+\ell)^2}} \delta\left(O^T O - I_{N+\ell}\right)\mathrm{d}\,O = \prod_{j=1}^{N+\ell}\frac{\pi^{j/2}}{\Gamma\left(j/2\right)}, \tag{2.2}$$

is the volume of the orthogonal group (see Proposition A.2). We decompose $O \in O\left(N + \ell\right)$ as (cf. (1.5))

$$O = \begin{pmatrix} M_N & B_{N\times\ell} \\ C_{\ell\times N} & D_\ell \end{pmatrix}.$$

Firstly, we compute the induced measure for the ensemble of the top-left minor M_N. We denote the space of all possible matrices M_N by $\mathcal{O}_N^{(\ell)}$. One can integrate out (see Lemma 4.1 for details) the variables $B_{N\times\ell}$ and D_ℓ to obtain the joint distribution of M_N and $C_{N\times\ell}$. Then the probability distribution on $\mathcal{O}_N^{(\ell)}$ is written as

$$\mathrm{d}\,P\left(M_N\right)=\frac{v_\ell}{v_{N+\ell}}\Big(\int_{\mathbb{R}^{\ell N}}\delta\left(M_N^TM_N+C_{\ell\times N}^TC_{\ell\times N}-I_N\right)\mathrm{d}\,C_{\ell\times N}\Big)\mathrm{d}\,M_N, \tag{2.3}$$

where $\mathrm{d}\,M_N$ is Lebesgue measure on $\mathbb{R}^{N^2}$ and $\mathrm{d}\,C_{\ell\times N}$ is the Lebesgue measure on $\mathbb{R}^{N\ell}$. The constraint imposed by the δ-function yields $M_N^TM_N=I_N-C_{\ell\times N}^TC_{\ell\times N}$, which implies that all eigenvalues of M_N belong to the unit disk $\mathbb{D}=\{z|\,|z|\leqslant 1\}$. We are now left with the integration over $C_{\ell\times N}$. The dimension of the δ-function inside the integrand is $\frac{N(N+1)}{2}$, i.e. the number of constraints imposed on the matrix in the integrand of the δ-function, and the number of independent variables in $C_{\ell\times N}^TC_{\ell\times N}$ is equal to $\frac{\ell(\ell+1)}{2}$. This means that in the case of $\ell\leqslant N$ the integration over $C_{\ell\times N}$ will lead to a singular measure concentrated on the boundary of the matrix ball $M_N^TM_N\leqslant I_N$ (see [38]). We concentrate on the case of fixed ℓ and $N\to\infty$, so the measure will be singular. If $\ell\geqslant N$ then the integration over $C_{\ell\times N}$ gives [38]

$$\mathrm{d}\,P\left(M_N\right)=\frac{v_\ell^2}{v_{N+\ell}v_{\ell-N}}\det\,\left(I-M_N^TM_N\right)_+^{\frac{\ell-N-1}{2}}\mathrm{d}\,M_N,$$

where we write M_+ to denote M, when M is positive definite and 0 otherwise.

In the singular case, i.e. $\ell\leqslant N$, we calculate the eigenvalue distribution by following [38] and fill missing details. We start by noting that there are two types of eigenvalues for matrix M_N: they can be either real or come in pairs of complex conjugate ones. We introduce disjoint subsets of $\mathcal{O}_N^\ell$ given by (see [9])

$$\mathcal{X}_{L,M}=\left\{\,M_N\in\mathcal{O}_N^\ell\right|M_N \text{ has } L \text{ real and } M \text{ pairs of complex eiv's}\big\}\,,$$

for all pairs $L,M\in\mathbb{Z}_+$ such that $L+2M=N$. For all $M_N\in\mathcal{O}_N^\ell$ we define a lexicographical order of the $2M+L$ eigenvalues:

$$\lambda_1,\ldots,\lambda_L,x_1+iy_1,x_1-iy_1,\ldots,x_M+iy_M,x_M-iy_M,$$

where $\lambda_i,x_i,y_i\in\mathbb{R}$ and

$$\lambda_i\geqslant\lambda_{i+1},\ x_i>x_{i+1} \text{ or } x_i=x_{i+1} \text{ with } y_i>y_{i+1}.$$

In what follows, we restrict ourselves to even N and therefore L is even as well. In the case of odd N we can obtain similar results, but the expressions would become lengthy and we omit this case.

It was suggested by Borodin and Sinclair in their seminal study [9] on the Real Ginibre ensemble that, instead of the full distribution, one should rather compute the distribution of the eigenvalues conditioned on $M_N\in\mathcal{X}_{L,M}$. We follow this idea and prove

Lemma 2.1. *Let M_N be the top left corner of an orthogonal matrix $O\in O(N+\ell)$ drawn randomly with respect to the measure* (2.1). *Then the density of the joint distribution of the ordered eigenvalues of M_N conditioned on having L real eigenvalues $\vec{\lambda}=(\lambda_1,\ldots,\lambda_L)$ and M pairs of complex conjugate eigenvalues $\vec{Z}=(z_1,\bar{z}_1,\ldots,z_M,\bar{z}_M)$ is*

$$p_\ell^{(L,M)}\left(\vec{\lambda},\vec{Z}\right)=2^M\frac{v_\ell v_N}{v_{N+\ell}}\left(\frac{(2\pi)^\ell}{\ell!}\right)^{N/2}\left|\Delta\left(\vec{\lambda}\cup\vec{Z}\right)\right|\prod_{j=1}^{L}w_\ell\left(\lambda_j\right)\prod_{j=1}^{M}w_\ell\left(z_j\right)w_\ell\left(\bar{z}_j\right),$$

where $\Delta(\zeta_1,\dots,\zeta_p)=\prod_{1\leqslant i,j\leqslant p}(\zeta_i-\zeta_j)$ *is the Vandermonde determinant and* $w_\ell(z)\equiv 0$ *outside the unit disk* $\mathbb{D}$ *while for* $z\in\mathbb{D}$

$$w_\ell^2(z)=\begin{cases}\dfrac{1}{2\pi}|1-z|^{-1}, & \ell=1,\\ \dfrac{\ell(\ell-1)}{2\pi}\left|1-z^2\right|^{\ell-2}\displaystyle\int_{\frac{2|\mathrm{Im}z|}{|1-z^2|}}^{1}\left(1-u^2\right)^{\frac{\ell-3}{2}}\mathrm{d}\,u, & \ell\geqslant 2.\end{cases} \tag{2.4}$$

We prove the above lemma in Section 4 that generalizes the corresponding result of [44] to any integer ℓ either larger than N or not. In the following we will use ω_ℓ a lot, which is of course the square root of the latter function. The above expression for the conditional distribution of the eigenvalues fits in the framework of point processes associated to weights developed by Borodin and Sinclair in [9]. We use their main result which can be stated as

Theorem 2.2. *[9, Thm 3.4] Let* $\mathcal{P}$ *be a point process of even size* N *in the complex plane containing either real or pairs of complex conjugate points. Assume that the probability distribution function for ordered configurations* $\Xi=(\xi_1,\xi_2,\dots,\xi_N)$ *conditioned on* $\Xi\in\mathcal{X}_{L,M}$ *is given by*

$$p^{(L,M)}(\Xi)=C_N 2^M|\Delta(\Xi)|\prod_{j=1}^{N}w(\xi_j)\,\mathbb{1}_{\Xi\in X_{L,M}}$$

with C_N *being a normalization constant.*

Then for any sequence of monic polynomials $\left\{p_k(z)=z^k+\dots\right\}_{k=0}^{N-1}$ *we obtain*

1. *For any function* $f:\mathbb{C}\to\mathbb{C}$

$$\left\langle\prod_{j=1}^{N}f(\xi_j)\right\rangle_{\mathcal{P}}=\frac{\mathrm{Pf}\,\left(U_{i,j}(f)\right)}{\mathrm{Pf}\,\left(U_{i,j}(1)\right)}, \tag{2.5}$$

   where $\langle\cdot\rangle_{\mathcal{P}}$ *is the average with respect to* $\mathcal{P}$ *and*

$$U_{i,j}(f)=\int_{\mathbb{C}}f(z)\,p_i(z)\,w(z)\left(\varepsilon\left[fp_jw\right]\right)(z)-f(z)\,p_j(z)\,w(z)\left(\varepsilon\left[fp_iw\right]\right)(z)\,\mathrm{d}^2z.$$

   Here Pf *denotes the Pfaffian of a matrix and* $\varepsilon:S(\mathbb{C})\to S(\mathbb{C})$ *is the operator defined on the class of Schwartz functions* $S(\mathbb{C})$ *by*

$$(\varepsilon[f])(z)=\begin{cases}\dfrac{1}{2}\displaystyle\int_{\mathbb{R}}f(t)\,\mathrm{sgn}\,(t-z)\,\mathrm{d}\,t, & z\in\mathbb{R},\\ i\,\mathrm{sgn}\,(\mathrm{Im}z)\,f(\overline{z}), & z\in\mathbb{C}\backslash\mathbb{R}.\end{cases}$$

2. *The point process* $\mathcal{P}$ *is a Pfaffian point process with kernel*

$$\mathbf{K}_N(z,w)=\begin{pmatrix}DS_N(z,w) & S_N(z,w)\\ -S_N(w,z) & IS_N(z,w)+\varepsilon(z,w)\end{pmatrix},$$

where

$$DS_N(z,w) = 2\sum_{i,j=0}^{N-1} v_{i,j} p_i(z) p_j(w) w(z) w(w),$$

$$S_N(z,w) = 2\sum_{i,j=0}^{N-1} v_{i,j} p_i(z) w(z) (\varepsilon[p_j w])(w),$$

$$IS_N(z,w) = 2\sum_{i,j=0}^{N-1} v_{i,j} (\varepsilon[p_i w])(z) (\varepsilon[p_j w])(w),$$

$$\varepsilon(z,w) = \begin{cases} \frac{1}{2}\operatorname{sgn}(z-w), & z,w \in \mathbb{R}, \\ 0, & \textit{otherwise.} \end{cases}$$

and $v_{i,j}$ *are the matrix elements of* $\left(U_{i,j}^{-1}(1)\right)^T$.

Remark 2.1. *The matrix elements* $U_{i,j}(f)$ *can be written in terms of the skew-product*

$$(f,g)^{(w)} = \int_{\mathbb{C}} f(z) w(z) (\varepsilon[gw])(z) - g(z) w(z) (\varepsilon[fw])(z) \,\mathrm{d}^2 z. \tag{2.6}$$

The product should be thought of consisting of two parts because the action of the operator ε *heavily depends on the argument being real or complex. More precisely one can introduce "real" and "complex" parts of the skew-product*

$$(f,g)_{\mathbb{R}}^{(w)} = \int_{\mathbb{R}} f(x) w(x) (\varepsilon[gw])(x) - g(x) w(x) (\varepsilon[fw])(x) \,\mathrm{d}x,$$

$$(f,g)_{\mathbb{C}}^{(w)} = i\int_{\mathbb{R}^2} (f(x+iy) g(x-iy) - g(x+iy) f(x-iy)) w(x+iy) w(x-iy) \operatorname{sgn}(y) \,\mathrm{d}x\mathrm{d}y.$$

We see that the distribution found in Lemma 2.1, satisfies the assumptions of the latter theorem with

$$C_N = \frac{v_\ell v_N}{v_{N+\ell}} \left(\frac{(2\pi)^\ell}{\ell!}\right)^{N/2}, \quad \omega = \omega_\ell,$$

defined in (2.4). The latter formulas can be further simplified by choosing a family of monic polynomials skew-orthogonal with respect to the skew-product (2.6) with $w = w_\ell$, for which we use notation $(f,g)^{(\ell)}$ (and $(f,g)_{\mathbb{R}}^{(\ell)}, (f,g)_{\mathbb{C}}^{(\ell)}$ for its parts). Such a family of skew-orthogonal polynomials was found in [23] and later generalized in [24] to the case of products of truncated orthogonal matrices. This family reads

Lemma 2.3. *Let* $k \in \mathbb{N}_0$. *Then the polynomials defined by*

$$\pi_{2k}(z) = z^{2k}, \qquad \pi_{2k+1}(z) = z^{2k+1} - \frac{2k}{2k+\ell} z^{2k-1}, \tag{2.7}$$

are skew-orthogonal with respect to the skew-product (2.6), *i.e. for* $i,j \in \mathbb{N}_0$ *with* $i < j$

$$(\pi_i, \pi_j)^{(\ell)} = \begin{cases} \frac{\ell!\,(2k)!}{(2k+\ell)!}, & i = 2k, j = 2k+1, \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover for any $i, j \in \mathbb{N}_0$

$$(\pi_i, \pi_j)_{\mathbb{R}}^{(\ell)} = \begin{cases} \dfrac{\ell!}{2^{\ell-1}\Gamma^2\left(\frac{\ell}{2}\right)(2q+\ell)} B\left(p+q+\dfrac{1}{2}, \ell\right), & i = 2p, j = 2q+1 \\ 0, & i - j \text{ even.} \end{cases}$$

We prove the above lemma in Section 5. Our final step is to apply the result of (2.5) to the function

$$f(\zeta) = 1 - \chi_{\mathbb{R}}(\zeta) = \begin{cases} 1, & \zeta \in \mathbb{C}\backslash\mathbb{R}, \\ 0, & \zeta \in \mathbb{R}. \end{cases} \tag{2.8}$$

Then the average $\left\langle \prod_{j=1}^{N} f(\xi_j) \right\rangle_{M_{2n}}$ with respect to the random matrices ensemble coincides with $p_{2n}^{(\ell)}$ and using (2.5) we obtain

$$p_{2n}^{(\ell)} = \frac{\operatorname{Pf}\left\{(\pi_i, \pi_j)_{\mathbb{C}}^{(\ell)}\right\}_{i,j=0}^{2n-1}}{\operatorname{Pf}\left\{(\pi_i, \pi_j)^{(\ell)}\right\}_{i,j=0}^{2n-1}} = \frac{\operatorname{Pf}\left\{(\pi_i, \pi_j)^{(\ell)} - (\pi_i, \pi_j)_{\mathbb{R}}^{(\ell)}\right\}_{i,j=0}^{2n-1}}{\operatorname{Pf}\left\{(\pi_i, \pi_j)^{(\ell)}\right\}_{i,j=0}^{n-1}}. \tag{2.9}$$

The matrix in the denominator is block-diagonal and the one in the numerator has a check board pattern. Both Pfaffians can be reduced to determinants by the use of

Proposition 2.4. *Let* $A = \{a_{i,j}\}_{i,j=0}^{2n-1}$ *be a skew-symmetric matrix with* $a_{i,j} = 0$ *whenever* i *and* j *have the same parity. Then*

$$\operatorname{Pf} A = \det\left\{a_{2i,2j+1}\right\}_{i,j=0}^{n-1}.$$

We prove this proposition in Appendix B. Inserting this in (2.9), implies

$$p_{2n}^{(\ell)} = \frac{\det\left\{(\pi_{2p}, \pi_{2q+1})^{(\ell)} - (\pi_{2p}, \pi_{2q+1})_{\mathbb{R}}^{(\ell)}\right\}_{p,q=0}^{n-1}}{\det\left\{(\pi_{2p}, \pi_{2q+1})^{(\ell)}\right\}_{p,q=0}^{n-1}}. \tag{2.10}$$

The multiplicative property of the determinant, Lemma 2.3 and the diagonal structure of the matrix in the denominator yields

$$p_{2n}^{(\ell)} = \det\left\{I - \frac{(\pi_{2p}, \pi_{2q+1})_{\mathbb{R}}^{(\ell)}}{(\pi_{2p}, \pi_{2q+1})^{(\ell)}}\right\}_{p,q=0}^{n-1} = \det\left\{I - \frac{B\left(p+q+\frac{1}{2}, \ell\right)}{2^{\ell-1}\Gamma^2\left(\frac{\ell}{2}\right)} \frac{(2p+\ell)!}{(2p)!\,(2q+\ell)}\right\}_{p,q=0}^{n-1}.$$

To obtain (1.6), we conjugate the latter matrix by the diagonal matrix Q given by $Q_{p,p} = \sqrt{\frac{(2p)!}{(2p+\ell)!(2p+\ell)}}$, i.e. multiply by Q from the left and Q^{-1} from the right. This proves Theorem 1.1. □

Proof of Proposition 1.2. Identity (1.9) follows along the same lines as (1.6). We change the test function (2.8) to

$$g(z) = 1 - (1 - e^s)\,\chi_{\mathbb{R}}(z).$$

One can easily see that $g(z) = e^s$ if $z \in \mathbb{R}$ and 1 otherwise. Therefore,

$$\left\langle \prod_{z \in \operatorname{spec} M_{2n}} g(z) \right\rangle_{M_{2n}} = \left\langle e^{s\mathcal{N}_n^{(\ell)}(\mathbb{R})} \right\rangle_{M_{2n}}.$$

Applying (2.5) to the above function, we obtain, compare also with (2.10), that

$$\left\langle e^{s\mathcal{N}_n^{(\ell)}(\mathbb{R})}\right\rangle_{M_{2n}} = \frac{\mathrm{Pf}\left\{(\pi_i,\pi_j)^{(\ell)} - \left(1-e^{2s}\right)(\pi_i,\pi_j)_{\mathbb{R}}^{(\ell)}\right\}_{i,j=0}^{2n-1}}{\mathrm{Pf}\left\{(\pi_i,\pi_j)^{(\ell)}\right\}_{i,j=0}^{n-1}}.$$

Mimicking the analysis at the end of the proof of Theorem 1.1, gives the result. □

Proof of Proposition 1.3. Expanding the expectation according to

$$\left\langle e^{s\mathcal{N}_n^{(\ell)}}\right\rangle_{M_{2n}} = \sum_{k=0}^{2n} e^{sk}\mathbb{P}\big(\mathcal{N}_n^{(\ell)} = k\big), \tag{2.11}$$

implies that the probability $p_{2n,2n}^{(\ell)}$ is given by the coefficient in front of e^{2ns} in this expansion. The latter expectation can be expressed in terms of the determinant (1.9). Expanding this determinant in the same way as (2.11) in terms of powers of e^{sk} using Leibniz formula, we obtain

$$p_{2n,2n}^{(\ell)} = \mathrm{coeff}\left[\det\left(I_n - \mathcal{D}_n^{(\ell)}H_n^{(\ell)}\mathcal{D}_n^{(\ell)} + e^{2s}\mathcal{D}_n^{(\ell)}H_n^{(\ell)}\mathcal{D}_n^{(\ell)}\right), e^{2sn}\right],$$

where coeff$[\cdot, e^{2sn}]$ stands here for the coefficient in front of e^{2sn} in this expansion. The power e^{2sn} is the maximum power of the exponent in the expansion and thus writing down the latter determinant using the Leibniz rule, one sees that

$$p_{2n,2n}^{(\ell)} = \det\mathcal{D}_n^{(\ell)}H_n^{(\ell)}\mathcal{D}_n^{(\ell)} = \det H_n^{(\ell)}\cdot\left(\det\mathcal{D}_n^{(\ell)}\right)^2.$$

For the determinant of $H_n^{(\ell)}$ we use a well-known identity for Hankel matrix determinants. Let $\mathcal{H} = \{h_{j+k}\}_{j,k=0}^{n-1}$ be a Hankel matrix with

$$h_n = \int x^n\mu(x)\,\mathrm{d}\,x, \tag{2.12}$$

for some weight function $\mu(x)$. Then using the symmetry of the integrand

$$\begin{aligned}
\det\mathcal{H} &= \det\left\{\int x_j^{j+k}\mu(x_j)\,\mathrm{d}\,x_j\right\}_{j,k=0}^{n-1} = \iint\prod_{j=0}^{n-1}x_j^j\det\left\{x_j^k\right\}_{j,k=0}^{n-1}\prod_{j=0}^{n-1}\mu(x_j)\,\mathrm{d}\,x_j\\
&= \frac{1}{n!}\iint\sum_{\sigma}\prod_{j=0}^{n-1}x_{\sigma(j)}^j\det\left\{x_{\sigma(j)}^k\right\}_{j,k=0}^{n-1}\prod_{j=0}^{n-1}\mu(x_j)\,\mathrm{d}\,x_j\\
&= \frac{1}{n!}\iint\prod_{1\leqslant j<k\leqslant n}(x_j-x_k)^2\prod_{j=0}^{n-1}\mu(x_j)\,\mathrm{d}\,x_j.
\end{aligned}$$

In the case of $H_n^{(\ell)}$ we take $\mu(x) = \dfrac{1}{2^{\ell-1}\Gamma^2\left(\frac{\ell}{2}\right)}x^{-1/2}(1-x)^{\ell-1}$, $x\in[0,1]$. Then

$$\begin{aligned}
\det H_n^{(\ell)} &= \frac{1}{2^{n(\ell-1)}\Gamma^{2n}\left(\frac{\ell}{2}\right)n!}\prod_{j=0}^{n-1}\frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma(j+\ell)\Gamma(j+2)}{\Gamma\left(j+n+\ell+\frac{1}{2}\right)}\\
&= \frac{1}{2^{n(\ell-1)}\Gamma^{2n}\left(\frac{\ell}{2}\right)}\prod_{j=0}^{n-1}\frac{\Gamma(j+\ell)\Gamma\left(j+\frac{1}{2}\right)\Gamma(j+1)}{\Gamma\left(j+n+\ell-\frac{1}{2}\right)},
\end{aligned}$$

where we used the value of Selberg's integral [50]. Using definition (1.7) of $\mathcal{D}_n^{(\ell)}$ and the value of its determinant, together with the Gamma function duplication formula, we get

$$p_{2n,2n}^{(\ell)} = \Gamma^{-2n}\left(\frac{\ell}{2}\right)\prod_{j=0}^{n-1}\frac{\Gamma\left(j+\frac{\ell}{2}\right)\Gamma\left(j+\frac{\ell+1}{2}\right)\Gamma\left(j+\ell\right)}{\Gamma\left(j+n+\ell-\frac{1}{2}\right)}.$$

The former can be rewritten as in (1.10) by using

$$\prod_{j=0}^{n-1}\Gamma\left(j+a\right) = \frac{G\left(n+a\right)}{G\left(a\right)},$$

with G being the Barnes G-function. The asymptotic expansion of Barnes G-function at infinity reads

$$\log G\left(z\right) = \frac{1}{2}z^2\log z - \frac{3}{4}z^2 - z\log z + z\left(1+\frac{\log 2\pi}{2}\right) + \frac{5}{12}\log z + O\left(1\right), \quad z\to\infty.$$

Let $\alpha_n = \frac{\ell}{n}$ be bounded by strictly positive constants from above and below when $n\to\infty$. Applying the latter asymptotic expansion to (1.10) and α_n, we only keep the first two terms as the rest will contribute to lower order terms. We then obtain

$$\begin{aligned}\log p_{2n,2n}^{(\ell)} = n^2\Bigg(&-\log 2 - \alpha_n\log\alpha_n\left(1+\frac{3}{4}\alpha_n\right) - \frac{\alpha_n}{2}\\ &+\left(1+\alpha_n\right)^2\log\left(1+\alpha_n\right) - \left(1+\frac{\alpha_n}{2}\right)^2\log\left(2+\alpha_n\right)\Bigg) + o\left(n^2\right), n\to\infty,\end{aligned}$$

where we also used

$$\log\Gamma\left(z\right) = \left(z-\frac{1}{2}\right)\log z - z + o\left(z\right), z\to\infty,$$

to analyse the denominator of (1.10). The result derived above is valid even for $\alpha_n = o\left(1\right), n\to\infty$, and thus for $\ell \ll n$ we get

$$\log p_{2n,2n}^{(\ell)} = -2\log 2n^2 + o\left(n^2\right).$$

One can also argue that for $\alpha_n\to\infty$ the coefficient in front of n^2 converges to $-\log 2$, which gives the answer. However, to prove rigorously the asymptotic result in the regime of large truncations with $\ell \gg n$ we need to take all non-constant terms of the Barnes G-function expansion into account. While expanding (1.10) with respect to ℓ one can see that terms containing $\ell^2\log\ell, \ell\log\ell$ and $\log\ell$ vanish and for $\beta_n = n/\ell = o\left(1\right), n\to\infty$ we get

$$\begin{aligned}\log p_{2n,2n}^{(\ell)} = &\left(\left(1+\beta_n\right)^2\log\left(1+\beta_n\right) - \left(\beta_n+\frac{1}{2}\right)^2\log\left(1+2\beta_n\right) - \frac{\beta_n}{2} - \beta_n^2\log 2\right)\ell^2\\ &+\left(-\frac{5}{2}\left(1+\beta_n\right)\log\left(1+\beta_n\right) + \frac{3}{4}\left(1+2\beta_n\right)\log\left(1+2\beta_n\right) + \beta_n\left(1+\log 2^{3/2}\pi\right)\right)\ell\\ &+\frac{7}{24}\left(2\log\left(1+2\beta_n\right) - 5\log\left(1+\beta_n\right)\right) + o\left(1\right), \quad \ell\to\infty.\end{aligned}$$

For small β_n the first bracket is dominated by $\beta_n^2\log 2$, this together with the definition of β_n gives the result. The second and third brackets are of order $O\left(\beta_n\right)$ and therefore will not contribute to the leading order. This finishes the proof of (1.11). $\square$

3 Asymptotic analysis of $\det\left(I_n - H_n^{(1)}\right)$

In this section we use the short hand notation $H_n = H_n^{(1)}$. The asymptotic analysis of the determinant $\det\,(I_n - H_n)$ follows along similar ideas used in [36], where a similar persistence problem led to an analysis of a determinant of identity minus a weighted Hankel matrix as well. We use the trace-log expansion of the determinant which leads to analysing traces of powers of the Hilbert matrix $H_n^{(1)}$. The asymptotic behaviour of the trace of a fixed power of the Hilbert matrix are studied in [57]. However, to obtain the asymptotic behaviour of the infinite series, we need to work a bit harder to obtain proper upper and a lower bound. The upper bound is simple and follows from truncating the trace-log expansion to a finite number of terms. For the lower bound we change $H_n^m \to H_n H_\infty^{m-1}$, where H_∞ denotes the infinite Hilbert matrix, as it was suggested in [36]. Finally, using the explicit diagonalization of H_∞ found in [35], we conclude the proof.

Proof of Theorem 1.4. In what follows we use the notation $D_n = \det\,(I_n - H_n)$ and $H = H_\infty$ for the infinite Hilbert matrix acting on $\ell_2\,(\mathbb{N}_0)$. From [35, Sec. 4] it follows that $\sigma(H) = [0,1]$ and consists of purely absolutely continuous spectrum only. In particular, $\|H\| = 1$, where $\|\cdot\|$ denotes the operator norm. For the $n \times n$ restriction we have H_n of H we have

Lemma 3.1. *For all $n \in \mathbb{N}$ we obtain $\|H_n\| < 1$.*

We prove this lemma in Subsection 6.1. Our analysis relies on the well-known series expansion

$$\log(1-x) = -\sum_{m\in\mathbb{N}} \frac{x^m}{m} \tag{3.1}$$

valid for all $|x| < 1$ and hence the latter lemma implies

$$\log D_n = -\sum_{m\in\mathbb{N}} \frac{\mathrm{Tr}\,(H_n^m)}{m}.$$

The above series is convergent because of Lemma 3.1 and consists of negative terms only. Thus any finite truncation of the series gives an upper bound on the series.

For the upper bound we rely on the following result proved in [57, Thm. 4.3 and Cor. 4.4].

Lemma 3.2. *[57, Theorem 4.3] For any fixed $m \in \mathbb{N}$ the moments of H_n satisfy*

$$\lim_{n\to\infty} \frac{\mathrm{Tr}\,(H_n^m)}{\log n} = \frac{1}{\pi}\int_0^\infty \mathrm{sech}^m\,(u\pi)\,\mathrm{d}\,u =: \mu_m. \tag{3.2}$$

Remark 3.1. *Compared to the result stated in [57], we have an additional prefactor $1/2\pi$ in (3.2). This was missed in [57] and pointed out to us by A. Pushnitski and E. Fedele, see [22].*

From the latter lemma we readily obtain an upper bound on the determinant D_n:

Corollary 3.3 (Upper bound)**.** *We obtain the bound*

$$\limsup_{n\to\infty} \frac{\log D_n}{\log n} \leqslant -\sum_{m\in\mathbb{N}} \frac{\mu_m}{m} = \frac{1}{\pi}\int_0^\infty \log\left(1-\mathrm{sech}(\pi u)\right)\mathrm{d}\,u.$$

Next, we prove a lower bound. Since the matrix elements of H are all positive we obtain the inequality

$$\mathrm{Tr}\,\big(H_n^m\big) \leqslant \mathrm{Tr}\,\big(1_n H^m 1_n\big),$$

where 1_n denotes the projection on $\ell^2(\{0,1,...n-1\}) \subset \ell^2(\mathbb{N}_0)$. Hence we obtain for all $\varepsilon > 0$

$$\begin{aligned}\log D_n &\geqslant -\sum_{m\in\mathbb{N}} \frac{\mathrm{Tr}\,\big(1_n H^m 1_n\big)}{m} \\ &\geqslant -\sum_{m\in\mathbb{N}} \frac{\mathrm{Tr}\,\big(1_n H^m 1_{>\varepsilon}(H) 1_n\big)}{m} - \sum_{m\in\mathbb{N}} \frac{\mathrm{Tr}\,\big(1_n H^m 1_{\leqslant\varepsilon}(H) 1_n\big)}{m}. \end{aligned} \tag{3.3}$$

Here $1_{>\varepsilon}(H)$ is short for the spectral projection of H on the set $\{x : x > \varepsilon\}$. We first estimate the second term.

Lemma 3.4. *We obtain*

$$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{\log n} \sum_{m\in\mathbb{N}} \frac{\mathrm{Tr}\,\big(1_n H^m 1_{\leqslant\varepsilon}(H) 1_n\big)}{m} = 0.$$

We prove this lemma in Subsection 6.1. In order to treat the first term on the left hand side of (3.3) we use the explicit diagonalization of the operator H found in [35, Sec. 4].

Lemma 3.5. *We define for $l \in \mathbb{N}_0$*

$$\hat{P}_l(x^2) := \frac{4^l \big(\frac14\big)_{(l)} \big(\frac12\big)_{(l)} \big(\frac34\big)_{(l)}}{l! \big(\frac12\big)_{(2l)}} {}_4F_3\Big(-l, l+\frac12, i\frac{x}{2}, -i\frac{x}{2}; \frac14, \frac12, \frac34; 1\Big), \tag{3.4}$$

where ${}_pF_q$ denotes the generalized hypergeometric function. Moreover, let

$$\rho(x) := 2\,\mathrm{sech}(\pi x).$$

and $\mathcal{H} := L^2((0,\infty), d\rho)$. Then $U : \ell^2(\mathbb{N}_0) \to \mathcal{H}$ defined by $(Ue_l)(x) := \hat{P}_l(x^2)$ is a unitary and

$$(UHU^* f)(x) = \mathrm{sech}(\pi x) f(x), \qquad x > 0,$$

i.e. U diagonalizes the operator H.

The latter lemma, Fubini's theorem and (3.1) readily imply

$$\begin{aligned}&\sum_{m\in\mathbb{N}} \frac{1}{m} \mathrm{Tr}\,\big(1_n H^m 1_{>\varepsilon}(H) 1_n\big) \\ &= 2\sum_{m\in\mathbb{N}} \frac{1}{m} \int_0^\infty \big(\mathrm{sech}(\pi x)\big)^m 1_{>\varepsilon}(\mathrm{sech}(\pi x))\,\mathrm{sech}(\pi x) \sum_{l=0}^{n-1} \big|\hat{P}_l(x^2)\big|^2 \mathrm{d}\,x \\ &= -\int_0^{\frac{\mathrm{sech}^{-1}(\varepsilon)}{\pi}} \log\big(1-\mathrm{sech}(\pi x)\big) 2\,\mathrm{sech}(\pi x) \sum_{l=0}^{n-1} \big|\hat{P}_l(x^2)\big|^2 \mathrm{d}\,x. \end{aligned} \tag{3.5}$$

Next we are interested in the asymptotic behaviour of $\hat{P}_l(x^2)$ as $l \to \infty$.

Lemma 3.6. *Let $x > 0$ be fixed. Then as $l \to \infty$ the asymptotics*

$$\hat{P}_l(x^2) = \sqrt{\frac{\cosh(\pi x)}{\pi l}} \cos\left(x \log l + \arg A(ix/2)\right) \left(1 + O(l^{-1})\right) + l^{-3/2} R_l\left(x\right)$$

holds, where arg *is the argument function,*

$$A(ix/2) := \frac{2^{2ix-3/2} \cosh(\pi x)^{1/2}}{\pi^{3/2}}$$

and the O-term is independent of $x > 0$ and

$$\sup_{x \in [0,M]} \sup_{l \in \mathbb{N}_0} \left| R_l\left(x\right) \right| \leqslant r\left(M\right)$$

for some constant $r(M)$ depending on $M > 0$.

We prove this in Subsection 6.2.

Lemma 3.7. *Let $\varepsilon > 0$. Then we obtain the upper bound*

$$\limsup_{n \to \infty} \frac{\sum_{m \in \mathbb{N}} \frac{1}{m} \operatorname{Tr}\left(1_n 1_{>\varepsilon}(H) H^m 1_n\right)}{\log n} \leqslant -\frac{1}{\pi} \int_0^\infty \log(1 - \operatorname{sech}(\pi x)) \mathrm{d}\, x.$$

We prove this in Subsection 6.2. The bound (3.3) implies that

$$\begin{aligned} \liminf_{n \to \infty} \frac{\log D_n}{\log n} \geqslant & - \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{\log n} \\ & \times \Big(\sum_{m \in \mathbb{N}} \frac{\operatorname{Tr}\left(1_n H^m 1_{>\varepsilon}(H) 1_n\right)}{m} + \sum_{m \in \mathbb{N}} \frac{\operatorname{Tr}\left(1_n H^m 1_{\leqslant\varepsilon}(H) 1_n\right)}{m} \Big). \end{aligned}$$

From Lemma 3.4 and Lemma 3.7 the assertion follows.

□

4 Proof of Lemma 2.1

In this Section we assume that if the region of integration is not specified, it is over whole space of corresponding dimension.

Lemma 4.1. *Let $O \in O\left(N + \ell\right)$ be an orthogonal matrix drawn randomly according to (2.1). Then the joint distribution of the top left minor M_N and bottom left minor $C_{\ell \times N}$ (cf. (1.5)) is given by*

$$\mathrm{d}\, P\left(M_N, C_{\ell \times N}\right) = \frac{v_\ell}{v_{N+\ell}} \delta\left(M_N^T M_N + C_{\ell \times N}^T C_{\ell \times N} - I_N\right) \mathrm{d}\, M_N \mathrm{d}\, C_{\ell \times N} \tag{4.1}$$

where $\mathrm{d}\, M_N$ and $\mathrm{d}\, C_{\ell \times N}$ denote Lebesgue measure on $\mathbb{R}^{N \times N}$, respectively $\mathbb{R}^{\ell \times N}$.

Proof of Lemma 4.1. Block decomposition of the matrix O yields

$$O^T O = \begin{pmatrix} M_N^T M_N + C_{\ell \times N}^T C_{\ell \times N} & M_N^T B_{N \times \ell} + C_{\ell \times N}^T D_\ell \\ B_{N \times \ell}^T M_N + D_\ell^T C_{\ell \times N} & B_{N \times \ell}^T B_{N \times \ell} + D_\ell^T D_\ell \end{pmatrix},$$

and the corresponding measure on the set of tuples $M_N, C_{\ell\times N}$ now can be rewritten as

$$\frac{\mathrm{d}\, P\left(M_N, C_{\ell\times N}\right)}{\mathrm{d}\, M_N \mathrm{d}\, C_{\ell\times N}} = \frac{1}{v_{N+\ell}} \int \delta\left(M_N^T M_N + C_{\ell\times N}^T C_{\ell\times N} - I_N\right)$$
$$\delta\left(M_N^T B_{N\times\ell} + C_{\ell\times N}^T D_\ell\right) \delta\left(B_{N\times\ell}^T B_{N\times\ell} + D_\ell^T D_\ell - I_\ell\right) \mathrm{d}\, B_{N\times\ell} \mathrm{d}\, D_\ell.$$

Integration over $dB_{N\times\ell}$ gives

$$\frac{\mathrm{d}\, P\left(M_N, C_{\ell\times N}\right)}{\mathrm{d}\, M_N \mathrm{d}\, C_{\ell\times N}} = \frac{1}{v_{N+\ell}} \int \delta\left(M_N^T M_N + C_{\ell\times N}^T C_{\ell\times N} - I_N\right)$$
$$\delta\left(D_\ell^T C_{\ell\times N} M_N^{-1} M_N^{-T} C_{\ell\times N}^T D_\ell + D_\ell^T D_\ell - I_\ell\right) \det{}^{-\ell} M_N \mathrm{d}\, D_\ell.$$

A change of variables $D_\ell = \left(I_\ell + C_{\ell\times N} M_N^{-1} M_N^{-T} C_{\ell\times N}^T\right)^{-1/2} \hat{D}_\ell$ yields

$$\frac{\mathrm{d}\, P\left(M_N, C_{\ell\times N}\right)}{\mathrm{d}\, M_N \mathrm{d}\, C_{\ell\times N}} = \frac{1}{v_{N+\ell}} \int \delta\left(M_N^T M_N + C_{\ell\times N}^T C_{\ell\times N} - I_N\right)$$
$$\delta\left(\hat{D}_\ell^T \hat{D}_\ell - I_\ell\right) \det{}^{-\ell/2}\left(I_\ell + C_{\ell\times N} M_N^{-1} M_N^{-T} C_{\ell\times N}^T\right) \det{}^{-\ell} M_N \mathrm{d}\, \hat{D}_\ell.$$

Integrating over $\hat{D}_\ell$ gives v_ℓ, see (2.2). The condition $M_N^T M_N + C_{\ell\times N}^T C_{\ell\times N} = I_N$ implies that the two determinants in the latter integrand cancel which can be seen as follows

$$\begin{aligned}
\det\left(I_\ell + C_{\ell\times N} M_N^{-1} M_N^{-T} C_{\ell\times N}^T\right) &= \det\left(I_\ell + C_{\ell\times N}(I_N - C_{\ell\times N}^T C_{\ell\times N})^{-1} C_{\ell\times N}^T\right) \\
&= \det\left(I_\ell - C_{\ell\times N} C_{\ell\times N}^T\right)^{-1} \\
&= \det{}^{-1}\left(I_N - C_{\ell\times N}^T C_{\ell\times N}\right) = \det{}^{-2} M_N.
\end{aligned}$$

And the statement is proved. □

Proof of Lemma 2.1. We start with the distribution of M_N given by (2.3) (cf. (4.1)) and try to integrate out all variables of M_N except of its eigenvalues. Below we follow a method described by Edelman [19].

We first note that any matrix can be uniquely written as a special product called real Schur decomposition. Let

$$\operatorname{spec} M_N = \{\lambda_1, \lambda_2, \ldots \lambda_L, z_1, \overline{z_1}, z_2, \overline{z_2}, \ldots, z_M, \overline{z_M}\},$$

be the ordered set of eigenvalues of $M_N \in \mathcal{X}_{L,M}$. Any such matrix can be uniquely decomposed into a product of real matrices

$$M_N = O \begin{pmatrix}
\lambda_1 & R_{1,2} & .. & R_{1,L} & R_{1,L+1} & .. & .. & R_{1,L+M} \\
0 & \lambda_2 & .. & R_{2,L} & R_{2,L+1} & .. & .. & R_{2,L+M} \\
\vdots & .. & .. & \vdots & .. & .. & .. & \vdots \\
0 & .. & 0 & \lambda_L & R_{L,L+1} & .. & .. & R_{L,L+M} \\
0 & .. & .. & 0 & \Lambda_1 & R_{L+1,L+2} & .. & R_{L+1,L+M} \\
0 & .. & .. & 0 & 0 & \Lambda_2 & .. & R_{L+2,L+M} \\
\vdots & .. & .. & \vdots & \vdots & .. & .. & \vdots \\
0 & .. & .. & 0 & 0 & .. & 0 & \Lambda_M
\end{pmatrix} O^T,$$

for an orthogonal matrix $O \in O(N)$. Here $\Lambda_1, \Lambda_2, \dots, \Lambda_M$ are 2×2 blocks of the form

$$\begin{pmatrix} \alpha_j & \beta_j \\ -\gamma_j & \alpha_j \end{pmatrix}$$

with $\beta_j \geqslant \gamma_j$ and $\beta_j \gamma_j > 0$, corresponding to complex eigenvalues $z_j, \overline{z_j} = \alpha_j \pm i\sqrt{\beta_j \gamma_j}$ ordered in lexicographical order, and finally

$$R_{j,k} = \begin{cases} 1 \times 1 & \text{blocks} \quad j < k \leqslant L, \\ 1 \times 2 & \text{blocks} \quad j \leqslant L < k, \\ 2 \times 2 & \text{blocks} \quad L < j < k. \end{cases}$$

Remark 4.1. *In fact the decomposition is not unique. One can change $O \mapsto O \cdot O_d$, where O_d is a diagonal orthogonal matrix. However, we will make it unique by assuming that all elements in the first row of O are positive.*

Next we use a result of Edelman regarding Schur decomposition

Theorem 4.2. *[Theorem 5.1, [19]] Let M_N be written in form of the Schur decomposition $M_N = O(Z + R)O^T$. The Jacobian for the change of variables is given by*

$$\mathrm{d}\, M_N = 2^M \prod_{1 \leqslant j < k \leqslant L} |\lambda_j - \lambda_k| \prod_{\substack{1 \leqslant j \leqslant L \\ 1 \leqslant k \leqslant M}} |\lambda_j - z_k|^2 \prod_{1 \leqslant j < k \leqslant M} |z_j - z_k| \, |z_j - \overline{z_k}|$$

$$\prod_{1 \leqslant j \leqslant M} (\beta_j - \gamma_j) \prod_{j=1}^{L} \mathrm{d}\, \lambda_j \prod_{k=1}^{M} \mathrm{d}\, \Lambda_k \mathrm{d}\, R \mathrm{d}\,_H O,$$

where $z_j, \overline{z_j}$ is the pair of eigenvalues corresponding to the block Λ_j,

$$\mathrm{d}\, \Lambda_j = \mathrm{d}\, \alpha_j \mathrm{d}\, \beta_j \mathrm{d}\, \gamma_j,$$

$\mathrm{d}\, R$ is the product over all $N^2/2 - N/2 - M$ real parameters of R and $\mathrm{d}\,_H O$ is the product over all independent elements of the antisymmetric matrix $O^T \mathrm{d}\, O$, which is the natural element of integration (for Haar measure) over the space of orthogonal matrices.

For any function $\widehat{F} : \mathcal{M}(\mathbb{R}^N) \to \mathbb{C}$, where $\mathcal{M}(\mathbb{R}^N)$ is the set of all $N \times N$ matrices, its average over $\mathcal{O}_N^\ell$ is given by

$$\langle \widehat{F} \rangle = \frac{v_\ell}{v_{N+\ell}} \int \widehat{F}(M_N)\, \delta\left(M_N^T M_N + C_{\ell \times N}^T C_{\ell \times N} - I_N\right) \mathrm{d}\, M_N \mathrm{d}\, C_{\ell \times N}.$$

This average can be naturally splitted into averages over the disjoint sets $\mathcal{X}_{L,M}$. If one assumes that $\widehat{F}$ depends only on the eigenvalues, i.e. it can be written as

$$\widehat{F}(M_N) = F(\lambda_1, \dots, \lambda_L, x_1 + iy_1, \dots, x_M + iy_M)$$

for some function $F : \mathbb{R}^L \times \mathbb{C}_+^M \to \mathbb{C}$, then the joint conditional eigenvalue distribution $p_\ell^{(L,M)}$ is defined by the identity

$$\langle \widehat{F} \rangle_{\mathcal{X}_{L,M}} = \int_{\mathbb{R}^L} \int_{\mathbb{C}_+^M} F\left(\vec{\lambda}, \vec{Z}\right) p_\ell^{(L,M)} \left(\vec{\lambda}, \vec{Z}\right) \mathrm{d}\, \vec{\lambda} \mathrm{d}\, \vec{Z}.$$

We start with changing variables from M_N to the triplet (O, Z, R) and use Theorem 4.2 to get

$$\left\langle \widehat{F} \right\rangle_{\mathcal{X}_{L,M}} = 2^M \frac{v_\ell}{v_{N+\ell}} \int\limits_{O(N)} \mathrm{d}_H O \int\limits_{\mathbb{R}^{\binom{N}{2}-M}} \mathrm{d}\, R \int\limits_{\mathbb{R}^{\ell N}} \mathrm{d}\, C_{\ell\times N} \int\limits_{\mathbb{R}^L_>} \mathrm{d}\, \vec{\lambda} \int\limits_{\mathbb{R}^M_> \times (\mathbb{R}^2_>)^M} \mathrm{d}\, \vec{\Lambda}$$
$$\prod_{k=1}^{M} (\beta_k - \gamma_k)\, F(Z)\, \Delta(Z)\, \delta\left(\left(Z^T + R^T\right)(Z+R) + C^T_{\ell\times N} C_{\ell\times N} - I_N \right), \tag{4.2}$$

where $\mathbb{R}^L_> := \left\{(x_1, ..., x_L) \subset \mathbb{R}^L : x_1 \leqslant x_2 \leqslant ... \leqslant x_L\right\}$ and $(\mathbb{R}^2_>)^M$ is defined accordingly and

$$\Delta(Z) = \prod_{1\leqslant j<k\leqslant L} |\lambda_j - \lambda_k| \prod_{\substack{1\leqslant j\leqslant L\\ 1\leqslant k\leqslant M}} |\lambda_j - z_k|^2 \prod_{1\leqslant j<k\leqslant M} |z_j - z_k|\, |z_j - \overline{z_k}|\,.$$

Now integration over the orthogonal group gives the prefactor v_N. Next we integrate over R.

Proposition 4.3. *Integration over R in* (4.2) *gives*

$$\left\langle \widehat{F} \right\rangle_{\mathcal{X}_{L,M}} = 2^M \frac{v_\ell v_N}{v_{N+\ell}} \int\limits_{\mathbb{R}^L_>} \mathrm{d}\, \vec{\lambda} \int\limits_{\mathbb{R}^{\ell N}} \mathrm{d}\, C_{\ell\times N} \int\limits_{\mathbb{R}^M_> \times (\mathbb{R}^2_>)^M} \mathrm{d}\, \vec{\Lambda} F(Z)\, \Delta(Z)$$
$$\prod_{j=1}^{L} |\lambda_j|^{j-N}\, \delta\left(\lambda_j^2 + C_j^T X_j C_j - 1\right) \prod_{k=1}^{M} (\beta_k - \gamma_k) \left(\det{}^{-2} \Lambda_k\right)^{M-k}$$
$$\prod_{k=1}^{M} \delta\left(\Lambda_k^T \Lambda_k + \begin{pmatrix} C^T_{L+2k-1} \\ C^T_{L+2k} \end{pmatrix} Y_k \begin{pmatrix} C_{L+2k-1} & C_{L+2k} \end{pmatrix} - I_2 \right), \tag{4.3}$$

where C_i is the i's column of the matrix $C_{\ell\times N}$ and X_j, Y_j are $\ell \times \ell$ real symmetric, positive definite matrices defined recursively by

$$\begin{cases} X_1 &= I_\ell, \\ X_{k+1} &= X_k + X_k P_k X_k, \quad k = \overline{1, L-1}, \end{cases}$$

respectively

$$\begin{cases} Y_1 &= I_\ell + \sum_{j=1}^{L} X_j P_j X_j, \\ Y_{k+1} &= Y_k + Y_k Q_k Y_k, \quad k = \overline{1, M-1} \end{cases}$$

with $P_j = \frac{C_j C_j^T}{\lambda_j^2}$ and $Q_j = \begin{pmatrix} C_{L+2j-1} & C_{L+2j} \end{pmatrix} \Lambda_j^{-1} \Lambda_j^{-T} \begin{pmatrix} C_{L+2j-1} & C_{L+2j} \end{pmatrix}^T$. Moreover,

$$\det X_1 = 1, \quad \det X_{k+1} = \lambda_k^{-2} \det X_k.$$

$$\det Y_{k+1} = \det{}^{-2} \Lambda_k \det Y_k.$$

Proof of Proposition 4.3. We start by rewriting the δ-function in (4.2) element wise. The corresponding term can be written as a product of lower dimensional, e.g. 1×1,1×2, 2×2, δ-functions. For the top left corner we have L δ-functions of the form

$$\delta\left(\sum_{j<k} R^2_{j,k} + \lambda_k^2 + C_k^T C_k - 1 \right), \quad 1 \leqslant k \leqslant L, \tag{4.4}$$

and $\binom{L-1}{2}$ δ-functions of the form

$$\delta\left(\sum_{j=1}^{k-1} R_{j,k}R_{j,m} + \lambda_k R_{k,m} + C_k^T C_m\right), \quad 1 \leqslant k < m \leqslant L. \tag{4.5}$$

By solving equations corresponding to restrictions (4.5) inductively one can see that $R_{k,m}$ can be expressed as $-\lambda_k^{-1}C_k^T X_k C_m$ for some matrix X_k. Applying the ansatz one gets

$$\sum_{j=1}^{k-1} X_j^T P_j X_j - X_k + I_\ell = 0,$$

and the former is solved by the X_j's defined above and therefore

$$R_{k,m} = -\lambda_k^{-1} C_k^T X_k C_m.$$

The top right corner δ-functions, corresponding to 1×2 matrices, can be written as

$$\delta\left(\sum_{j=1}^{k-1} R_{j,k}R_{j,L+m} + \lambda_k R_{k,L+m} + \left[C_k^T C_{L+2m-1}\, C_k^T C_{L+2m}\right]\right), \; 1 \leqslant m \leqslant M.$$

These conditions can be resolved inductively in a similar way and one gets

$$R_{k,L+m} = -\frac{1}{\lambda_k} C_k^T X_k \left[C_{L+2m-1} \quad C_{L+2m}\right], \quad 1 \leqslant k \leqslant L, 1 \leqslant m \leqslant M.$$

Finally, the bottom right corner δ-functions, corresponding to 2×2 blocks, are given by

$$\delta\left(\sum_{j=1}^{L} R_{j,L+k}^T R_{j,L+m} + \sum_{j=1}^{k-1} R_{L+j,L+k}^T R_{L+j,L+m} + \Lambda_k^T R_{L+k,L+m} \right.$$
$$\left. + \begin{pmatrix} C_{L+2k-1}^T C_{L+2m-1} & C_{L+2k-1}^T C_{L+2m} \\ C_{L+2k}^T C_{L+2m-1} & C_{L+2k}^T C_{L+2m} \end{pmatrix}\right), \quad 1 \leqslant k < m \leqslant M.$$

with diagonal ones

$$\delta\left(\sum_{j=1}^{L} R_{j,L+m}^T R_{j,L+m} + \sum_{j=1}^{m-1} R_{L+j,L+m}^T R_{L+j,L+m} + \Lambda_m^T \Lambda_m \right.$$
$$\left. + \begin{pmatrix} C_{L+2m-1}^T C_{L+2m-1} & C_{L+2m-1}^T C_{L+2m} \\ C_{L+2m}^T C_{L+2m-1} & C_{L+2m}^T C_{L+2m} \end{pmatrix} - I_2\right), \quad 1 \leqslant m \leqslant M.$$

The "non-diagonal equations" are solved by

$$R_{L+k,L+m} = -\Lambda_k^{-T} \begin{pmatrix} C_{L+2k-1}^T \\ C_{L+2k}^T \end{pmatrix} Y_k \left(C_{L+2m-1} \quad C_{L+2m}\right), \; 1 \leqslant k < m \leqslant M.$$

where the Y_k are defined above. Positiveness of the matrices X_k, Y_k is obvious from their definitions. Finally, we integrate out the variables R from (4.2) to obtain the first statement. We just mention that all δ-functions constraints contained $R_{k,m}$ elements with a prefactor of the form either λ_k or Λ_k and this is the source of Jacobian

$$\prod_{j=1}^{L} |\lambda_j|^{j-N} \prod_{k=1}^{M} \left(\det{}^{-2}\Lambda_k\right)^{M-k},$$

appearing in the final result. For the determinants we note

$$\det X_{k+1} = \det\left(X_k + X_k P_k X_k\right) = \det X_k \det\left(I + P_k X_k\right).$$

P_k is up to a constant the projection onto C_k, and therefore the matrix $I + P_k X_k$ can only have one eigenvalue different from 1 with corresponding eigenvector C_k. Therefore,

$$\det\left(I + P_k X_k\right) = 1 + \lambda_k^{-2}\langle C_k, X_k C_k\rangle.$$

Finally, we note that there is a δ-function of the form (4.4) left in the integration with argument $\lambda_k^2 - 1 + \langle C_k, X_k C_k\rangle$. Thus

$$\det\left(I + P_k X_k\right) = 1 + \frac{1-\lambda_k^2}{\lambda_k^2} = \lambda_k^{-2}.$$

For Y_k the proof is analogous. $\square$

In the next step we integrate out the C variables in (4.3). We first change coordinates

$$\begin{aligned} &C_j \to X_j^{-1/2} C_j, \quad 1 \leqslant j \leqslant L,\\ &\left[C_{L+2j-1}\, C_{L+2j}\right] \to Y_j^{-1/2}\left[C_{L+2j-1}\, C_{L+2j}\right], \quad 1 \leqslant j \leqslant M. \end{aligned}$$

It follows from Proposition 4.3, that all determinants coming from Jacobians cancel and we obtain

$$\begin{aligned} \left\langle \widehat{F}\right\rangle_{\mathcal{X}_{L,M}} = 2^M \frac{v_\ell v_N}{v_{N+\ell}} \int_{\mathbb{R}_>^L} \mathrm{d}\vec{\lambda} \int_{\mathbb{R}_>^M \times (\mathbb{R}_>^2)^M} \mathrm{d}\vec{\Lambda} \int_{\mathbb{R}^{\ell N}} \mathrm{d}\, C_{\ell\times N} F(Z)\,\Delta(Z) \\ \prod_{k=1}^M (\beta_k - \gamma_k) \prod_{j=1}^L \delta\left(\lambda_j^2 + C_j^T C_j - 1\right) \\ \prod_{m=1}^M \delta\left(\Lambda_m^T \Lambda_m + \begin{pmatrix} C_{L+2m-1}^T \\ C_{L+2m}^T \end{pmatrix} \begin{pmatrix} C_{L+2m-1} & C_{L+2m} \end{pmatrix} - I_2\right), \end{aligned}$$

Now we are ready to integrate out the variables C. The first L integrals are given by Proposition A.1 and using the corresponding result we integrate over the variables C_i for $i = \overline{1, L}$

$$\begin{aligned} \left\langle \widehat{F}\right\rangle_{\mathcal{X}_{L,M}} = 2^M \frac{v_\ell v_N \pi^{L\ell/2}}{v_{N+\ell}\Gamma^L\left(\frac{\ell}{2}\right)} \int_{\mathbb{R}_>^L} \mathrm{d}\vec{\lambda} \int_{\mathbb{R}_>^M \times (\mathbb{R}_>^2)^M} \mathrm{d}\vec{\Lambda} \int_{\mathbb{R}^{\ell(N-L)}} \mathrm{d}\, C_{\ell\times(N-L)} F(Z)\,\Delta(Z) \\ \int_{\mathbb{R}^{\ell(N-L)}} \mathrm{d}\, C_{\ell\times(N-L)} \prod_{j=1}^L \left(1-\lambda_j^2\right)_+^{\frac{\ell}{2}-1} \prod_{k=1}^M (\beta_k - \gamma_k) \\ \prod_{m=1}^M \delta\left(\Lambda_m^T \Lambda_m + \begin{pmatrix} C_{L+2m-1}^T \\ C_{L+2m}^T \end{pmatrix} \begin{pmatrix} C_{L+2m-1} & C_{L+2m} \end{pmatrix} - I_2\right), \end{aligned}$$

For every 2×2 dimensional δ-function we have 2ℓ variables of integration and 3 δ-functions. Therefore we need to distinguish two different cases: $\ell = 1$ (singular) and $\ell \geqslant 2$. In the first case we have

$$I(\Lambda) := \int \mathrm{d}\, x \mathrm{d}\, y \delta\left(x^2 - \Lambda_{11}\right) \delta\left(y^2 - \Lambda_{22}\right) \delta\left(xy - \Lambda_{12}\right) \delta\left[\det\left(I_2 - \Lambda^T \Lambda\right)\right],$$

where $\begin{pmatrix}\Lambda_{11} & \Lambda_{12}\\ \Lambda_{12} & \Lambda_{22}\end{pmatrix} = I_2 - \Lambda^T\Lambda$. For $\ell \geqslant 2$ we have a non-singular integral of the form

$$I(\Lambda) := \int \mathrm{d}\,\vec{x}\mathrm{d}\,\vec{y}\delta\left(\|\vec{x}\|^2 - A\right)\delta\left(\|\vec{y}\|^2 - C\right)\delta\left(\langle\vec{x},\vec{y}\rangle - B\right).$$

We change variables $\vec{y}$ to $t = \|P_{\vec{x}}\vec{y}\| = \frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{x}\|^2}$ and $\vec{r} = (r_1, r_2, \ldots, r_{\ell-1})$ such that

$$\vec{y} = t\vec{x} + \Big(r_1, \ldots, r_{\ell-1}, -\frac{1}{x_\ell}\sum_{j=1}^{\ell-1} x_j r_j\Big).$$

Then the corresponding Jacobian is given by $J = \frac{\|\vec{x}\|^2}{x_\ell}$. Applying this to the above, yields

$$\begin{aligned}I(\Lambda) &= \int \mathrm{d}\,\vec{x}\mathrm{d}\,\vec{r}\mathrm{d}\,t\delta\big(\|\vec{x}\|^2 - \Lambda_{11}\big)\delta\big(t\|\vec{x}\|^2 - \Lambda_{12}\big)\frac{\|\vec{x}\|^2}{|x_\ell|}\delta\Big(t^2\|\vec{x}\|^2 + \|\vec{r}\|^2 + x_\ell^{-2}\Big(\sum_{j=1}^{\ell-1} x_j r_j\Big)^2 - \Lambda_{22}\Big)\\ &= \int \mathrm{d}\,\vec{x}\mathrm{d}\,\vec{r}\delta\big(\|\vec{x}\|^2 - \Lambda_{11}\big)\delta\left(\langle\vec{r}, V\vec{r}\rangle - U\right)|x_\ell|^{-1}.\end{aligned}$$

where $V = I_{\ell-1} + x_\ell^{-2}(x_1, x_2, \ldots, x_{\ell-1})^T(x_1, x_2, \ldots, x_{\ell-1})$ is a positive definite matrix of size $(\ell-1)\times(\ell-1)$ and $U = \Lambda_{22} - \frac{\Lambda_{12}^2}{\|\vec{x}\|^2}$. If $U < 0$, then the integral with respect to $\vec{r}$ vanishes. Otherwise we get after changing variables $\vec{r} = V^{-1/2}\vec{s}$ and applying the result of Proposition A.1

$$I(\Lambda) = \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma\left(\frac{\ell-1}{2}\right)}\int \mathrm{d}\,\vec{x}\delta\left(\|\vec{x}\|^2 - \Lambda_{11}\right)U_+^{\frac{\ell-3}{2}}\det{}^{-1/2}V\,|x_\ell|^{-1}.$$

It is easy to see that

$$\det V = 1 + \Big(\sum_{j=1}^{\ell-1} x_j^2\Big)/x_\ell^2 = \frac{\|\vec{x}\|^2}{x_\ell^2}.$$

Then

$$\begin{aligned}I(\Lambda) &= \frac{\pi^{\frac{\ell-1}{2}}}{\Gamma\left(\frac{\ell-1}{2}\right)}\int \mathrm{d}\,\vec{x}\delta\left(\|\vec{x}\|^2 - \Lambda_{11}\right)\left(\Lambda_{22}\|\vec{x}\|^2 - \Lambda_{12}^2\right)_+^{\frac{\ell-3}{2}}\|\vec{x}\|^{2-\ell}\\ &= \frac{2^{\ell-2}\pi^{\ell-1}}{\Gamma(\ell-1)}\det\left(I_2 - \Lambda^T\Lambda\right)_+^{\frac{\ell-3}{2}},\end{aligned}$$

where we used one more time Proposition A.1. Summarizing the above, we have shown for $\ell = 1$

$$\begin{aligned}\langle\widehat{F}\rangle_{\mathcal{X}_{L,M}} =& 2^M\frac{v_1 v_N}{v_{N+1}}\int_{\mathbb{R}_>^L}\mathrm{d}\,\vec{\lambda}F(Z)\,\Delta(Z)\prod_{j=1}^{L}\left(1-\lambda_j^2\right)_+^{-\frac{1}{2}}\\ &\int_{\mathbb{R}_>^M\times(\mathbb{R}_>^2)^M}\mathrm{d}\,\vec{\Lambda}\prod_{k=1}^{M}(\beta_k - \gamma_k)\,\delta\left(\det\left(I_2 - \Lambda_k^T\Lambda_k\right)\right),\end{aligned}$$

while for $\ell \geqslant 2$

$$\left\langle \widehat{F} \right\rangle_{\mathcal{X}_{L,M}} = \frac{2^{M(\ell-1)} \pi^{N\ell/2-M} v_\ell v_N}{v_{N+\ell} \Gamma^L\left(\frac{\ell}{2}\right) \Gamma^M(\ell-1)} \int_{\mathbb{R}^L_>} \mathrm{d}\,\vec{\lambda} F(Z) \Delta(Z) \prod_{j=1}^{L} \left(1-\lambda_j^2\right)_+^{\frac{\ell}{2}-1}$$

$$\int_{\mathbb{R}^M_> \times (\mathbb{R}^2_\geq)^M} \mathrm{d}\,\vec{\Lambda} \prod_{k=1}^{M} (\beta_k - \gamma_k) \det{}^{\frac{\ell-3}{2}} \left(I_2 - \Lambda_k^T \Lambda_k\right)_+ .$$

Any real symmetric 2×2 matrix is non-negative iff $\operatorname{Tr} M \geqslant 0$, and $\det M \geqslant 0$. Therefore, condition $\Lambda_k^T \Lambda_k \leqslant I_2$ is equivalent to

$$\begin{cases} 2\left(1-\alpha_k^2-\beta_k\gamma_k\right) - (\beta_k-\gamma_k)^2 \geqslant 0 \\ \left(1-\alpha_k^2-\beta_k\gamma_k\right)^2 - (\beta_k-\gamma_k)^2 \geqslant 0. \end{cases}$$

It is easy to see that the second condition implies the first one and therefore we suppress the first one. The last step is a change of variables from $(\alpha_k, \beta_k, \gamma_k)$ to $z_k = x_k + iy_k$. Let

$$\begin{cases} x_k &= \alpha_k, \\ y_k &= \sqrt{\beta_k\gamma_k}, \\ \delta_k &= \beta_k - \gamma_k. \end{cases}$$

This change of variables is two to one and its Jacobian is equal to

$$J_k = \frac{2y_k}{\sqrt{4y_k^2 + \delta_k^2}}.$$

Finally, for $\ell = 1$ integrating out the variables δ_k we obtain

$$\left\langle \widehat{F} \right\rangle_{\mathcal{X}_{L,M}} = \frac{2^M v_N}{v_{N+1}} \int \mathrm{d}\,\vec{\lambda} \mathrm{d}\,\vec{x} \mathrm{d}\,\vec{y} F(Z) \Delta(Z) \prod_{j=1}^{L} \left(1-\lambda_j^2\right)_+^{-\frac{1}{2}} \prod_{k=1}^{M} \frac{2y_k}{\left|1-z_k^2\right|}.$$

For $\ell \geqslant 2$ let $\Omega_k = \left\{(\alpha_k, \beta_k, \gamma_k) : \beta_k \geqslant \gamma_k \wedge (\beta_k-\gamma_k)^2 \leqslant \left(1-\alpha_k^2-\beta_k\gamma_k\right)^2\right\}$, then

$$\int_{\Omega_k} (\beta_k-\gamma_k) \left(\left(1-\alpha_k^2-\beta_k\gamma_k\right)^2 - (\beta_k-\gamma_k)^2\right)^{\frac{\ell-3}{2}} \mathrm{d}\,\alpha_k \mathrm{d}\,\beta_k \mathrm{d}\,\gamma_k$$

$$= \int_{\mathbb{R}^M_>} \mathrm{d}\,\vec{x} \int_{\mathbb{R}^M_+} \mathrm{d}\,\vec{y} \prod_{k=1}^{M} \int_0^{\left|1-x_k^2-y_k^2\right|} \mathrm{d}\,\delta_k \frac{4y_k\delta_k}{\sqrt{4y_k^2+\delta_k^2}} \left(\left(1-x_k^2-y_k^2\right)^2 - \delta_k^2\right)^{\frac{\ell-3}{2}}$$

$$= 4y_k \int_{-\infty}^{\infty} \mathrm{d}\,x_k \int_0^{\infty} \mathrm{d}\,y_k \left|1-z_k^2\right|^{\ell-2} \int_{\frac{2y_k}{\left|1-z_k^2\right|}}^{1} \left(1-u^2\right)^{\frac{\ell-3}{2}} \mathrm{d}\,u,$$

where we made the substitution $u = \sqrt{4y_k^2+\delta_k^2}\left|1-z_k^2\right|^{-1}$ in the last integral. Introducing

$$w_\ell^2(z) = \begin{cases} \dfrac{1}{2\pi}\left|1-z^2\right|^{-1}, & \ell = 1, \\ \dfrac{\ell(\ell-1)}{2\pi}\left|1-z^2\right|^{\ell-2} \displaystyle\int_{\frac{2|\operatorname{Im} z|}{\left|1-z^2\right|}}^{1} \left(1-u^2\right)^{\frac{\ell-3}{2}} \mathrm{d}\,u, & \ell \geqslant 2, \end{cases}$$

one can rewrite the answer as stated in Lemma 2.1. We like to stress that for real z the weight function can be written for any ℓ as

$$w_\ell^2(x) = \frac{\Gamma(\ell+1)}{2^\ell \Gamma^2\left(\frac{\ell}{2}\right)} \left|1-x^2\right|^{\ell-2},$$

which follows from calculating the integral

$$\int_0^1 \left(1-u^2\right)^{\frac{\ell-3}{2}} \mathrm{d}\, u = \frac{1}{2} B\left(\frac{\ell-1}{2}, \frac{1}{2}\right) = \frac{\pi 2^{1-\ell}\Gamma(\ell-1)}{\Gamma^2\left(\frac{\ell}{2}\right)}.$$

□

5 Proof of Lemma 2.3

Proof of Lemma 2.3. First we note that the skew-product of polynomials with indexes of the same parity is zero: The real part of the skew-product changes sign after changing $x \to -x$ and $y \to -y$ in the integral and hence has to vanish. For the complex part we first note that $w_\ell(x+iy)\, w_\ell(x-iy)$ is an even function of y, therefore only odd powers (because of the additional $\mathrm{sgn}(y)$ factor) contribute to the integral when expanding $\pi_j(x+iy)\, \pi_k(x-iy)$ in powers of x and y. However, these term will contain odd powers of x at the same time and thus will vanish when integrated over $[-1,1]$. For indexes with different parities we begin by calculating $\left(\varepsilon\left[w\pi_i\right]\right)(x)$ for even and odd values of i and real x. For complex argument there is no need to calculate anything, as the transformed polynomials are different from the original ones by a factor $i\mathrm{Im}(z)$ and complex conjugation only. Let $|x| \leqslant 1$ and $\ell \geqslant 2$, then

$$\begin{aligned}\left(\varepsilon\left[w_\ell z^{2k}\right]\right)(x) &= \frac{1}{2}\int_{\mathbb{R}} t^{2k} w_\ell(t)\, \mathrm{sgn}(t-x)\, \mathrm{d}\, t = -\,\mathrm{sgn}(x)\left(\frac{\ell!}{2^\ell\Gamma^2\left(\frac{\ell}{2}\right)}\right)^{1/2} \int_0^{|x|} t^{2k}\left(1-t^2\right)^{\frac{\ell}{2}-1} \mathrm{d}\, t \\ &= -\frac{1}{2}\left(\frac{\ell!}{2^\ell\Gamma^2\left(\frac{\ell}{2}\right)}\right)^{1/2} \mathrm{sgn}(x)\, B\left(x^2; k+\frac{1}{2}, \frac{\ell}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\varepsilon\left[w_\ell z^{2k+1}\right]\right)(x) &= \frac{1}{2}\int_{\mathbb{R}} t^{2k+1} w_\ell(t)\, \mathrm{sgn}(t-x)\, \mathrm{d}\, t = \left(\frac{\ell!}{2^\ell\Gamma^2\left(\frac{\ell}{2}\right)}\right)^{1/2} \int_{|x|}^{1} t^{2k+1}\left(1-t^2\right)^{\frac{\ell}{2}-1} \mathrm{d}\, t \\ &= \frac{1}{2}\left(\frac{\ell!}{2^\ell\Gamma^2\left(\frac{\ell}{2}\right)}\right)^{1/2} \left[B\left(k+1, \frac{\ell}{2}\right) - B\left(x^2; k+1, \frac{\ell}{2}\right)\right].\end{aligned}$$

Using definitions of polynomials $\pi_j(z)$ together with the identity

$$B(t; p+1, q) - \frac{p}{p+q} B(t; p, q) = -\frac{t^p(1-t)^q}{p+q},$$

$$(\varepsilon [w_\ell \pi_{2k}])(z) = \begin{cases} -\frac{1}{2}\left(\frac{\ell!}{2^\ell \Gamma^2\left(\frac{\ell}{2}\right)}\right)^{1/2} \operatorname{sgn}(x)\, B\left(x^2; k+\frac{1}{2}, \frac{\ell}{2}\right), & x \in \mathbb{R}, \\ i \operatorname{sgn}(\mathrm{Im} z)\, \overline{z}^{2k}, & z \in \mathbb{C}\backslash\mathbb{R}. \end{cases} \tag{5.1}$$

$$(\varepsilon [w_\ell \pi_{2k+1}])(z) = \begin{cases} \frac{1}{2k+\ell}\left(\frac{\ell!}{2^\ell \Gamma^2\left(\frac{\ell}{2}\right)}\right)^{1/2} x^{2k}\left(1-x^2\right)^{\frac{\ell}{2}}, & x \in \mathbb{R}, \\ i \operatorname{sgn}(\mathrm{Im} z)\, \overline{z}^{2k+1}, & z \in \mathbb{C}\backslash\mathbb{R}. \end{cases} \tag{5.2}$$

Now we proceed with the calculation of the skew-product.

$$(\pi_{2j}, \pi_{2k+1})^\ell = (\pi_{2j}, \pi_{2k+1})^\ell_{\mathbb{R}} + (\pi_{2j}, \pi_{2k+1})^\ell_{\mathbb{C}}.$$

For the real part we use (5.1) and (5.2) to write

$$\begin{aligned}(\pi_{2j}, \pi_{2k+1})^\ell_{\mathbb{R}} =& \frac{\ell!}{2^\ell \Gamma^2\left(\frac{\ell}{2}\right)} \int_{-1}^{1} x^{2j}\left(1-x^2\right)^{\frac{\ell}{2}-1} \frac{x^{2k}}{2k+\ell}\left(1-x^2\right)^{\frac{\ell}{2}} \mathrm{d}\, x \\ &+ \frac{\ell!}{2^\ell \Gamma^2\left(\frac{\ell}{2}\right)} \int_{-1}^{1} \frac{1}{2}\left(x^{2k+1} - \frac{2k}{2k+\ell} x^{2k-1}\right)\left(1-x^2\right)^{\frac{\ell}{2}-1} \operatorname{sgn}(x)\, B\left(x^2; j+\frac{1}{2}, \frac{\ell}{2}\right) \mathrm{d}\, x.\end{aligned}$$

The first integral is given by $\frac{B\left(j+k+\frac{1}{2},\ell\right)}{2k+\ell}$, while for the second one we use the important observation

$$\frac{\mathrm{d}}{\mathrm{d}\, x}\left(x^{2k}\left(1-x^2\right)^{\frac{\ell}{2}}\right) = -(2k+\ell)\left(x^{2k+1} - \frac{2k}{2k+\ell} x^{2k-1}\right)\left(1-x^2\right)^{\frac{\ell}{2}-1}.$$

Applying integration by parts to the second integral, we obtain

$$\begin{aligned}(\pi_{2j}, \pi_{2k+1})^\ell_{\mathbb{R}} &= \frac{\ell!}{2^\ell \Gamma^2\left(\frac{\ell}{2}\right)(2k+\ell)}\left(B\left(j+k+\frac{1}{2}, \ell\right) + 2\int_0^1 x^{2k+2j}\left(1-x^2\right)^{\ell-1} \mathrm{d}\, x\right) \\ &= \frac{\ell!}{2^{\ell-1}\Gamma^2\left(\frac{\ell}{2}\right)(2k+\ell)} B\left(j+k+\frac{1}{2}, \ell\right).\end{aligned}$$

For the complex part of the skew-product and $\ell \geqslant 2$ symmetry of the integrand yields

$$(\pi_{2j}, \pi_{2k+1})^\ell_{\mathbb{C}} = 2i\frac{\ell(\ell-1)}{2\pi}\int_{\mathbb{D}} z^{2j}\left(\overline{z}^{2k+1} - \frac{2k}{2k+\ell}\overline{z}^{2k-1}\right) \operatorname{sgn}(\mathrm{Im} z)\left|w_\ell^2(z)\right| \mathrm{d}^2 z.$$

One can check that

$$\begin{aligned}\frac{\partial}{\partial x}(x-iy)^{2k}\left(1-(x-iy)^2\right)^{\frac{\ell}{2}} &= i\frac{\partial}{\partial y}(x-iy)^{2k}\left(1-(x-iy)^2\right)^{\frac{\ell}{2}} \\ &= -(2k+\ell)\,\pi_{2k+1}(x-iy)\left(1-(x-iy)^2\right)^{\frac{\ell}{2}-1}.\end{aligned}$$

Using this, integration by parts with respect to x implies

$$\begin{aligned}(\pi_{2j}, \pi_{2k+1})^\ell_{\mathbb{C}} =& \frac{2i}{2\pi(2k+\ell)}\int_{\mathbb{D}} \overline{z}^{2k}\left(1-\overline{z}^2\right)^{\frac{\ell}{2}} \frac{\partial}{\partial x}\left(z^{2j}\left(1-z^2\right)^{\frac{\ell}{2}-1}\right) \operatorname{sgn}(\mathrm{Im} z)\, \widehat{w}_\ell(z)\, \mathrm{d}^2 z \\ &+ \frac{2i}{2\pi(2k+\ell)}\int_{\mathbb{D}} \overline{z}^{2k}\left(1-\overline{z}^2\right)^{\frac{\ell}{2}} z^{2j}\left(1-z^2\right)^{\frac{\ell}{2}-1} \operatorname{sgn}(\mathrm{Im} z)\frac{\partial}{\partial x}\widehat{w}_\ell(z)\, \mathrm{d}^2 z,\end{aligned} \tag{5.3}$$

where

$$\widehat{w}_\ell(z) = \ell(\ell-1) \int\limits_{\frac{2|\mathrm{Im}z|}{|1-z^2|}}^{1} \left(1-u^2\right)^{\frac{\ell-3}{2}} \mathrm{d}\,u.$$

Integration by parts with respect to y gives

$$\begin{aligned}
(\pi_{2j}, \pi_{2k+1})_{\mathbb{C}}^{\ell} = &- \frac{2}{2\pi(2k+\ell)} \int\limits_{\mathbb{D}} \overline{z}^{2k} \left(1-\overline{z}^2\right)^{\frac{\ell}{2}} \frac{\partial}{\partial y} \left(z^{2j}\left(1-z^2\right)^{\frac{\ell}{2}-1}\right) \mathrm{sgn}\,(\mathrm{Im}z)\, \widehat{w}_\ell(z)\, \mathrm{d}^2 z \\
&- \frac{2}{2\pi(2k+\ell)} \int\limits_{\mathbb{D}} \overline{z}^{2k} \left(1-\overline{z}^2\right)^{\frac{\ell}{2}} z^{2j} \left(1-z^2\right)^{\frac{\ell}{2}-1} \frac{\partial}{\partial y} (\,\mathrm{sgn}\,(\mathrm{Im}z)\, \widehat{w}_\ell(z))\, \mathrm{d}^2 z \\
&- 4\frac{\ell(\ell-1)}{2\pi(2k+\ell)} \int\limits_{-1}^{1} x^{2j+2k}\left(1-x^2\right)^{\ell-1} \int\limits_{0}^{1} \left(1-u^2\right)^{\frac{\ell-3}{2}} \mathrm{d}\,u. \qquad (5.4)
\end{aligned}$$

We now add together (5.3) and (5.4) to obtain twice the skew-product. When adding the first terms of the above expressions, we note that the integrand can be written as $F(z)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) P(z)$, for some function F and a polynomial P. This obviously vanishes since $\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy) = 0$. The third term in (5.4) can be calculated explicitly, while the second terms of (5.3) and (5.4) can be merged together to obtain

$$\begin{aligned}
(\pi_{2j}, \pi_{2k+1})_{\mathbb{C}}^{\ell} = &\frac{1}{2\pi(2k+\ell)} \int\limits_{\mathbb{D}} \overline{z}^{2k} \left(1-\overline{z}^2\right)^{\frac{\ell}{2}} z^{2j} \left(1-z^2\right)^{\frac{\ell}{2}-1} \left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) (\,\mathrm{sgn}\,(\mathrm{Im}z)\, \widehat{w}_\ell(z))\, \mathrm{d}^2 z \\
&- \frac{\ell!}{2^{\ell-1}\Gamma^2\left(\frac{\ell}{2}\right)(2k+\ell)} B\left(j+k+\frac{1}{2}, \ell\right).
\end{aligned}$$

The former term coincides with $(\pi_{2j}, \pi_{2k+1})_{\mathbb{R}}^{\ell}$. For the first one we note

$$\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(\,\mathrm{sgn}\,(\mathrm{Im}z)\, \widehat{w}_\ell(z)) = 2\ell(\ell-1)\frac{\left(1-|z|^2\right)^{\ell-2}}{|1-z^2|^{\ell}}\left(1-z^2\right).$$

This yields

$$\begin{aligned}
(\pi_{2j}, \pi_{2k+1})_{\mathbb{C}}^{\ell} &= -(\pi_{2j}, \pi_{2k+1})_{\mathbb{R}}^{\ell} + \frac{2\ell(\ell-1)}{2\pi(2k+\ell)} \int\limits_{\mathbb{D}} \overline{z}^{2k} z^{2j} \left(1-|z|^2\right)^{\ell-2} \mathrm{d}^2 z \\
&= -(\pi_{2j}, \pi_{2k+1})_{\mathbb{R}}^{\ell} + \frac{1}{\binom{2k+\ell}{\ell}} \delta_{k,j}.
\end{aligned}$$

The derivation for $\ell = 1$ is even simpler and one should just put $\widehat{w}_\ell(z) \equiv 1$ for all z. ☐

6 Auxiliary results used in Section 3

6.1 Proof of Lemma 3.1 and Lemma 3.4

Proof of Lemma 3.1. Assume by contradiction that there exists an $n \in \mathbb{N}$ such that $\|H_n\| = 1$. Hence, there exists a $\varphi \in \ell^2(\mathbb{N}_0)$ with $\|\varphi\|_2 = 1$ such that $\langle\varphi, H\varphi\rangle = 1$. Since $\|H\| = 1$ this implies that 1 is a boundary point of the numerical range of H but this implies 1 is a proper eigenvalue [17]. This contradicts purely absolutely continuous spectrum of the operator H and the assertion follows. □

Proof of Lemma 3.4. The operator inequality $H^m 1_{\leqslant\varepsilon}(H) \leqslant \varepsilon^{m-1} H$ implies

$$\operatorname{Tr}\big(1_n H^m 1_{\leqslant\varepsilon}(H) 1_n\big) \leqslant \varepsilon^{m-1} \operatorname{Tr}\big(1_n H 1_n\big).$$

Hence, we obtain

$$\begin{aligned}\sum_{m\in\mathbb{N}} \frac{\operatorname{Tr}\big(1_n H^m 1_{\leqslant\varepsilon}(H) 1_n\big)}{m} &\leqslant \sum_{m\in\mathbb{N}} \frac{\varepsilon^{m-1}}{m} \operatorname{Tr}\big(1_n H 1_{\leqslant\varepsilon}(H) 1_n\big) \\ &\leqslant \frac{1}{\varepsilon} \log(1-\varepsilon) \operatorname{Tr}\big(1_n H h_\varepsilon(H)\big),\end{aligned}$$

where $h_\varepsilon \in C([0,1])$ is a continuous function such that $1_{<\varepsilon} \leqslant h_\varepsilon \leqslant 1_{<2\varepsilon}$. It follows from Lemma 3.2 that

$$\limsup_{n\to\infty} \frac{\operatorname{Tr}\big(1_n H^m 1_n\big)}{\log n} = \frac{1}{\pi} \int_0^\infty \operatorname{sech}^m(u\pi)\,\mathrm{d}u.$$

From this and a Stone-Weierstraß argument we infer that

$$\limsup_{n\to\infty} \frac{\operatorname{Tr}\big(1_n H g(H) 1_n\big)}{\log n} = \frac{1}{\pi} \int_0^\infty g(\operatorname{sech}(u\pi)) \operatorname{sech}(u\pi)\,\mathrm{d}\,u$$

for all $g \in C([0,1])$. Therefore, we obtain that

$$\limsup_{n\to\infty} \frac{\operatorname{Tr}\big(1_n H h_\varepsilon(H)\big)}{\log n} = \frac{1}{\pi} \int_0^\infty h_\varepsilon(\operatorname{sech}(u\pi)) \operatorname{sech}(u\pi)\,\mathrm{d}u$$

and from dominated convergence the assertion

$$\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\operatorname{Tr}\big(1_n H h_\varepsilon(H)\big)}{\log n} = 0.$$

□

6.2 Proof of Lemma 3.6 and Lemma 3.7

Proof of Lemma 3.6. For $x > 0$ and $l \in \mathbb{N}_0$, we define

$$F_l(x) := \Big(\frac{1}{4}\Big)_l \Big(\frac{1}{2}\Big)_l \Big(\frac{3}{4}\Big)_l {}_4F_3\Big(-l, l+\frac{1}{2}, i\frac{x}{2}, -i\frac{x}{2}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1\Big)$$

Then, it follows from [58, Eq. (2.5)] that we have the asymptotics

$$F_l(x) = (2\pi)^{3/2} e^{-3l} l^{3l} \big(2|A(ix/2)| \cos\big(x\log l + \arg(A(ix/2)\big) + l^{-1}\widetilde{R}_l(x)\big) \tag{6.1}$$

as $l \to \infty$ with

$$\sup_{x\in[0,M]} \sup_{l\in\mathbb{N}_0} \big|\widetilde{R}_l(x)\big| \leqslant \widetilde{r}(M)$$

for some constant $\widetilde{r}(M)$ depending on M. We recall $|A(ix/2)| = \dfrac{\cosh(\pi x)^{1/2}}{(2\pi)^{3/2}}$. On the other hand, the asymptotics

$$\frac{e^{-3l} l^{3l} 4^l}{l! \left(\frac{1}{2}\right)_{2l}} = \frac{1}{2\sqrt{\pi}\sqrt{l}} + O\Big(\frac{1}{l^{3/2}}\Big) \tag{6.2}$$

holds as $l \to \infty$. To see this, we use that $\left(\frac{1}{2}\right)_{2l} = \dfrac{\Gamma(\frac{1}{2}+2l)}{\Gamma(\frac{1}{2})}$ and Stirling's formula. This impies

$$\frac{e^{-3l} l^{3l} 4^l}{l! \left(\frac{1}{2}\right)_{2l}} = \frac{e^{-2l} l^{2l} 4^l \sqrt{\pi}}{\sqrt{2\pi l}\,\Gamma(2l+\frac{1}{2})} \Big(\frac{1}{1+O(1/l)}\Big). \tag{6.3}$$

Now, the identity $\Gamma(2l+\frac{1}{2}) = \dfrac{(4l)!\sqrt{\pi}}{4^{2l}(2l)!}$ and Stirling's formula give

$$\begin{aligned}
(6.3) &= \frac{e^{-2l} l^{2l} 4^{3l} \sqrt{4\pi l}(2l/e)^{2l}}{\sqrt{2\pi l}\sqrt{8\pi l}(4l/e)^{4l}} \big(1+O(1/l)\big) \\
&= \frac{1}{2\sqrt{\pi}} \frac{n^{4l}}{\sqrt{l} l^{4l}} \\
&= \frac{1}{2\sqrt{\pi}\sqrt{l}} \big(1+O(1/l)\big).
\end{aligned}$$

This proves (6.2). Inserting (6.1) in (3.4), (6.2), gives the assertion. ☐

Proof of Lemma 3.7. Using (3.5) and Lemma 3.6, we obtain

$$\begin{aligned}
&\sum_{m\in\mathbb{N}} \frac{1}{m} \operatorname{Tr}\big(1_n 1_{>\varepsilon}(H) H^m 1_n\big) \\
&= -\frac{2}{\pi} \int_0^{\frac{\operatorname{sech}^{-1}(\varepsilon)}{\pi}} \log\big(1-\operatorname{sech}(x\pi)\big) \\
&\qquad \times \Big(\sum_{l=0}^{n-1} \frac{1}{l} \cos^2\big(x\log l + \arg(A(ix/2))\big) + l^{-3/2}\hat{R}_l(x)\Big) \mathrm{d}\,x,
\end{aligned}$$

where the error term satisfies $\sup_{l\in\mathbb{N}_0} |\hat{R}_l(x)| < r(\varepsilon)$. Since the latter is integrable and $l^{-3/2}\hat{R}_l(x)$ is summable in l, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\sum_{m\in\mathbb{N}} \frac{1}{m} \operatorname{Tr}\big(1_n 1_{>\varepsilon}(H) H^m 1_n\big)}{\log n} \\
&= \limsup_{n\to\infty} -\frac{2}{\pi\log n} \int_0^{\frac{\operatorname{sech}^{-1}(\varepsilon)}{\pi}} \log\big(1-\operatorname{sech}(x\pi)\big) \\
&\qquad \times \Big(\sum_{l=0}^{n-1} \frac{1}{l} \cos^2\big(x\log l + \arg(A(ix/2))\big)\Big) \mathrm{d}\,x.
\end{aligned} \tag{6.4}$$

The identity

$$\begin{aligned}\cos^2\big(x\log n+\arg(A(ix/2)\big) &= \frac{1}{2}+\frac{1}{2}\cos\left(2x\log l\right)\cos\left(2\arg A(ix/2)\right)\\ &\quad-\frac{1}{2}\sin\left(2x\log l\right)\sin\left(2\arg A(ix/2)\right)\\ &=: \frac{1}{2}+F(x,l)\end{aligned}\tag{6.5}$$

and the asymptotics of the harmonic series yield

$$\begin{aligned}(6.4)\leqslant&-\frac{1}{\pi}\int_0^\infty \log(1-\operatorname{sech}(x\pi))\mathrm{d}\,x\\ &+\limsup_{n\to\infty}\frac{-2}{\pi\log n}\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}\log(1-\operatorname{sech}(x\pi))\sum_{l=1}^{n-1}\frac{1}{l}F(x,l)\mathrm{d}\,x.\end{aligned}\tag{6.6}$$

We are left with estimating the second term. To do this, we split the latter integral further. For fixed $\delta>0$ let $0\leqslant g_\delta\in C_c^\infty\big([0,\frac{\operatorname{sech}^{-1}(\varepsilon)}{\pi}]\big)$ and $g_\delta(x)=1$ for all $x\in\big(\delta,\operatorname{sech}^{-1}(\varepsilon)-\delta\big)$. Set

$$G(x):=\log(1-\operatorname{sech}(x\pi))\sum_{l=1}^{n-1}\frac{1}{l}F(x,l)$$

and we split the latter integral in the following way

$$\frac{2}{\pi}\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}G(x)\mathrm{d}\,x=\frac{2}{\pi}\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}g_\delta(x)G(x)\mathrm{d}\,x+\frac{2}{\pi}\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}(1-g_\delta(x))G(x)\mathrm{d}\,x.\tag{6.7}$$

The function $g_\delta(\,\cdot\,)\log(1-\operatorname{sech}(\pi\,\cdot\,))e^{2i\arg(A(i(\cdot)/2))}$ is smooth, compactly supported and exponentially decaying for any fixed $\varepsilon,\delta>0$. Hence, integration by parts implies that for any $k\in\mathbb{N}$

$$\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}g_\delta(x)\log(1-\operatorname{sech}(x\pi))e^{\pm 2i(x\log l+\arg(A(ix/2)))}\mathrm{d}\,x=O(1/(\log l)^k)\tag{6.8}$$

as $l\to\infty$. Writing the sin and cos terms in $F(x,n)$, see (6.5), in terms of exponentials and using the latter with $k=2$, we obtain that

$$\frac{2}{\pi}\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}g_\delta(x)G(x)\mathrm{d}\,x=O\Big(\sum_{l=0}^{n-1}\frac{1}{l(\log l)^2}\Big)=O(1),\tag{6.9}$$

as $l\to\infty$. For the second integral in (6.7), we note that

$$\sup_{x\in(0,\infty)}\sup_{n\in\mathbb{N}_0}|F(n,x)|\leqslant 1.$$

Therefore, using the asymptotics of the harmonic series, we obtain as $n\to\infty$

$$\Big|\frac{2}{\pi}\int_0^{\frac{\sinh^{-1}(\varepsilon)}{\pi}}\big(1-g_\delta(x)\big)G(x)\mathrm{d}\,x\Big|\leqslant\log n\int_0^\infty\big|\big(1-g_\delta(x)\big)\log(1-\operatorname{sech}(x\pi))\big|\mathrm{d}\,x+O(1).\tag{6.10}$$

We obtain from (6.6), (6.7), (6.9) and (6.10) that for all $\delta > 0$

$$\limsup_{n\to\infty} \frac{\sum_{m\in\mathbb{N}} \frac{1}{m} \operatorname{tr}\left(1_n H^m 1_{>\varepsilon}(H) 1_n\right)}{\log n} \leqslant -\frac{1}{\pi}\int_0^\infty \log(1-\operatorname{sech}(x\pi))dx + \int_0^{\frac{\operatorname{sech}^{-1}(\varepsilon)}{\pi}} \left|\left(1-g_\delta(x)\right)\log(1-\operatorname{sech}(x\pi))\right| \mathrm{d}\, x.$$

Taking the limit $\delta \to 0$, the last term in the latter vanishes by dominated convergence using that $\log(1-\operatorname{sech}(\cdot\,\pi))$ is integrable. This gives the assertion. ☐

7 Open problems and conjectures

In this paper we derived an explicit expression (1.6) for the "persistence" probability of truncations of random orthogonal matrices of size ℓ. In the case $\ell = 1$ we were also able to perform an asymptotic analysis of this probability in (1.14). It is natural to ask what happens for $\ell > 1$ or even for ℓ growing with n. These questions have their own applications to the distribution of roots of random, matrix valued, polynomials with coefficients given by Real Ginibre matrices of size $\ell \times \ell$, see [23] for details. Looking at (1.6), one has to analyse the determinant of identity minus a weighted Hankel matrix in the case of $\ell > 1$. We claim that the methods described in this paper are also applicable to this case and a similar analysis using the results of [53], gives

Conjecture 7.1. *Let ℓ be a fixed integer number and the ensemble of random matrices M_{2n} be defined as in Theorem 1.1. Then the corresponding persistence probability decays as*

$$\lim_{n\to\infty} \frac{\log p_{2n}^{(\ell)}}{\log n} = -2\theta(\ell), \quad \text{with} \quad \theta(\ell) = -\frac{1}{2\pi}\int_0^\infty \log\left(1-\left|\frac{\Gamma\left(\frac{\ell}{2}+ix\right)}{\Gamma\left(\frac{\ell}{2}\right)}\right|^2\right)\mathrm{d}\, x. \tag{7.1}$$

A rigorous proof of the above will be the content of a future work. So far we were not successful in finding a closed form of the integral (7.1). However, we could rewrite the above in terms of a random walk. More precisely, we obtain

Lemma 7.2. *Let $\{\xi_j\}_{j=1}^\infty$ be a family of i.i.d. random variables having probability density function*

$$\rho_\ell(x) = \frac{1}{2B\left(\frac{\ell}{2},\frac{1}{2}\right)}\operatorname{sech}^\ell\left(\frac{x}{2}\right),$$

and $S_k = \sum_{j=1}^k \xi_j$ be a random walk with corresponding steps. Let τ be the first hitting time of the origin, then

$$\theta(\ell) = \frac{1}{4}\mathbb{P}\left[S_\tau \in \mathrm{d}\,0\right],$$

where by $\mathbb{P}\left[\zeta \in \mathrm{d}\,0\right]$ we mean the probability density function of the random variable ζ evaluated at the origin.

This lemma will also be proved in a future work. As discussed in the introduction, the persistence probability of rank-one truncations of random orthogonal matrices has evident connections to the persistence probability of sech correlated Gaussian Stationary Processes (GSP). The analysis of the corresponding GSP led us in [47] to the study of the related persistence problem for the latter random walk with the parameter ℓ set to one. One may expect that for general $\ell > 1$

there should be a connection of (7.1) to GSP with $\operatorname{sech}^{\ell}\left(\frac{x}{2}\right)$ correlated process. However, an accurate comparison of our numerical results with the one found in [45] shows some mismatch.

Another intriguing and challenging question is to study the asymptotics of the persistence problem for our ensemble of random matrices when the parameter ℓ is growing in n. Here one would expect some phase transition from the weak non-orthogonality universality class, corresponding to $\ell/n = o(1)$, to the Real Ginibre universality class when $\ell/n \to \infty$ (compare to Proposition 1.3). In full generality the problem is yet to be solved, but some partial results can be already obtained given the above conjecture.

Conjecture 7.3. *For large integers ℓ the decay exponent $\theta(\ell)$ behaves as*

$$\theta(\ell) = \frac{1}{4}\sqrt{\frac{\ell}{2\pi}}\zeta(3/2)(1 + o(1)), \quad \ell \to \infty. \tag{7.2}$$

This can be either confirmed by an asymptotic analysis of the Gamma-function or by approximating the random walk described above by a random walk with Gaussian $N\left(0, \frac{4}{\ell}\right)$ distributed steps. By formally taking $\ell = 2n$, corresponding to a transition from singular to non-singular measure in (2.3), one gets half of the corresponding result for the Real Ginibre ensemble (see, [36, Thm. 1.1]). The factor one half originates from the fact that truncated orthogonal matrix can not have eigenvalues outside the unit disk, but the Real Gibre random matrix can.

Apart from studying the probability of having no real eigenvalues for a random matrix, one can also look at the probability $p_{2n,2k}^{(\ell)}$ of having $2k$ real eigenvalues. For $k = 2n$ we computed this probability in Proposition 1.3. In the intermediate regime, $0 < k < 2n$ we expect that the answer doesn't change until the point when k changes from 0 to roughly the average number of real roots, see similar results [36]. For k being of order of n, analogously to the result of [15], we expect that the probability will decay exponentially in terms of n^2 with some non-trivial coefficient depending on a ratio k/n. For other values of k the problem seem very challenging and technical.

A Volume of orthogonal group

Proposition A.1. *Let A be a real number. Then*

$$I_m(A) = \int\limits_{\mathbb{R}^m} \delta\left(\vec{x}^T\vec{x} - A\right) \mathrm{d}\,x = \frac{\pi^{\frac{m}{2}} A_+^{\frac{m-2}{2}}}{\Gamma\left(\frac{m}{2}\right)}, \text{ where } \quad x_+ = \max\{0, x\}.$$

Proof of Proposition A.1. For $m = 1$ the statement is obvious. For $m \geqslant 2$ we change to polar coordinates The integral above is now equal to

$$\begin{aligned} I_m(A) &= \int\limits_0^\infty r^{m-1}\delta\left(r^2 - A\right)\mathrm{d}\,r \int\limits_0^{2\pi} \mathrm{d}\,\phi_{m-1} \prod_{j=1}^{m-2} \int\limits_0^\pi \sin^{m-1-j}\phi_j \mathrm{d}\,\phi_j \\ &= \frac{1}{2}A_+^{\frac{m-2}{2}} 2\pi \prod_{j=1}^{m-2} B\left(\frac{m-j}{2}, \frac{1}{2}\right) = \frac{\pi^{\frac{m}{2}} A_+^{\frac{m-2}{2}}}{\Gamma\left(\frac{m}{2}\right)}. \end{aligned}$$

□

Proposition A.2. *The volume of the orthogonal group $O(N)$ is equal to*

$$v_N = \int\limits_{\mathbb{R}^{N^2}} \delta\left(O^T O - I_N\right) \mathrm{d}\,O = \prod_{j=1}^{N} \frac{\pi^{j/2}}{\Gamma\left(\frac{j}{2}\right)},$$

where $\mathrm{d}\,O$ *is the flat Lebesgue measure on* $\mathbb{R}^{N^2}$.

Remark A.1. *This is different to what was stated in [38].*

Proof of Proposition A.2. We proof the statement by induction. For $N = 1$ one can easily check that

$$\int\limits_{\mathbb{R}} \delta\left(x^2 - 1\right) \mathrm{d}\,x = 1.$$

Let now $N = k + 1$ and we split every matrix into blocks of the following form

$$O_{k+1} = \begin{pmatrix} m & \vec{b}^T \\ \vec{c} & D \end{pmatrix},$$

where $\vec{b}, \vec{c}$ are k-dimensional column vectors and D is a $k \times k$ real matrix. The corresponding integral can now be written as

$$v_{k+1} = \int \delta\left(m^2 + \vec{c}^T \vec{c} - 1\right) \delta\left(m\vec{b}^T + \vec{c}^T D\right) \delta\left(\vec{b}\vec{b}^T + D^T D - I_k\right) \mathrm{d}\,m \mathrm{d}\,\vec{b} \mathrm{d}\,\vec{c} \mathrm{d}\,D.$$

Integrating out $\vec{b}$, we obtain

$$v_{k+1} = \int \mathrm{d}\,m \mathrm{d}\,\vec{c} \mathrm{d}\,D \delta\left(m^2 + \vec{c}^T \vec{c} - 1\right) \delta\left(D^T \frac{\vec{c}\vec{c}^T}{m^2} D + D^T D - I_k\right) |m|^{-k}.$$

Integration over D can be performed by using the induction hypothesis. We define

$$V = I_k + \frac{\vec{c}\vec{c}^T}{m^2},$$

which is a real symmetric, positive definite rank one perturbation of the identity with determinant $\det V = I + \frac{\vec{c}^T \vec{c}}{m^2}$. Changing variables with

$$D = V^{-1/2} \hat{D},$$

one gets

$$\begin{aligned} v_{k+1} &= \int \delta\left(m^2 + \vec{c}^T \vec{c} - 1\right) \delta\left(\hat{D}^T \hat{D} - I_k\right) |m|^{-k} \det{}^{-k/2} V \mathrm{d}\,m \mathrm{d}\,\vec{c} \mathrm{d}\,\hat{D} \\ &= v_k \int \delta\left(m^2 + \vec{c}^T \vec{c} - 1\right) \left(m^2 + \vec{c}^T \vec{c}\right)^{-k/2} \mathrm{d}\,m \mathrm{d}\,\vec{c} \\ &= v_k \int \delta\left(m^2 + \vec{c}^T \vec{c} - 1\right) \mathrm{d}\,m \mathrm{d}\,\vec{c}. \end{aligned}$$

Finally, we integrate over $\vec{c}$ using Proposition A.1 and obtain

$$v_{k+1} = v_k \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \int\limits_{\mathbb{R}} \mathrm{d}\,m \left(1 - m^2\right)_+^{\frac{k}{2} - 1} = v_k \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} B\left(\frac{k}{2}, \frac{1}{2}\right) = v_k \frac{\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)},$$

and the statement follows. □

B Properties of Pfaffians and Proof of Proposition 2.4

The Pfaffian is an analogue of the determinant defined for skew-symmetric matrices of even size. Let $A = \{a_{j,k}\}_{j,k=1}^{2n}$ be a skew-symmetric matrix with entries $a_{j,k} = -a_{k,j}$, $j,k = 1,\dots,2n$. Then its Pfaffian is defined by

$$\mathrm{Pf}\, A = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}\,(\sigma) \prod_{j=1}^{n} a_{\sigma(2j-1)} a_{\sigma(2j)},$$

where the sum is taken over all permutations of elements $(1,2,\dots,2n)$. For skew-symmetric matrices of odd size the Pfaffian is defined to be zero. The Pfaffian can be thought as a square root of the determinant because of an identity

$$\mathrm{Pf}^{\,2} A = \det A,$$

valid vor any skew-symmetric matrix. Below we also use another definition of the Pfaffian via integration over Grassmann (anticommuting) variables. Let $(\phi_1,\phi_2,\dots,\phi_j,\dots)$ and $(\psi_1,\psi_2,\dots,\psi_j,\dots)$ be two families of anticommuting variables

$$\phi_j\phi_k = -\phi_k\phi_j\,, \phi_j\psi_k = -\psi_k\phi_j\,, \psi_j\psi_k = -\psi_k\psi_j\,. \tag{B.1}$$

Functions of Grassmann variables are defined by the corresponding Taylor series, which are always finite because of Grassmann variables being nilpotent. The Berezin integral with respect to these variables is formally defined using the identities

$$\int \mathrm{d}\,\phi_j = \int \mathrm{d}\,\psi_j = 0, \int \mathrm{d}\,\phi_j\phi_j = \int \mathrm{d}\,\psi_j\psi_j = 1, \tag{B.2}$$

and a multiple integral is defined to be a repeated one. Then for any matrix M of size $n \times n$ one can see that

$$\int \mathrm{d}\,\phi_1 \mathrm{d}\,\psi_1 \mathrm{d}\,\phi_2 \mathrm{d}\,\psi_2 \dots \mathrm{d}\,\phi_n \mathrm{d}\,\psi_n \exp\left\{ -\sum_{j,k=1}^{n} M_{j,k}\phi_j\psi_k \right\} = \det M.$$

The above follows from two simple observations: Expanding the exponential function and using (B.1) and (B.2), one sees that the integral on the left is given by the coefficient in front of the monomial $\phi_1\psi_1\phi_2\psi_2\dots\phi_n\psi_n$. This term comes only from expanding $\frac{1}{n!}\Big(-\sum_{j,k=1}^{n} M_{j,k}\phi_j\psi_k \Big)^n$. Analogously, one can also write the Pfaffian in terms of the Berezin integral. Let M be a skew-symmetric matrix of size $2n \times 2n$, then the result reads

$$\int \mathrm{d}\,\phi_1 \dots \mathrm{d}\,\phi_{2n} \exp\left\{ -\frac{1}{2} \sum_{j,k=1}^{2n} M_{j,k}\phi_j\phi_k \right\} = \mathrm{Pf}\, M.$$

This also follows from finding the coefficient in front of the monomial $\phi_1\phi_2\dots\phi_{2n}$ that in its turn comes from expanding $\frac{1}{n!}\Big(-\frac{1}{2} \sum_{j,k=1}^{2n} M_{j,k}\phi_j\phi_k \Big)^n$. For more information about Berezin integrals and Grassmann variables we refer to [7] and about Pfaffians to [28].

Proof of Proposition 2.4. Writing $\operatorname{Pf} A$ as a Berezin integral over Grassmann variables, we obtain

$$\operatorname{Pf} A = \int \mathrm{d}\,\psi_0 \ldots \mathrm{d}\,\psi_{2n-1} \exp\left\{-\frac{1}{2}\sum_{j,k=0}^{2n-1} A_{j,k}\psi_j\psi_k\right\}.$$

A has checkboard pattern, and therefore there are no terms in the exponent containing $\psi_j\psi_k$ with even $j-k$. Let us split the Grassmann variables into two groups: with even and odd indexes which do not "interact". Then

$$\begin{aligned}\operatorname{Pf} A &= \int \mathrm{d}\,\psi_0 \ldots \mathrm{d}\,\psi_{2n-1} \exp\left\{-\frac{1}{2}\sum_{j,k=0}^{n-1} \psi_{2j}\psi_{2k-1}\left(A_{2j,2k-1} - A_{2k-1,2j}\right)\right\} \\ &= \int \mathrm{d}\,\psi_0 \ldots \mathrm{d}\,\psi_{2n-1} \exp\left\{-\sum_{j,k=0}^{n-1} \psi_{2j}\psi_{2k-1}A_{2j,2k-1}\right\} = \det A',\end{aligned}$$

where we used determinant representation via Grassmann variables with $A' = \{a_{2i,2j+1}\}_{i,j=0}^{n-1}$. $\square$

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