ITÔ'S FORMULA FOR JUMP PROCESSES IN L_p-SPACES

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ABSTRACT. We present an Itô formula for the L_p-norm of jump processes having stochastic differentials in L_p-spaces. The main results extend well-known theorems of Krylov to the case of processes with jumps, and which can be used to prove existence and uniqueness theorems in L_p-spaces for SPDEs driven by Lévy processes.

1. INTRODUCTION

Itô formulas for semimartingales taking values in function spaces play important roles in the theory of stochastic partial differential equations (SPDEs). To get a priori estimates in the L_2-theory of SPDEs, driven by a cylindrical Wiener process $(w_t^1, w_t^2, ...)_{t\in[0,T]}$, one usually needs a suitable formula for $|u_t|_H^2$, the square of H-valued solutions $(u_t)_{t\in[0,T]}$ to SPDEs, when H is a Hilbert space. In the framework of L_2-theory there is a Banach space V, embedded continuously and densely in H, such that $u_t \in V$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, and from the definition of the solution it follows that for some processes $(v_t^*)_{t\in[0,T]}$ and $(g_t^r)_{t\in[0,T]}$, with values in V^* and H, respectively, for $r = 1, 2, ...$,

$$du_t = v_t^*\, dt + g_t^r\, dw_t^r \quad \text{for } P \otimes dt\text{-a.e. } (\omega, t) \in \Omega \times [0, T], \tag{1.1}$$

where V^* is the adjoint of V. (Here, and later on, the summation convention with respect to repeated integer valued indices is used, i.e., $(g_t^r, \varphi)\, dw_t^r$ means $\sum_r (g_t^r, \varphi)\, dw_t^r$.) A basic example for such couple of spaces V and H is the couple of Hilbert spaces W_2^1 and L_2 of real functions defined on the whole Euclidean space $\mathbb{R}^d$. In this case equation (1.1) can be rewritten as

$$du_t = D_\alpha f_t^\alpha\, dt + g_t^r\, dw_t^r \tag{1.2}$$

with some L_2-valued processes $(f_t^\alpha)_{t\in[0,T]}$, $\alpha = 0, 1, 2, ..., d$, where $D_\alpha = \frac{\partial}{\partial x^\alpha}$ for $\alpha = 1, 2, ..., d$, and D_α is the identity operator for $\alpha = 0$. It was first proved in [13] that if (1.1) holds and u, f and g satisfy appropriate measurability and integrability conditions then u has a continuous H-valued modification, denoted also by u, such that $|u_t|_H^2$ has the stochastic differential

$$d|u_t|_H^2 = \big(2 < u_t, v_t^* > + \|g_t\|_H^2\big)\, dt + 2(u_t, g_t^r)\, dw_t^r, \tag{1.3}$$

where $\|h\|_H^2 = \sum_r |h^r|_H^2$, and $(\ ,\ )$ and $<\ ,\ >$ denote the inner product in H and the duality product of V and V^*, respectively. The proof of this result in [13] was combined with the theory of SPDEs developed there. A direct proof was first given in [11], see also [15] and [16], and a very nice shorter proof is presented in [12] when V is a Hilbert space. To study SPDEs driven by (possibly discontinuous) semimartingales, processes u satisfying (1.1) with dA_t and dM_t in place of dt and dw_t were considered, and a theorem on Itô's formula was

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2010 *Mathematics Subject Classification.* Primary 60H05, 60H15; Secondary 35R60.

Key words and phrases. Itô formula, random measures, Lévy processes.

proved in [6], when $A=(A_t)_{t\in[0,T]}$ and $M=(M^1_t,M^2_t,...)_{t\in[0,T]}$ are (possibly discontinuous) increasing processes and martingales, respectively. In this situation a further generalisation was given in [7]. In the special case when $V=W^1_2$, $H=L_2$ and equation (1.2) holds, Itô's formula (1.3) has the form

$$d|u_t|^2_{L_2}=\big(2(D^*_\alpha u_t,f^\alpha_t)+\|g_t\|^2_{L_2}\big)\,dt+2(u_t,g^r_t)\,dw^r_t, \tag{1.4}$$

where $D^*_\alpha=-D_\alpha$ for $\alpha=1,2,...,d$ and D^*_α is the identity operator for $\alpha=0$. This formula is an important tool in the proof of existence and uniqueness of solutions in W^m_2 Sobolev spaces for SPDEs driven by Wiener processes. To have the corresponding tool for solvability in W^m_p spaces, for $p\geq 2$ a theorem on Itô formula for $|u_t|^p_{L_p}$ is proved in [10] when $(u_t)_{t\in[0,T]}$ is a W^1_p-valued process and $f^\alpha=(f^\alpha_t)_{t\in[0,T]}$ and $(g^r_t)_{t\in[0,T]}$ in equation (1.2) are L_p-valued processes. Our aim is to present an Itô formula for $d|u_t|^p_{L_p}$ when instead of (1.2) we have

$$du_t=D_\alpha f^\alpha_t\,dt+g^r_t\,dw^r_t+\int_Z h_t(z)\,d\tilde\pi_t(dz), \tag{1.5}$$

where $(\tilde\pi_t(dz))_{t\in[0,T]}$ is a Poisson martingale measure with structure measure $\mu(dz)$ on a measurable space $(Z,\mathcal{Z})$, and $h=(h_t)_{t\in[0,T]}$ is a process with values in

$$L_p(\mathbb{R}^d,L_2(Z,\mu))\cap L_p(\mathbb{R}^d,L_p(Z,\mu)).$$

Our motivation is to present an Itô formula to study solvability in L_p-spaces of SPDEs driven by Wiener processes and Poisson martingale measures. Our main theorems on Itô's formula, Theorem 2.1 and Theorem 2.2 generalise Lemma 5.1 and Theorem 2.1, respectively, from [10]. We use them to prove an existence and uniqueness theorem for a class of stochastic integro-differential equations in [8]. In [8] we need an Itô's formula for $d|\langle u_t\rangle|^p_{L_p}$, where $\langle u_t\rangle=(\sum_{i=1}^M|u^i_t|^2)^{1/2}$ and $(u^i_t)_{t\in[0,T]}$ is a W^1_p-valued process having a stochastic differential of the type (1.5) for each $i=1,2,...,M$. Therefore in Theorem 2.1 of the present paper we consider a system of stochastic differentials instead of a single one.

There are well-known theorems in the literature on Itô's formula for semimartingales with values in separable Banach spaces, see for example [2], [17] and [18]. In some aspects these results are more general than our main theorems, but do not cover them. In [2] and [18] only continuous semimartingales are considered and their differential does not contain $D_if^i\,dt$ terms. The semimartingales $(u_t)_{t\in[0,T]}$ in [17] contains stochastic integrals with respect to Poisson random measures and random martingale measures, but the theorems on Itô's formula in this paper cannot be applied to $|u_t|^p_{L_p}$.

The structure of the paper is the following. In the next section we formulate the main results, Theorems 2.1 and 2.2. In Section 3 we present a suitable Itô's formula, Theorem 3.1 for the p-th power of the norm of an $\mathbb{R}^M$-valued semimartingale, together with a stochastic Fubini theorem and a very simple tool, Lemma 3.8, which allow us to prove our main results in Section 4, by adapting the ideas and methods of [10].

In conclusion we present some notions and notations. All random elements are given on a fixed complete probability space $(\Omega,\mathcal{F},P)$ equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all P-zero sets of $\mathcal{F}$. The σ-algebra of the predictable subsets of $\Omega\times[0,\infty)$ is denoted by $\mathcal{P}$. We are given a sequence $w=(w^1_t,w^2_t,...)_{t\geq 0}$ of $\mathcal{F}_t$-adapted independent Wiener processes $w^r=(w^r_t)_{t\geq 0}$, such that w_t-w_s is independent of $\mathcal{F}_s$ for any $0 \leq s \leq t$. We are given also a Poisson random measure $\pi(dz, dt)$ on $[0, \infty) \times Z$, with intensity measure $\mu(dz)dt$, where μ is a σ-finite measure on a measurable space $(Z, \mathcal{Z})$ with a countably generated σ-algebra $\mathcal{Z}$. We assume that the process $\pi_t(\Gamma) := \pi((0, t] \times \Gamma)$, $t \geq 0$, is $\mathcal{F}_t$-adapted and $\pi_t(\Gamma) - \pi_s(\Gamma)$ is independent of $\mathcal{F}_s$ for any $0 \leq s \leq t$ and $\Gamma \in \mathcal{Z}$ such that $\mu(\Gamma) < \infty$. We use the notation $\tilde{\pi}(dz, dt) = \pi(dz, dt) - \mu(dz)dt$ for the *compensated Poisson random measure*, and set $\tilde{\pi}_t(\Gamma) = \tilde{\pi}(\Gamma, (0, t]) = \pi_t(\Gamma) - t\mu(\Gamma)$ for $t \geq 0$ and $\Gamma \in \mathcal{Z}$ such that $\mu(\Gamma) < \infty$. For basic results concerning stochastic integrals with respect π and $\tilde{\pi}$ we refer to [1] and [4]. The Borel σ-algebra of a topological space V is denoted by $\mathcal{B}(V)$.

The space of smooth functions $\varphi = \varphi(x)$ with compact supports on the d-dimensional Euclidean space $\mathbb{R}^d$ is denoted by C_0^∞. For $p, q \geq 1$ we denote by $L_p = L_p(\mathbb{R}^d, \mathbb{R}^M)$ and $\mathcal{L}_q = \mathcal{L}_q(Z, \mathbb{R}^M)$ the Banach spaces of $\mathbb{R}^M$-valued Borel-measurable functions of $f = (f^i(x))_{i=1}^M$ and $\mathcal{Z}$-measurable functions $h = (h^i(z))_{i=1}^M$ of $x \in \mathbb{R}^d$ and $z \in Z$, respectively such that

$$|f|_{L_p}^p = \int_{\mathbb{R}^d} |f(x)|^p \, dx \quad \text{and} \quad |h|_{\mathcal{L}_q}^q = \int_{\mathbb{R}^d} |h(z)|^q \, \mu(dz) < \infty,$$

where $|v|$ means the Euclidean norm for vectors v from Euclidean spaces. The notation $\mathcal{L}_{p,q}$ means the space $\mathcal{L}_p \cap \mathcal{L}_q$ with the norm

$$|v|_{\mathcal{L}_{p,q}} = \max(|v|_{\mathcal{L}_p}, |v|_{\mathcal{L}_q}) \quad \text{for } v \in \mathcal{L}_p \cap \mathcal{L}_q.$$

As usual W_p^1 denotes the space of functions $u \in L_p$ such that $D_i u \in L_p$ for every $i = 1, 2, ..., d$, where $D_i v$ means the generalised derivative of v in x^i for locally integrable functions v on $\mathbb{R}^d$. The norm of $u \in W_p^1$ is defined by

$$|u|_{W_p^1} = |u|_{L_p} + \sum_{i=1}^d |D_i u|_{L_p}.$$

The space of sequences $\nu = (\nu^1, \nu^2, ...)$ of vectors $\nu^k \in \mathbb{R}^M$ with finite norm

$$|\nu|_{\ell_2} = \big(\sum_{k=1}^\infty |\nu^k|^2 \big)^{1/2}$$

is denoted by $\ell_2 = \ell_2(\mathbb{R}^M)$, and by l_2 when $M = 1$. We use the notation $L_p = L_p(\ell_2)$ for $L_p(\mathbb{R}^d, \ell_2)$, the space of Borel-measurable functions $g = (g^{ir})$ on $\mathbb{R}^d$ with values in ℓ_2 such that

$$|g|_{L_p}^p = \int_{\mathbb{R}^d} |g(x)|_{\ell_2}^p \, dx < \infty.$$

We denote by $L_p = L_p(\mathcal{L}_{p,q})$ and $L_p = L_p(\mathcal{L}_q)$ the Banach spaces of Borel-measurable functions $h = (h^i(x, z))$ and $\tilde{h} = (\tilde{h}^i(x, z))$ of $x \in \mathbb{R}^d$ with values in $\mathcal{L}_{p,q}$ and $\mathcal{L}_q$, respectively, such that

$$|h|_{L_p}^p = \int_{\mathbb{R}^d} |h(x, \cdot)|_{\mathcal{L}_{p,q}}^p \, dx < \infty \quad \text{and} \quad |\tilde{h}|_{L_p}^p = \int_{\mathbb{R}^d} |\tilde{h}(x, \cdot)|_{\mathcal{L}_q}^p \, dx < \infty.$$

For fixed $T > 0$, $p \geq 2$ and for a separable real Banach space V we denote by $\mathbb{L}_p = \mathbb{L}_p(V)$ the space of predictable V-valued functions $f = (f_t)$ of $(\omega, t) \in \Omega \times [0, T]$ such that

$$|f|_{\mathbb{L}_p}^p = E \int_0^T |f_t|_V^p \, dt < \infty.$$

In the sequel V will be $L_p(\mathbb{R}^d, \mathbb{R}^M)$, or $L_p(\mathbb{R}^d, \ell_2)$, or $L_p(\mathbb{R}^d, \mathcal{L}_{p,2})$. When $V = L_p(\mathbb{R}^d, \mathcal{L}_{p,2})$ then for $\mathbb{L}_p(V)$ the notation $\mathbb{L}_{p,2}$ is also used. Recall that the summation convention with respect to integer valued indices is used throughout the paper.

2. Formulation of the results

Assumption 2.1. Let $u = (u^i_t)_{t\in 0,T}$ be a progressively measurable L_p-valued process such that there exist $f = (f^i_t(x)) \in \mathbb{L}_p$, $g = (g^{ir}_t(x)) \in \mathbb{L}_p$, $h = (h^i_t(x,z)) \in \mathbb{L}_{p,2}$, and an L_p-valued $\mathcal{F}_0$-measurable random variable $\psi = (\psi^i(x))_{i=1}^M$, such that for every $\varphi \in C_0^\infty$

$$(u^i_t, \varphi) = (\psi, \varphi) + \int_0^t (f^i_s, \varphi)\, ds + \int_0^t (g^{ir}_s, \varphi)\, dw^r_s + \int_0^t \int_Z (h^i_s(z), \varphi)\, \tilde{\pi}(dz, ds) \tag{2.1}$$

for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$ for $i = 1, 2, ..., M$.

In equation (2.1), and later on, we use the notation (v, ϕ) for the Lebesgue integral over $\mathbb{R}^d$ of the product $v\phi$ for functions v and ϕ on $\mathbb{R}^d$ when their product and its integral are well-defined.

Theorem 2.1. *Let Assumption 2.1 hold with $p \geq 2$. Then there is an L_p-valued adapted cadlag process $\bar{u} = (\bar{u}^i_t)_{t\in[0,T]}$ such that equation (2.1), with $\bar{u}$ in place of u, holds for each $\varphi \in C_0^\infty$ almost surely for all $t \in [0, T]$. Moreover, $u = \bar{u}$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, and almost surely*

$$\begin{aligned}|\bar{u}_t|^p_{L_p} = |\psi|^p_{L_p} &+ p\int_0^t \int_{\mathbb{R}^d} |\bar{u}_s|^{p-2}\bar{u}^i_s g^{ir}_s\, dx\, dw^r_s \\ &+ \tfrac{p}{2}\int_0^t \int_{\mathbb{R}^d} \big(2|\bar{u}_s|^{p-2}\bar{u}^i_s f^i_s + (p-2)|\bar{u}_s|^{p-4}|\bar{u}^i_s g^{i\cdot}_s|^2_{l_2} + |\bar{u}_s|^{p-2}|g_s|^2_{\ell_2}\big)\, dx\, ds \\ &+ p\int_0^t \int_Z \int_{\mathbb{R}^d} |\bar{u}_{s-}|^{p-2}\bar{u}^i_{s-} h^i_s\, dx\, \tilde{\pi}(dz, ds) \\ &+ \int_0^t \int_Z \int_{\mathbb{R}^d} (|\bar{u}_{s-} + h_s|^p - |\bar{u}_{s-}|^p - p|\bar{u}_{s-}|^{p-2}\bar{u}^i_{s-} h^i_s)\, dx\, \pi(dz, ds)\end{aligned} \tag{2.2}$$

for all $t \in [0, T]$, where $\bar{u}_{s-}$ means the left-hand limit in L_p at s of $\bar{u}$.

Notice that for $M = 1$ equation (2.2) has the simpler form

$$\begin{aligned}|\bar{u}_t|^p_{L_p} = |\psi|^p_{L_p} &+ p\int_0^t \int_{\mathbb{R}^d} |\bar{u}_s|^{p-2}\ \bar{u}_s g^r_s\, dx\, dw^r_s \\ &+ \tfrac{p}{2}\int_0^t \int_{\mathbb{R}^d} \big(2|\bar{u}_s|^{p-2}\bar{u}_s f_s + (p-1)|\bar{u}_s|^{p-2}|g_s|^2_{l_2}\big)\, dx\, ds \\ &+ p\int_0^t \int_Z \int_{\mathbb{R}^d} |\bar{u}_{s-}|^{p-2}\bar{u}_{s-} h_s\, dx\, \tilde{\pi}(dz, ds) \\ &+ \int_0^t \int_Z \int_{\mathbb{R}^d} \big(|\bar{u}_{s-} + h_s|^p - |\bar{u}_{s-}|^p - p|\bar{u}_{s-}|^{p-2}\bar{u}_{s-} h_s\big)\, dx\, \pi(dz, ds).\end{aligned} \tag{2.3}$$

To formulate our second main theorem we take $M = 1$ and make the following assumption.

Assumption 2.2. Let $u = (u_t)_{t\in 0,T}$ be a progressively measurable W^1_p-valued process such that the following conditions hold:
(i)

$$E\int_0^T |u_t|^p_{W^1_p}\,dt < \infty\,;$$

(ii) there exist $f^\alpha = (f^\alpha_t(x)) \in \mathbb{L}_p$ for $\alpha \in \{0, 1, ..., d\}$, $g = (g^r_t(x)) \in \mathbb{L}_p$, $h = (h_t(x, z)) \in \mathbb{L}_{p,2}$, and an L_p-valued $\mathcal{F}_0$-measurable random variable $\psi = (\psi(x))$, such that for every $\varphi \in C^\infty_0$ we have

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t (f^\alpha_s, D^*_\alpha \varphi)\,ds + \int_0^t (g^r_s, \varphi)\,dw^r_s + \int_0^t \int_Z (h_s(z), \varphi)\,\tilde\pi(dz, ds) \tag{2.4}$$

for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, where $D^*_\alpha = -D_\alpha$ for $\alpha = 1, 2, .., d$, and D^*_α is the identity operator for $\alpha = 0$.

Theorem 2.2. *Let Assumption 2.2 hold with $p \geq 2$. Then there is an L_p-valued adapted cadlag process $\bar u = (\bar u_t)_{t\in[0,T]}$ such that for each $\varphi \in C^\infty_0$ equation (2.4) holds with $\bar u$ in place of u almost surely for all $t \in [0, T]$. Moreover, $u = \bar u$ for $P\otimes dt$-almost every $(\omega, t) \in \Omega\times[0, T]$, and almost surely*

$$\begin{aligned}
|\bar u_t|^p_{L_p} = |\psi|^p_{L_p} + p\int_0^t \int_{\mathbb{R}^d} |u_s|^{p-2}u_s g^r_s\,dx\,dw^r_s \\
+\tfrac{p}{2}\int_0^t \int_{\mathbb{R}^d} \big(2|u_s|^{p-2}u_s f^0_s - 2(p-1)|u_s|^{p-2}f^i_s D_i u_s + (p-1)|u_s|^{p-2}|g_s|^2_{l_2}\big)\,dx\,ds \\
+ \int_0^t \int_Z \int_{\mathbb{R}^d} p|\bar u_{s-}|^{p-2}\bar u_{s-}h_s\,dx\,\tilde\pi(dz, ds) \\
+ \int_0^t \int_Z \int_{\mathbb{R}^d} \big(|\bar u_{s-} + h_s|^p - |\bar u_{s-}|^p - p|\bar u_{s-}|^{p-2}\bar u_{s-}h_s\big)\,dx\,\pi(dz, ds)
\end{aligned} \tag{2.5}$$

for all $t \in [0, T]$, where $\bar u_{s-}$ denotes the left-hand limit in $L_p(\mathbb{R}^d)$ of $\bar u$ at $s \in (0, T]$. Furthermore, there is a constant $N = N(d, p)$ such that

$$\begin{aligned}
E\sup_{t\leq T} |\bar u_t|^p_{L_p} \leq 2E|\psi|^p_{L_p} + NT^{p-1}E\int_0^T |f^0_t|^p_{L_p}\,dt + NE\int_0^T |h_t|^p_{L_p(\mathcal{L}_p)}\,dt \\
+ NT^{(p-2)/2}E\int_0^T |g|^p_{L_p} + \sum_{i=1}^d |f^i_t|^p_{L_p} + |Du_t|^p_{L_p}\,dt + NT^{(p-2)/2}E\int_0^T |h_t|^p_{L_p(\mathcal{L}_2)}\,dt.
\end{aligned} \tag{2.6}$$

3. Preliminaries

First we present an Itô formula for an $\mathbb{R}^M$-valued semimartigale $X = (X^1_t, ..., X^M_t)_{t\in[0,T]}$ given by

$$\begin{aligned}
X_t = X_0 + \int_0^t f_s\,ds + \int_0^t g^r_s\,dw^r_s \\
+ \int_0^t \int_Z \bar h_s(z)\,\pi(dz, ds) + \int_0^t \int_Z h_s(z)\,\tilde\pi(dz, ds), \quad \text{for } t \in [0, T],
\end{aligned} \tag{3.1}$$

where X_0 is an $\mathbb{R}^M$-valued $\mathcal{F}_0$-measurable random variable, $f = (f^i_t)_{t\in[0,T]}$ and $g = (g^{ir}_t)_{t\in[0,T]}$ are predictable processes with values in $\mathbb{R}^M$ and $\ell_2 = \ell_2(\mathbb{R}^M)$, respectively, $\bar h = (\bar h^i_t(z))_{t\in[0,T]}$

and $h = (h^i_t(z))_{t\in[0,T]}$ are $\mathbb{R}^M$-valued $\mathcal{P}\otimes\mathcal{Z}$-measurable functions on $\Omega\times[0,T]\times Z$ such that almost surely

$$\bar{h}^i_t(z)h^j_t(z) = 0 \quad \text{for } i,j = 1,2,...,M, \text{ for all } t\in[0,T] \text{ and } z\in Z, \tag{3.2}$$

and

$$\int_0^T\int_Z |\bar{h}_s(z)|\,\pi(dz,dt) < \infty, \qquad \int_0^T |f_t| + |g_t|^2_{\ell_2} + |h_t(\cdot)|^2_{\mathcal{L}_2}\,dt < \infty. \tag{3.3}$$

Theorem 3.1. *Let conditions* (3.2) *and* (3.3) *hold, and let ϕ from $C^2(\mathbb{R}^M)$, the space of continuous real functions on $\mathbb{R}^M$ whose derivatives up to second order are continuous functions on $\mathbb{R}^M$. Then $\phi(X_t)$ is a semimartingale such that*

$$\begin{aligned}\phi(X_t) =& \phi(X_0) + \int_0^t D_i\phi(X_s)g^{ir}_s\,dw^r_s + \int_0^t D_i\phi(X_s)f^i_s + \tfrac{1}{2}D_iD_j\phi(X_s)g^{ir}_s g^{jr}_s\,ds \\ &+ \int_0^t\int_Z \phi(X_{s-}+\bar{h}_s(z)) - \phi(X_{s-})\,\pi(dz,ds) + \int_0^t\int_Z D_i\phi(X_{s-})h^i_s(z)\,\tilde{\pi}(dz,ds) \\ &+ \int_0^t\int_Z \phi(X_{s-}+h_s(z)) - \phi(X_{s-}) - D_i\phi(X_{s-})h^i_s(z)\,\pi(dz,ds)\end{aligned} \tag{3.4}$$

almost surely for all $t\in[0,T]$.

In this paper we need the following corollary of this theorem.

Corollary 3.2. *Let conditions* (3.2) *and* (3.3) *hold. Then for any $p\geq 2$ the process $|X_t|^p$ is a semimartingale such that*

$$\begin{aligned}|X_t|^p =& |\psi|^p + p\int_0^t |X_s|^{p-2}X^i_s g^{ir}_s\,dw^r_s \\ &+ \tfrac{p}{2}\int_0^t \left(2|X_s|^{p-2}X^i_s f^i_s + (p-2)|X_s|^{p-4}|X^i_s g^{i\cdot}_s|^2_{l_2} + |X_s|^{p-2}|g_s|^2_{l_2}\right)\,ds \\ &+ p\int_0^t\int_Z |X_{s-}|^{p-2}X^i_{s-}h^i_s(z)\,\tilde{\pi}(dz,ds) + \int_0^t\int_Z (|X_{s-}+\bar{h}_s|^p - |X_{s-}|^p)\,\pi(dz,ds) \\ &+ \int_0^t\int_Z \left(|X_{s-}+h_s|^p - |X_{s-}|^p - p|X_{s-}|^{p-2}X^i_{s-}h^i_s\right)\,\pi(dz,ds)\end{aligned} \tag{3.5}$$

almost surely for all $t\in[0,T]$.

Proof. Since the function $\phi(x) = |x|^p$ for $p\geq 2$ belongs to $C^2(\mathbb{R}^M)$ with

$$D_i|x|^p = p|x|^{p-2}x^i, \quad D_jD_i|x|^p = p(p-2)|x|^{p-4}x^ix^j + p|x|^{p-2}\delta_{ij},$$

it is easy to see that Theorem 3.1 for $\phi(x) = |x|^p$ gives the corollary. Here and in the sequel $0/0 := 0$. □

We obtain Theorem 3.1 from the following well-known theorem on Itô's formula.

Theorem 3.3. *Besides conditions* (3.2) *and* (3.3) *assume there is a constant K such that $|h|\leq K$ for all $\omega\in\Omega$, $t\in[0,T]$ and $z\in Z$. Then for any $\phi\in C^2(\mathbb{R}^M)$ the process*

$(\phi(X_t))_{t\in[0,T]}$ *is a semimartingale such that*

$$\phi(X_t)=\phi(X_0)+\int_0^t f^i_sD_i\phi(X_s)+\tfrac{1}{2}g^{ir}_sg^{jr}_sD_iD_j\phi(X_s)\,ds+\int_0^t g^{ir}_sD_i\phi(X_s)\,dw^r_s$$
$$+\int_0^t\int_Z\phi(X_{s-}+\bar h_s(z))-\phi(X_{s-})\,\pi(dz,ds)+\int_0^t\int_Z\phi(X_{s-}+h_s(z))-\phi(X_{s-})\,\tilde\pi(dz,ds)$$
$$+\int_0^t\int_Z\big(\phi(X_s+h_s(z))-\phi(X_s)-h^i_s(z)D_i\phi(X_s)\big)\,\mu(dz)\,ds. \tag{3.6}$$

Proof. This theorem, with a finite dimensional Wiener process in place of an infinite sequence of independent Wiener processes is proved, for example, in [4]. The extension of it to our setting is a simple exercise left for the reader. □

Notice that for $\phi(x)=|x|^p$ the last two integrals in (3.6) may not exist without the additional condition that h is bounded. Thus Itô's formula (3.6) does not hold in general for $\phi(x)=|x|^p$, $p\geq 2$, under the conditions (3.2) and (3.3).

We prove Theorem 3.1 by rewriting Itô formula (3.6) into equation (3.4) under the additional condition that h is bounded, and we dispense with this condition by approximating h by bounded functions.

Proof of Theorem 3.1. First in addition to the conditions (3.2) and (3.3) assume there is a constant K such that $|h|\leq K$. By Taylor's formula for

$$I^a\phi(v):=\phi(v+a)-\phi(v)\quad\text{and}\quad J^a\phi(v):=I^a\phi(v)-D_i\phi(v)a^i,$$

for each $v,a\in\mathbb{R}^M$ we have

$$|I^a\phi(v)|\leq\sup_{|x|\leq|a|+|v|}|D\phi(x)||a|,\quad |J^a\phi(v)|\leq\sup_{|x|\leq|a|+|v|}|D^2\phi(x)||a|^2, \tag{3.7}$$

where $|D\phi|^2:=\sum_{i=1}^M|D_i\phi|^2$ and $|D^2\phi|^2:=\sum_{i=1}^M\sum_{j=1}^M|D_iD_j\phi|^2$. Since $(X_t)_{t\in[0,T]}$ is a cadlag process, $R:=\sup_{t\leq T}|X_t|$ is a finite random variable. Thus we have

$$\int_0^T\int_Z|J^{h_t(z)}\phi(X_{t-})|\mu(dz)\,dt\leq\sup_{|x|\leq R+K}|D^2\phi(x)|\int_0^T\int_Z|h_t(z)|^2\,\mu(dz)\,dt<\infty \tag{3.8}$$

and

$$\int_0^T\int_Z|J^{h_t(z)}\phi(X_{t-})|^2\mu(dz)\,dt\leq\sup_{|x|\leq R+K}|D^2\phi(x)|^2K^2\int_0^T\int_Z|h_t(z)|^2\,\mu(dz)\,dt<\infty \tag{3.9}$$

almost surely. Clearly,

$$\int_0^T\int_Z|D_i\phi(X_{t-})h^i_t(z)|^2\,\mu(dz)\,dt\leq\sup_{|x|\leq R}|D\phi(x)|^2\int_0^T\int_Z|h_t(z)|^2\,\mu(dz)\,dt<\infty\ \text{(a.s.)}.$$

Hence, by virtue of (3.9) the stochastic Itô integral

$$\int_0^t\int_Z\phi(X_{t-}+h_t(z))-\phi(X_t)\,\tilde\pi(dz,dt)=\int_0^t\int_Z I^{h_t(z)}\phi(X_{t-})\tilde\pi(dz,dt)$$

can be decomposed as

$$\int_0^t\int_Z I^{h_t(z)}\phi(X_{t-})\tilde\pi(dz,dt)=\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\tilde\pi(dz,dt)+\int_0^t\int_Z D_i\phi(X_{t-})h^i_t(z)\,\tilde\pi(dz,dt),$$

and by virtue of (3.8) and (3.9),

$$\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\tilde\pi(dz,dt)+\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\mu(dz)\,dt=\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\pi(dz,dt).$$

Hence

$$\int_0^t\int_Z I^{h_t(z)}\phi(X_{t-})\,\tilde\pi(dz,dt)+\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\mu(dz)\,dt$$
$$=\int_0^t\int_Z D_i\phi(X_{t-})h^i_t(z)\,\tilde\pi(dz,dt)+\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\pi(dz,dt),$$

which shows that Theorem 3.1 holds under the additional condition that $|h|$ is bounded. To prove the theorem in full generality we approximate h by $h^{(n)}=(h^{1n},...,h^{Mn})$, where $h^{in}_t=-n\vee h^i_t\wedge n$ for integers $n\geq 1$, and define

$$X^{(n)}_t:=X_0+\int_0^t f_s\,ds+\int_0^t g^r_s\,dw^r_s+\int_0^t\int_Z \bar h_s(z)\,\pi(dz,ds)+\int_0^t\int_Z h^{(n)}_s(z)\,\tilde\pi(dz,ds),\quad t\in[0,T].$$

Clearly, for all (ω,t,z)

$$|h^{(n)}|\leq\min(|h|,nM)\quad\text{and}\quad h^{(n)}\to h\quad\text{as }n\to\infty.\tag{3.10}$$

Therefore Theorem 3.1 for $X^{(n)}$ holds, and

$$\lim_{n\to\infty}\int_0^T\int_Z|h^{(n)}_t(z)-h_t(z)|^2\,\mu(dz)\,dt=0\text{ (a.s.)},$$

which implies

$$\sup_{t\leq T}|X^{(n)}_t-X_t|\to 0\quad\text{in probability as }n\to\infty.$$

Thus there is a strictly increasing subsequence of positive integers $(n_k)_{k=1}^\infty$ such that

$$\lim_{k\to\infty}\sup_{t\leq T}|X^{(n_k)}_t-X_t|=0\quad\text{(a.s.)},$$

which implies

$$\rho:=\sup_{k\geq1}\sup_{t\leq T}|X^{(n_k)}_t|<\infty\quad\text{(a.s.)}.$$

Hence it is easy to pass to the limit $k\to\infty$ in $\phi(X^{(n_k)}_t)$ and in the first two integral terms in the equation for $\phi(X^{(n_k)}_t)$ in Theorem 3.1. To pass to the limit in the other terms in this equation notice that since $\pi(dz,dt)$ is a counting measure of a point process, from the condition for $\bar h$ in (3.3) we get

$$\xi:=\pi\text{-}\operatorname{ess\,sup}|\bar h|<\infty\text{ (a.s.)},\tag{3.11}$$

where π-ess sup denotes the essential supremum operator with respect to the measure $\pi(dz,dt)$ over $Z\times[0,T]$. Similarly, from the condition for h we have

$$\eta:=\pi\text{-}\operatorname{ess\,sup}|h|<\infty\text{ (a.s.)}.\tag{3.12}$$

This can be seen by noting that for the sequence of predictable stopping times

$$\tau_j = \inf\left\{t \in [0,T] : \int_0^t \int_Z |h_s(z)|^2\, \mu(dz)\, ds \geq j\right\}, \quad j = 1, 2, ...,$$

we have

$$E\int_0^T \int_Z \mathbf{1}_{t\leq\tau_j} |h_t(z)|^2\, \pi(dz, dt) = E\int_0^T \int_Z \mathbf{1}_{t\leq\tau_j} |h_t(z)|^2\, \mu(dz)\, dt \leq j < \infty,$$

which gives

$$\int_0^T \int_Z |h_t(z)|^2\, \pi(dz, dt) < \infty \quad \text{almost surely on } \Omega_j = \{\omega \in \Omega : \tau_j \geq T\} \text{ for each } j \geq 1.$$

Since $(\tau_j)_{j=1}^{\infty}$ is an increasing sequence converging to infinity, we have $P(\cup_{j=1}^{\infty}\Omega_j) = 1$, i.e.,

$$\int_0^T \int_Z h_t^2(z)\, \pi(dz, dt) < \infty \text{ (a.s.)} \tag{3.13}$$

which implies (3.12). By (3.11) and the first inequality in (3.7), we have

$$|I^{\bar{h}_t(z)}\phi(X_{t-}^{(n_k)})| + |I^{\bar{h}_t(z)}\phi(X_{t-})| \leq 2 \sup_{|x|\leq\rho+\xi} |D\phi(x)||\bar{h}_t(z)| < \infty$$

almost surely for $\pi(dz, dt)$-almost every $(z, t) \in Z \times [0, T]$. Hence by Lebesgue's theorem on dominated convergence we get

$$\lim_{k\to\infty} \int_0^T \int_Z |I^{\bar{h}_s(z)}\phi(X_{s-}^{(n_k)}) - I^{\bar{h}_s(z)}\phi(X_{s-})|\, \pi(dz, ds) = 0 \quad \text{(a.s.)},$$

which implies that for $k \to \infty$

$$\int_0^t \int_Z I^{\bar{h}_s(z)}\phi(X_{s-}^{(n_k)})\, \pi(dz, ds) \to \int_0^t \int_Z I^{\bar{h}_s(z)}\phi(X_{s-})\, \pi(dz, ds)$$

almost surely, uniformly in $t \in [0, T]$. Clearly,

$$|D_i\phi(X_{t-}^{(n_k)})h_t^{in_k}(z)|^2 + |D_i\phi(X_{t-})h_t^i(z)|^2 \leq 2 \sup_{|x|\leq\rho} |D\phi(x)|^2 |h_t(z)|^2$$

almost surely for all $(z, t) \in Z \times [0, T]$. Hence by Lebesgue's theorem on dominated convergence

$$\lim_{k\to\infty} \int_0^T \int_Z |D_i\phi(X_{t-}^{(n_k)})h_t^{in_k}(z) - D_i\phi(X_{t-})h_t^i(z)|^2\, \mu(dz)\, dt = 0 \quad \text{(a.s.)},$$

which implies that for $k \to \infty$

$$\int_0^t \int_Z D_i\phi(X_{t-}^{(n_k)})h_t^{in_k}(z)\, \tilde{\pi}(dz, dt) \to \int_0^t \int_Z D_i\phi(X_{t-})h_t^i(z)\, \tilde{\pi}(dz, dt)$$

in probability, uniformly in $t \in [0, T]$. Finally note that by using the second inequality in (3.7) together with (3.12) we have

$$|J^{h_t^{(n_k)}(z)}\phi(X_{t-}^{(n_k)})| + |J^{h_t(z)}\phi(X_{t-})| \leq 2 \sup_{|x|\leq\rho+\eta} |D^2\phi(x)||h_t(z)|^2$$

almost surely for $\pi(dz,dt)$-almost every $(z,t)\in Z\times[0,T]$. Hence, taking into account (3.13), by Lebesgue's theorem on dominated convergence we obtain

$$\lim_{k\to\infty}\int_0^T\int_Z |J^{h_t^{(n_k)}(z)}\phi(X_{t-}^{(n_k)})-J^{h_t(z)}\phi(X_{t-})|\,\pi(dz,dt)=0\quad\text{(a.s.)},$$

which implies that for $k\to\infty$

$$\int_0^t\int_Z J^{h_t^{(n_k)}(z)}\phi(X_{t-}^{(n_k)})\,\pi(dz,dt)\to\int_0^t\int_Z J^{h_t(z)}\phi(X_{t-})\,\pi(dz,dt)$$

almost surely, uniformly in $t\in[0,T]$, and finishes the proof of the theorem. □

Remark 3.1. One can give a different proof of Theorem (3.1) by showing that for finite measures μ, the Itô's formula for general semimartingales, Theorem VIII.27 in [3], applied to $(X_t)_{t\in[0,T]}$, can be rewritten as equation (3.4). Hence by an approximation procedure one can get the general case of σ-finite measures μ.

To obtain Theorem 2.1 from Theorem 3.1 besides well-known Fubini theorems for deterministic integrals and stochastic integrals with respect to Wiener processes, see [9], we need the following Fubini theorems for stochastic integrals with respect to Poisson random measures and Poisson martingale measures, where $(\Lambda,\mathcal{S},m)$ denotes a measure space, with a σ-finite measure m and a countably generated σ-algebra $\mathcal{S}$.

Theorem 3.4. *Let $f=f(\omega,t,z,\lambda)$ be a $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{S}$-measurable real function on $\Omega\times[0,T]\times Z\times\Lambda$ such that*

$$\int_0^T\int_Z |f(t,z,\lambda)|^2\,\mu(dz)\,dt<\infty \tag{3.14}$$

for every $\lambda\in\Lambda$ and $\omega\in\Omega$. Then there is an $\mathcal{F}\otimes\mathcal{B}([0,T])\otimes\mathcal{S}$-measurable function $F=F(t,\lambda)$ such that it is cadlag in $t\in[0,T]$ for every $(\omega,\lambda)\in\Omega\times\Lambda$, for each $\lambda\in\Lambda$ the process $(F(t,\lambda))_{t\in[0,T]}$ is a locally square-integrable $\mathcal{F}_t$-martingale and

$$F(t,\lambda)=\int_0^t\int_Z f(s,z,\lambda)\,\tilde\pi(dz,ds)\quad\textit{almost surely for all } t\in[0,T]. \tag{3.15}$$

Moreover, if almost surely

$$\int_\Lambda\Big(\int_0^T\int_Z |f(t,z,\lambda)|^2\,\mu(dz)\,dt\Big)^{1/2} m(d\lambda)<\infty, \tag{3.16}$$

then almost surely

$$\int_\Lambda F(t,\lambda)\,m(d\lambda)=\int_0^t\int_Z\int_\Lambda f(s,z,\lambda)\,m(d\lambda)\,\tilde\pi(dz,ds)\quad\textit{for all } t\in[0,T]. \tag{3.17}$$

Proof. The proof of this theorem is similar to that of Lemma 2.5 from [9]. Let us call the function F, whose existence is stated in the theorem, a regular version of the stochastic integral process defined in the right-hand side of (3.15). Assume first that μ and m are finite measures, and consider the space $\mathcal{H}$ of $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{S}$-measurable bounded real functions f such that the conclusions of the theorem hold. Then it is easy to see that $\mathcal{H}$ is a real vector space which contains the constants. Let $(f^n)_{n=1}^\infty$ be an increasing uniformly bounded sequence from

$\mathcal{H}$, and denote by F^n the regular version of the stochastic integral of f^n. Thus, in particular, for each $\lambda \in \Lambda$

$$F^n(t,\lambda) = \int_0^t \int_Z f^n(s,z,\lambda)\,\tilde{\pi}(dz,ds) \quad \text{almost surely for all } t \in [0,T], \tag{3.18}$$

and almost surely

$$\int_\Lambda F^n(t,\lambda)\,m(d\lambda) = \int_0^t \int_Z \int_\Lambda f^n(s,z,\lambda)\,m(d\lambda)\,\tilde{\pi}(dz,ds) \quad \text{for all } t \in [0,T]. \tag{3.19}$$

Set $f = \lim_{n\to\infty} f^n$. Then f is a bounded $\mathcal{P} \otimes \mathcal{Z} \otimes \mathcal{S}$-measurable function and

$$\lim_{n\to\infty} \int_0^T \int_Z (f^n(t,z,\lambda) - f(t,z,\lambda))^2\,\mu(dz)\,dt = 0 \quad \text{for every } \lambda \in \Lambda.$$

Consequently, for each $\lambda \in \Lambda$ the sequence $F^n(t,\lambda)$ converges in probability, uniformly in $t \in [0,T]$, and hence by a straightforward modification of Lemma 2.1 from [9] there is a $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{S}$-measurable function $F = F(t,\lambda)$ such that it is cadlag in $t \in [0,T]$ for every $(\omega,\lambda) \in \Omega \times \Lambda$, for each $\lambda \in \Lambda$ the process $(F(t,\lambda))_{t\in[0,T]}$ is a locally square-integrable $\mathcal{F}_t$-martingale, and (3.15) holds. Now we show that almost surely (3.17) also holds, by taking $n \to \infty$ in equation (3.19). Clearly, by Lebesgue's theorem on dominated convergence we have

$$\lim_{n\to\infty} \int_0^T \int_Z \left(\int_\Lambda (f^n - f)(t,z,\lambda)\,m(d\lambda) \right)^2 \mu(dz)\,dt = 0$$

for every $\omega \in \Omega$, which implies that for $n \to \infty$

$$\int_0^t \int_Z \int_\Lambda f^n(s,z,\lambda)\,m(d\lambda)\,\tilde{\pi}(dz,ds) \to \int_0^t \int_Z \int_\Lambda f(s,z,\lambda)\,m(d\lambda)\,\tilde{\pi}(dz,ds) \tag{3.20}$$

in probability, uniformly in $t \in [0,T]$. By the Davis inequality

$$E \int_\Lambda \sup_{t\in[0,T]} |F^n - F|(t,\lambda)\,m(d\lambda) \le 3 \int_\Lambda E \left(\int_0^T \int_Z |f^n(t,z) - f(t,z)|^2\,\mu(dz)\,dt \right)^{1/2} m(d\lambda),$$

and the right-hand side of this inequality converges to zero by virtue of Lebesgue's theorem on dominated convergence again. Hence for $n \to \infty$

$$\int_\Lambda F^n(t,\lambda)\,m(d\lambda) \to \int_\Lambda F(t,\lambda)\,m(d\lambda) \quad \text{in probability, uniformly in } t \in [0,T], \tag{3.21}$$

and equation (3.17) follows. Thus we have proved that if f is the limit of an increasing uniformly bounded sequence of functions f^n from $\mathcal{H}$ then f belongs to $\mathcal{H}$. Let $\mathcal{C}$ denote the class of functions f of the form $f(t,z,\lambda) = c\mathbf{1}_{(r,s]}\mathbf{1}_U\varphi(\lambda)$, for $0 \le r \le s \le T$, bounded $\mathcal{F}_r$-measurable random variables c, sets $U \in \mathcal{Z}$ and bounded $\mathcal{S}$-measurable real functions φ. Then

$$\int_0^t \int_Z f(s,z,\lambda)\,\tilde{\pi}(dz,ds) = c\varphi(\lambda)\tilde{\pi}((r\wedge t, s\wedge t] \times U) =: F(t,\lambda), \quad t \in [0,T],\ \lambda \in \Lambda$$

is a regular version of the stochastic integral of f, and it is easy to see that (3.17) holds. Notice that $\mathcal{C}$ is closed with respect to the multiplication of functions, and the σ-algebra generated by $\mathcal{C}$ on $\Omega \times [0,T] \times Z \times \Lambda$ is $\mathcal{P} \otimes \mathcal{Z} \otimes \mathcal{S}$. Consequently, by the well-known Monotone Class Theorem, $\mathcal{H}$ contains all $\mathcal{P} \otimes \mathcal{Z} \otimes \mathcal{S}$-measurable bounded real functions on $\Omega \times [0,T] \times Z \times \Lambda$.

Consider now a $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{S}$-measurable function f satisfying (3.14), and for every integer $n\geq 1$ define $f^n=-n\vee f\wedge n$. Then clearly, f^n is bounded, $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{S}$-measurable, and satisfies (3.14) and (3.16). Hence by virtue of what we have proved above, there is a regular version F^n of the stochastic integral process of f^n, i.e., in particular, with this F^n and $f^n=-n\vee f\wedge n$ equations (3.18) and (3.19) hold. Clearly, $\lim_{n\to\infty}f^n=f$ and $|f^n-f|\leq|f|$ for all $\omega\in\Omega$, $t\in[0,T]$, $z\in Z$ and $\lambda\in\Lambda$, which allow us to repeat the above arguments to show the existence of a regular version F for the stochastic integral process of f, and to get (3.20) if (3.16) also holds. To obtain also (3.21), under the additional assumption (3.16), we introduce the process

$$Q(t)=\int_\Lambda\left(\int_0^t\int_Z f^2(s,z,\lambda)\,\mu(dz)\,ds\right)^{1/2}m(d\lambda),\quad t\in[0,T],$$

and the random time $\tau_\delta=\inf\{t\in[0,T]:Q(t)\geq\delta\}$ for $\delta>0$. Then Q is a continuous $\mathcal{F}_t$-adapted process. Thus τ_δ is an $\mathcal{F}_t$-stopping time and for every $\omega\in\Omega$

$$\begin{aligned}Q_n(t)&:=\int_\Lambda\left(\int_0^t\int_Z|f-f^n|^2(s,z,\lambda)\,\mu(dz)\,ds\right)^{1/2}m(d\lambda)\\&\leq Q(t)\leq\delta\quad\text{for } t\leq T\wedge\tau_\delta.\end{aligned}\tag{3.22}$$

Hence for any $\varepsilon>0$ by the Markov and Davis inequalities

$$\begin{aligned}&P\left(\int_\Lambda\sup_{t\leq T}|F^n-F|(t,\lambda)\,m(d\lambda)\geq\varepsilon\right)\\&\leq P\left(\int_\Lambda\sup_{t\leq T}|F^n-F|(t\wedge\tau_\delta,\lambda)\,m(d\lambda)\geq\varepsilon\right)+P\left(\tau_\delta<T\right)\\&\leq\varepsilon^{-1}E(Q_n(T)\wedge\delta)+P\left(Q(T)\geq\delta\right).\end{aligned}\tag{3.23}$$

Letting here first $n\to\infty$ and then $\delta\to\infty$ we get

$$\lim_{n\to\infty}P\left(\int_\Lambda\sup_{t\leq T}|F^n-F|(t,\lambda)\,m(d\lambda)\geq\varepsilon\right)=0\quad\text{for any }\varepsilon>0,\tag{3.24}$$

which implies (3.21) in the new situation, and finishes the proof of the theorem under the additional condition that μ and m are finite measures.

In the general case of σ-finite measures μ and m for every integer $n\geq 1$ we define $\tilde\pi_n$, μ_n and m_n by

$$\tilde\pi_n(F)=\tilde\pi(F\cap(Z_n\times(0,T])),\quad\mu_n(A)=\mu(A\cap Z_n),\quad m_n(B)=m(B\cap\Lambda_n)$$

for $F\in\mathcal{Z}\otimes\mathcal{B}(0,T)$, $A\in\mathcal{Z}$ and $B\in\mathcal{S}$, where $Z_n\in\mathcal{Z}$ and $\Lambda_n\in\mathcal{S}$ are sets such that $\mu(Z_n)<\infty$, $m(\Lambda_n)<\infty$, $Z_n\subset Z_{n+1}$, $\Lambda_n\subset\Lambda_{n+1}$ for every $n\geq 1$, and $\cup_nZ_n=Z$ and $\cup_n\Lambda_n=\Lambda$. It is easy to see that $\tilde\pi_n$ is a Poisson martingale measure with characteristic measure μ_n. Let f be a $\mathcal{P}\otimes\mathcal{Z}\otimes\mathcal{S}$-measurable function satisfying (3.14). Then clearly, f satisfies (3.14) also with μ_n in place of μ and Z_n in place of Z. Hence by what we have already proved above, there is a regular version $F_n(t,\lambda)$ of the stochastic integral of f (over $Z_n \times (0,t]$) with respect to $\tilde{\pi}_n$, i.e., in particular, for each $\lambda \in \Lambda_n$

$$F_n(t,\lambda) = \int_0^t \int_{Z_n} f(s,z,\lambda)\,\tilde{\pi}_n(dz,ds) = \int_0^t \int_Z \mathbf{1}_{Z_n} f(s,z,\lambda)\,\tilde{\pi}(dz,ds)$$

almost surely for all $t \in [0,T]$, and if f satisfies also (3.16), then almost surely

$$\int_{\Lambda_n} F_n(t,\lambda)\,m_n(d\lambda) = \int_0^t \int_{Z_n} \int_{\Lambda_n} f(s,z,\lambda)\,m_n(d\lambda)\,\tilde{\pi}_n(dz,ds)$$

$$= \int_0^t \int_Z \int_\Lambda \mathbf{1}_{Z_n}\mathbf{1}_{\Lambda_n} f(s,z,\lambda)\,m(d\lambda)\,\tilde{\pi}(dz,ds) \quad \text{for all } t \in [0,T]. \tag{3.25}$$

Clearly, $\lim_{n\to\infty} f\mathbf{1}_{Z_n} = f$ and $|f - f\mathbf{1}_{Z_n}| \le |f|$ for all $\omega \in \Omega$, $t \in [0,T]$, $z \in Z$ and $\lambda \in \Lambda$. Hence

$$\lim_{n\to\infty} \int_0^T \int_Z (f - f\mathbf{1}_{Z_n})^2(s,z,\lambda)\,\mu(dz)\,ds = 0 \quad \text{for all } \omega \in \Omega, \text{ and } \lambda \in \Lambda,$$

which, just like before, implies the existence of a regular version $F(t,\lambda)$ of the stochastic integral of f with respect to $\tilde{\pi}$ (over $Z \times (0,t]$). If f satisfies also (3.16), then we have

$$\lim_{n\to\infty} \int_0^T \int_Z \left(\int_\Lambda (f - \mathbf{1}_{Z_n}\mathbf{1}_{\Lambda_n} f)(s,z,\lambda)\,m(d\lambda) \right)^2 \mu(dz)\,ds = 0 \quad \text{for all } \omega \in \Omega,$$

which implies

$$\int_0^t \int_Z \int_\Lambda \mathbf{1}_{Z_n}\mathbf{1}_{\Lambda_n} f(s,z,\lambda)\,m(d\lambda)\,\tilde{\pi}(dz,ds) \to \int_0^t \int_Z \int_\Lambda f(s,z,\lambda)\,m(d\lambda)\,\tilde{\pi}(dz,ds)$$

in probability, uniformly in $t \in [0,T]$, as $n \to \infty$. Introducing the stopping time τ_δ as before, we have (3.22), (3.23), and hence (3.24) with $F_n\mathbf{1}_{\Lambda_n}$ and $f\mathbf{1}_{\Lambda_n}\mathbf{1}_{Z_n}$ in place of F^n and f^n, respectively. Consequently, letting $n \to \infty$ in (3.25) we obtain (3.17), which finishes the proof of the theorem. ☐

Remark 3.2. There is a Fubini theorem for stochastic integrals with respect to semimartingales in [14], see Theorem 65 in Chapter lV. Its integrability condition applied to our situation reads as

$$\int_0^T \int_Z \int_\Lambda |f(t,z,\lambda)|^2\,m(d\lambda)\,\mu(dz)\,dt < \infty \quad (a.s.), \tag{3.26}$$

which for finite measures m is stronger than condition (3.16).

We also need a Fubini theorem for integrals against the Poisson random measure $\pi(dz,dt)$, that we formulate it as follows.

Theorem 3.5. *Let $g = g(\omega,t,z,\lambda)$ be a real-valued $\mathcal{P} \otimes \mathcal{Z} \otimes \mathcal{S}$-measurable function on $\Omega \times [0,T] \times Z \times \Lambda$ such that*

$$\int_0^T \int_Z |g(t,z,\lambda)|\,\mu(dz)\,dt < \infty \tag{3.27}$$

for each $\lambda \in \Lambda$ *and* $\omega \in \Omega$. *Then there exists an* $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{S}$*-measurable function* $G = G(t,\lambda)$ *such that it is cadlag in* $t \in [0,T]$ *for each* $(\omega,\lambda) \in \Omega \times \Lambda$, *for each* $\lambda \in \Lambda$ *the process* $(G(t,\lambda))_{t\in[0,T]}$ *is locally integrable and* $\mathcal{F}_t$*-adapted, and*

$$G(t,\lambda) = \int_0^t \int_Z g(s,z,\lambda)\,\pi(dz,ds)$$

almost surely for all $t \in [0,T]$. *Furthermore, if almost surely*

$$\int_\Lambda \int_0^T \int_Z |g(t,z,\lambda)|\,\mu(dz)\,dt\,m(d\lambda) < \infty,$$

then

$$\int_\Lambda G(t,\lambda)\,m(d\lambda) = \int_0^t \int_Z \int_\Lambda g(s,z,\lambda)\,m(d\lambda)\,\pi(dz,ds)$$

almost surely for all $t \in [0,T]$.

Proof. One knows, see e.g. [4] that

$$E\int_0^T \int_Z |f_s(z)|\,\pi(dz,ds) = E\int_0^T \int_Z |f_s(z)|\,\mu(dz)\,ds$$

for $\mathcal{P}\otimes\mathcal{S}$-measurable real-valued functions f on $\Omega\times[0,T]\times Z$, when the right-hand side of the above equation is finite. Using this identity, we can prove this theorem by a straightforward modification of the proof of Theorem 3.4 above. □

For σ-finite measure spaces $(\Lambda_i, \mathcal{S}_i, \mu_i)$, a separable real Banach space V and $p_i \in [1,\infty)$ for $i = 1,2$ let L_{p_1,p_2} denote the space of V-valued $\mathcal{S}_1 \otimes S_2$-measurable functions $f = f(x,y)$ of $(x,y) \in \Lambda_1 \times \Lambda_2$, such that

$$\int_{\Lambda_1} \left(\int_{\Lambda_2} |f(x,y)|_V^{p_2}\,\mu_2(dy) \right)^{p_1/p_2} \mu_1(dx) < \infty.$$

Assume that $(\Lambda_2, \mathcal{S}_2, \mu_2)$ is separable, and let $L_{p_1}(L_{p_2}(V))$ denote the space of $\mathcal{S}_1$-measurable functions f mapping Λ_1 into the space $L_{p_2}(V) = L_{p_2}((\Lambda_2,\mathcal{S}_2,\mu_2),V)$ equipped with the Borel σ-algebra, such that

$$\int_{S_1} |f(x)|^{p_1}_{L_{p_2}(V)}\,\mu_1(dx) < \infty.$$

Then we have the following lemma.

Lemma 3.6. *The spaces* L_{p_1,p_2} *and* $L_{p_1}(L_{p_2})$ *are the same in the sense that for each* f *from* $L_{p_1}(L_{p_2})$ *there is* $\bar{f} \in L_{p_1,p_2}$ *such that for every* $x \in \Lambda_1$ *we have* $\bar{f}(x,y) = f(x,y)$ *for* μ_2*-a.e.* $y \in \Lambda_2$, *and for each* $g \in L_{p_1,p_2}$ *there is* $\tilde{g} \in L_{p_1}(L_{p_2})$ *such that for* μ_1*-a.e.* $x \in \Lambda_1$ *we have* $g(x,y) = \tilde{g}(x,y)$ *for all* $y \in \Lambda_2$.

Proof. Due to the separability of $(\Lambda_2, \mathcal{S}_2, \mu_2)$ and V, there are countable subsets $\mathcal{S}_0 \subset \mathcal{S}_2$ and $V_0 \subset V$ such that the space $\mathcal{V}$ of functions g of the form

$$g(y) = \sum_{i=1}^N \mathbf{1}_{\Gamma_i}(y) v_i \quad \text{for } \Gamma_i \in \mathcal{S}_0,\ \mu_2(\Gamma_i) < \infty,\ v_i \in V_0,\ N = 1,2,...,$$

is a countable dense subspace of $L_{p_2}(V)$. Hence for any $\mathcal{S}_1$-measurable function

$$f:\Lambda_1\to L_{p_2}(V)$$

there is a sequence $(f^n)_{n=1}^{\infty}$ of $\mathcal{V}$-valued functions of the form

$$f^n(x)=\sum_{i=1}^{\infty}\mathbf{1}_{F_i^n}(x)g_i^n,$$

such that $F_i^n\in\mathcal{S}_1$, $F_i^n\cap F_j^n=\emptyset$ for $i\neq j$, $g_i^n\in\mathcal{V}$ and

$$|f^n(x)-f(x)|_{L_{p_2}(V)}<2^{-n-1}\quad\text{for all } x\in\Lambda_1$$

for $n\geq 1$. Thus for each $x\in\Lambda_1$ for the set

$$A_n(x)=\{y\in S_2:|f^{n+1}(x,y)-f^n(x,y)|_V\geq n^{-2}\}\in\mathcal{S}_2$$

we have $\mu_2(A_n(x))\leq n^{p_2}2^{-p_2n}$, which, due to $\sum_{n=1}^{\infty}\mu_2(A_n(x))<\infty$, implies that for each $x\in\Lambda_1$ the sequence $(f^n(x,y))_{n=1}^{\infty}$ is convergent in V for μ_2-almost every $y\in\Lambda_2$. Define

$$B=\{(x,y)\in\Lambda_1\times\Lambda_2:(f^n(x,y))_{n=1}^{\infty}\text{ is convergent in } V\},$$

$$\bar{f}(x,y)=\begin{cases}\lim_{n\to\infty}f^n(x,y) & \text{for } (x,y)\in B\\ 0\in V & \text{for } (x,y)\notin B.\end{cases}$$

Then $B\in\mathcal{S}_1\otimes\mathcal{S}_2$, and hence $\bar{f}$ is $\mathcal{S}_1\otimes\mathcal{S}_2$-measurable. Moreover, $\bar{f}(x,y)=f(x,y)$ for μ_2-almost every $y\in\Lambda_2$ for every $x\in\Lambda_1$. Assume now that $g\in L_{p_1,p_2}$. Then $|g(x,\cdot)|_{L_{p_2}(V)}$ is an $\mathcal{S}_1$-measurable function of $x\in\Lambda_1$, with values in $[0,\infty]$. In particular,

$$A:=\{x\in\Lambda_1:|g(x,\cdot)|_{L_{p_2}(V)}<\infty\}\in\mathcal{S}_1,$$

and $\mu_1(\Lambda_1\setminus A)=0$ by Fubini's theorem. For the function $\tilde{g}(x,y)=\mathbf{1}_A(x)g(x,y)$ by Fubini's theorem we have

$$\{x\in\Lambda_1:|\tilde{g}(x)-e|_{L_{p_2}(V)}<R\}\in\mathcal{S}_1$$

for any $e\in L_{p_2}(V)$ and $R>0$. Consequently, $\tilde{g}$ is an $\mathcal{S}_1$-measurable $L_{p_2}(V)$-valued function on Λ_1. In particular, $\tilde{g}\in L_{p_1}(L_{p_2})$, and clearly, for μ_1-almost every $x\in\Lambda_1$ we have $\tilde{g}(x,y)=g(x,y)$ for every $y\in\Lambda_2$. □

Recall that $\mathbb{L}_p(L_p)$, $\mathbb{L}_p(L_p(\ell_2))$ and $\mathbb{L}_p(L_p(\mathcal{L}_{p,2}))$ denote the spaces of predictable functions defined on $\Omega\times[0,T]$ and taking values in $L_p=L_p(\mathbb{R}^d,\mathbb{R}^M))$, $L_p(\ell_2)=L_p(\mathbb{R}^d,\ell_2)$ and in $L_p(\mathcal{L}_{p,2})=L_p(\mathbb{R}^d,\mathcal{L}_{p,2})$, respectively. For separable Banach spaces B and numbers $p,q\in[1,\infty)$ the notations

$$L_p(\Omega\times[0,T]\times\mathbb{R}^d,V)\quad\text{and}\quad L_{p,q}(\Omega\times[0,T]\times\mathbb{R}^d\times Z,V)$$

mean the space of $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions $f:\Omega\times[0,T]\times\mathbb{R}^d\to V$ and the space of $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{Z}$-measurable functions $g:\Omega\times[0,T]\times\mathbb{R}^d\times Z\to V$, respectively, such that

$$E\int_0^T\int_{\mathbb{R}^d}|f_t(x)|_V^p\,dx\,dt<\infty\quad E\int_0^T\int_{\mathbb{R}^d}\left(\int_Z|g_t(x,z)|_V^q\,\mu(dz)\right)^{p/q}dx\,dt<\infty.$$

Corollary 3.7. *The following identifications hold in the sense of Lemma 3.6:*

$$\mathbb{L}_p(L_p)=L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathbb{R}^M),\quad \mathbb{L}_p(L_p(\ell_2))=L_p(\Omega\times[0,T]\times\mathbb{R}^d,\ell_2(\mathbb{R}^M))$$

$$\begin{aligned}\mathbb{L}_p(L_p(\mathcal{L}_{p,2}))&=L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathcal{L}_{p,2})\\&=L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathcal{L}_p)\cap L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathcal{L}_2)\\&=L_{p,p}(\Omega\times[0,T]\times\mathbb{R}^d\times Z,\mathbb{R}^M)\cap L_{p,2}(\Omega\times[0,T]\times\mathbb{R}^d\times Z,\mathbb{R}^M).\end{aligned}$$

Proof. By definition of intersection spaces

$$L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathcal{L}_{p,2})=L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathcal{L}_p)\cap L_p(\Omega\times[0,T]\times\mathbb{R}^d,\mathcal{L}_2)$$

as vector spaces, and it is easy to see that their norms are equivalent. The other equalities can be obtained by repeated applications of Lemma 3.6. ☐

We conclude this section with a simple lemma, which plays a useful role in situations when we want to use Lebesgue's theorem on dominated convergence to pass to the limit in some expressions in the proof of the main theorems.

Lemma 3.8. *Let $(V,|\cdot|_V)$ be a real Banach space whose elements are real-valued functions on a set Λ such that when $f\in V$ then $|f|$, the absolute value of f, belongs to V as well, and the norms of f and $|f|$ are the same. Assume that the pointwise limit of every increasing sequence of non-negative functions $f_n\in V$ belongs to V if $\sup_n|f_n|_V<\infty$. Then for every convergent sequence $(g_n)_{n=1}^\infty$ in $(V,|\cdot|_V)$ there is a subsequence $(g_{n(k)})_{k=1}^\infty$ and an element G from V such that $|g_{n(k)}|\leq G$ for each k.*

Proof. If $(g_n)_{n=1}^\infty$ is a Cauchy sequence in $(V,|\cdot|_V)$ then there is a strictly increasing sequence of positive integers $(n(k))_{k=1}^\infty$ such that $|g_{n(k+1)}-g_{n(k)}|_V\leq 2^{-k}$ for each $k\geq 1$. Thus

$$G:=|g_{n(1)}|+\sum_{k=1}^{\infty}|g_{n(k+1)}-g_{n(k)}|\in V\quad\text{and}\quad|g_{n(k)}|\leq G\text{ for every }k\geq 1.$$

☐

4. Proof of the main results

We use ideas and methods from [10]. To prove the existence of the process $\bar{u}$ with the stated properties in Theorem 2.1, first we show that when φ runs through C_0^∞, then the integral processes of (f,φ), (g,φ) and (h,φ) in equation (2.1) define appropriate L_p-valued integral processes of f, g and h, respectively. To this end we introduce a class of functions $\mathcal{U}_p$, the counterpart of the class $\mathcal{U}_p$ introduced in [10].

Let $\mathcal{U}_p$ denote the set of $\mathbb{R}^M$-valued functions $u=u_t(x)=u_t(\omega,x)$ on $\Omega\times[0,T]\times\mathbb{R}^d$ such that

(i) u is $\mathcal{F}\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)$-measurable,
(ii) for each $x\in\mathbb{R}^d$, $u_t(x)$ is $\mathcal{F}_t$-adapted,
(iii) $u_t(x)$ is cadlag in $t\in[0,T]$ for each (ω,x),
(iv) $u_t(\omega,\cdot)$ as a function of (ω,t) is L_p-valued, $\mathcal{F}_t$-adapted and cadlag in t for every $\omega\in\Omega$.

The following two lemmas are obvious corollaries of Lemmas 4.3 and 4.4 in [10].

Lemma 4.1. *Let f be an $\mathbb{R}^M$-valued function from $\mathbb{L}_p$. Then there exists a function $m \in \mathcal{U}_p$ such that for each $\varphi \in C_0^\infty$ almost surely*

$$(m_t, \varphi) = \int_0^t (f_s, \varphi)\, ds$$

holds for all $t \in [0, T]$. Furthermore, we have

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |m_t(x)|^p \, dx \le N T^{p-1} E \int_0^T |f_s|_{L_p}^p \, ds,$$

with a constant $N = N(p, M)$.

Lemma 4.2. *Let g be an ℓ_2-valued function from $\mathbb{L}_p$. Then there exists a function $a \in \mathcal{U}_p$ such that for each $\varphi \in C_0^\infty$ almost surely*

$$(a_t, \varphi) = \sum_{r=1}^{\infty} \int_0^t (g_s^r, \varphi)\, dw_s^r$$

holds for all $t \in [0, T]$. Furthermore, we have

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |a_t(x)|^p \, dx \le N T^{(p-2)/2} E \int_0^T |g_s|_{L_p}^p \, ds,$$

with a constant $N = N(p, M)$.

Lemma 4.3. *Let $h \in \mathbb{L}_{p,2}$ for $p \ge 2$. Then there exists a function $b \in \mathcal{U}_p$ such that for each real-valued $\varphi \in L_q(\mathbb{R}^d)$ with $q = p/(p-1)$, almost surely*

$$(b_t, \varphi) = \int_0^t \int_Z (h_s, \varphi)\, \tilde{\pi}(dz, ds) \tag{4.1}$$

for all $t \in [0, T]$, and

$$E \sup_{t \le T} |(b_t, \varphi)| \le 3 T^{(p-2)/(2p)} |\varphi|_{L_q} \left(E \int_0^T |h_t|_{L_p(\mathcal{L}_2)}^p \, dt \right)^{1/p}. \tag{4.2}$$

Furthermore

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |b_t(x)|^p \, dx \le N E \int_0^T |h_t|_{L_p(\mathcal{L}_p)}^p \, dt + N T^{(p-2)/2} E \int_0^T |h_t|_{L_p(\mathcal{L}_2)}^p \, dt \le N' |h|_{\mathbb{L}_{p,2}}^p \tag{4.3}$$

with constants $N = N(p, M)$ and $N' = N'(p, M, T)$.

Proof. Let $\mathcal{H}$ denote the set of functions h of the form

$$h_t(z, x) = \sum_{i=1}^{k} \varphi_i(x) c_i \mathbf{1}_{(s_i, t_i]}(t) \mathbf{1}_{U_i}(z)$$

for integers $k \ge 1$, functions $\varphi_i \in C_0^\infty(\mathbb{R}^d)$, time points $0 \le s_i \le t_i$, $\mathcal{F}_{s_i}$-measurable bounded random vectors c_i and sets $U_i \in \mathcal{Z}$ such that $\mu(U_i) < \infty$. For this function h define b by

$$b_t(x) = \sum_i \varphi_i(x) c_i (\tilde{\pi}_{t_i \wedge t}(U_i) - \tilde{\pi}_{s_{i \wedge t}}(U_i)), \quad t \in [0, T],\ x \in \mathbb{R}^d.$$

Clearly, $b \in \mathcal{U}_p$, for every $\varphi \in L_q(\mathbb{R}^d, \mathbb{R})$

$$(b_t, \varphi) = \sum_{i=1}^{k} (\varphi_i, \varphi) c_i (\tilde{\pi}_{t_i \wedge t}(U_i) - \tilde{\pi}_{s_{i \wedge t}}(U_i)) = \int_0^t \int_{\mathcal{Z}} (h_s(z), \varphi) \, \tilde{\pi}(dz, ds)$$

almost surely for all $t \in [0, T]$, and for each $x \in \mathbb{R}^d$

$$b_t(x) = \int_0^t \int_Z h_s(x, z) \, \tilde{\pi}(dz, ds) \quad \text{almost surely for all } t. \tag{4.4}$$

By the Davis, Minkowski and Hölder inequalities,

$$E \sup_{t \le T} |(b_t, \varphi)| \le 3E \left(\int_0^T \int_Z |(h_s(z), \varphi)|^2 \, \mu(dz) \, ds \right)^{1/2} \le 3E \left(\int_0^T (|h_s|_{\mathcal{L}_2}, |\varphi|)^2 \, ds \right)^{1/2}$$

$$\le 3E \left(\int_0^T (|h_s|^2_{L_p(\mathcal{L}_2)} |\varphi|^2_{L_q} \, ds \right)^{1/2} \le 3|\varphi|_{L_q} \left(\int_0^T E|h_s|^2_{L_p(\mathcal{L}_2)} \, ds \right)^{1/2}$$

$$\le 3|\varphi|_{L_q} \left(\int_0^T (E|h_s|^p_{L_p(\mathcal{L}_2)})^{2/p} \, ds \right)^{1/2} \le 3|\varphi|_{L_q} T^{(p-2)/(2p)} \left(\int_0^T E|h_s|^p_{L_p(\mathcal{L}_2)} \, ds \right)^{1/p},$$

which proves (4.2) when $h \in \mathcal{H}$. From (4.4) by the Burkholder-Davis-Gundy inequality for Poisson martingale measures, see, e.g. [5], for $p \ge 2$ for each $x \in \mathbb{R}^d$ we have

$$E \sup_{t \in [0,T]} |b_t(x)|^p = E \sup_{t \le T} \left(\int_0^t \int_Z h_s(x, z) \, \tilde{\pi}(dz, ds) \right)^p$$

$$\le NE \int_0^T \int_Z |h_s(x, z)|^p \, \mu(dz) \, ds + NE \left(\int_0^T \int_Z |h_s(x, z)|^2 \, \mu(dz) \, ds \right)^{p/2}$$

with $N = N(p, M)$. Hence by Jensen's inequality and integrating over $\mathbb{R}^d$ we get (4.3) for $h \in \mathcal{H}$. It is not difficult to see that $\mathcal{H}$ is dense in $\mathbb{L}_{p,2}$. Thus for $h \in \mathbb{L}_{p,2}$ there is a sequence $h^n \in \mathcal{H}$ and $b^n \in \mathcal{U}_p$, such that $h^n \to h$ in $h \in \mathbb{L}_{p,2}$, and (4.1) and (4.3) hold with b^n and h^n in place of b and h, respectively. Therefore we can find a subsequence $h^{n(k)}$ and $b^{n(k)}$ such that

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |b_t^{n(k+1)}(x) - b_t^{n(k)}(x)|^p \, dx$$

$$\le N(E \int_0^T |h_t^{n(k+1)} - h_t^{n(k)}|^p_{L_p(\mathcal{L}_2)} \, dt + E \int_0^T |h_t^{n(k+1)} - h_t^{n(k)}|^p_{L_p(\mathcal{L}_p)} \, dt) \le \frac{N}{2^{kp}}.$$

Hence there is a set $\Theta \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ of full measure such that for $k \to \infty$ the sequence $b_t^{n(k)}(x)$ converges for $(t, \omega, x) \in [0, T] \times \Theta$, uniformly in $t \in [0, T]$. Define

$$\Gamma = \{x \in \mathbb{R}^d : P((\omega, x) \in \Theta) = 1\} \quad \text{and} \quad \bar{\Theta} = \Theta \cap (\Omega \times \Gamma).$$

By Fubini's theorem $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, and it is of full measure. Hence $\bar{\Theta} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$, and it is of full measure. If $x \in \Gamma$ then $\bar{\Theta}_x := \{\omega \in \Omega : (\omega, x) \in \bar{\Theta}\} = \{\omega \in \Omega : (\omega, x) \in \Theta\} =: \Theta_x$, i.e., $P(\bar{\Theta}_x) = P(\Theta_x) = 1$, which implies $\bar{\Theta}_x \in \mathcal{F}_0$, since $\mathcal{F}_0$ is complete. If $x \notin \Gamma$ then $\bar{\Theta}_x = \emptyset \in \mathcal{F}_0$. Thus $\bar{b}^{n(k)} := b^{n(k)} \mathbf{1}_{\bar{\Theta}}$ is $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and $\bar{b}^{n(k)}(t, x)$ is $\mathcal{F}_t$-measurable for each $(t, x) \in [0, T] \times \mathbb{R}^d$. Consequently, $b = \lim_{k \to \infty} \bar{b}^{n(k)}$ has these measurability properties as well. Since for every $(\omega, x) \in \Omega \times \mathbb{R}^d$ the functions $\bar{b}^{n(k)}$ are cadlag and converge to b, uniformly in $t \in [0,T]$, the limit b is a cadlag function of $t \in [0,T]$ for every $(\omega, x) \in \Omega \times \mathbb{R}^d$. Thus b satisfies the conditions (i), (ii) and (iii) in the definitions of $\mathcal{U}_p$. Letting $k \to \infty$ in

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |\bar{b}_t^{n(k)}(x)|^p \, dx \le N(E \int_0^T |h_t^{n(k)}|^p_{L_p(\mathcal{L}_p)} \, dt + T^{(p-2)/2} E \int_0^T |h_t^{n(k)}|^p_{L_p(\mathcal{L}_2)}) \, dt$$

and

$$E \sup_{t \le T} |(\bar{b}_t^{n(k)}, \varphi)| \le 3T^{(p-2)/(2p)} |\varphi|_{L_q} (E \int_0^T |h_t^{n(k)}|^p_{L_p(\mathcal{L}_2)} \, dt)^{1/p}$$

by Fatou's lemma we get (4.3) and (4.2). Letting $k \to \infty$ in

$$E \sup_{t \le T} |\bar{b}_t^{n(k)} - \bar{b}_t^{n(l)}|^p_{L_p} \le NE \int_0^T |h_t^{n(k)} - h_t^{n(l)}|^p_{L_p(\mathcal{L}_p)} \, dt + |h_t^{n(k)} - h_t^{n(l)}|^p_{L_p(\mathcal{L}_2)} \, dt$$

by Fatou's lemma we have

$$E \sup_{t \le T} |b_t - \bar{b}_t^{n(l)}|^p_{L_p} \le NE \int_0^T |h_t - h_t^{n(l)}|^p_{L_p(\mathcal{L}_p)} + |h_t - h_t^{n(l)}|^p_{L_p(\mathcal{L}_2)} \, dt,$$

which converges to zero as $l \to \infty$. Thus there is $\Omega' \subset \Omega$ of full probability such that $(\mathbf{1}_{\Omega'} b_t)_{t \in [0,T]}$ is an L_p-valued $\mathcal{F}_t$-adapted cadlag process. For $\varphi \in L_q(\mathbb{R}^d)$ using the Davis inequality, then Minkowski's and Hölder's inequalities we have

$$E \sup_{t \in [0,T]} |(\bar{b}_t^{n(k+1)}, \varphi) - (\bar{b}_t^{n(k)}, \varphi)|$$

$$\le N' |\varphi|_{L_q} \left(E \int_0^T |h^{n(k+1)} - h^{n(k)}|^p_{L_p(\mathcal{L}_2)} \, dt \right)^{1/p} \le N'' |\varphi|_{L_q} 2^{-k}$$

with constants $N' = N'(p,T)$ and $N'' = N''(p,T)$. Hence we can see that letting $k \to \infty$ in

$$(\bar{b}_t^{n(k)}, \varphi) = \int_0^t \int_Z (h_s^{n(k)}(z), \varphi) \, \tilde{\pi}(dz, ds),$$

both sides converge almost surely, uniformly in $t \in [0,T]$, and for each $\varphi \in L_q$ we get that there is $\Omega_\varphi \subset \Omega$ of full probability such that for $\omega \in \Omega_\varphi$

$$(b_t, \varphi) = \int_0^t \int_Z (h_s(z), \varphi) \, \tilde{\pi}(dz, ds)$$

for all $t \in [0,T]$, which completes the proof of the lemma. □

Following [10] we obtain Itô's formula (2.2) by mollifying $\bar{u}$ in $x \in \mathbb{R}^d$ and applying Itô's formula (3.5). To this end we take a nonnegative kernel $k \in C_0^\infty$ with unit integral, and for $\varepsilon \in (0,1)$ and for locally integrable functions v of $x \in \mathbb{R}^d$ we use the notation $v^{(\varepsilon)}$ for the mollifications of v,

$$v^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} v(x-y) k_\varepsilon(y) \, dy, \quad x \in \mathbb{R}^d, \tag{4.5}$$

where $k_\varepsilon(y) = \varepsilon^{-d} k(y/\varepsilon)$ for $y \in \mathbb{R}^d$. Note that if $v = v(x)$ is a locally Bochner-integrable function on $\mathbb{R}^d$, taking values in a Banach space, the mollification of v is defined as (4.5) in the sense of Bochner integral.

We will make use of well-known smoothness properties of mollifications and the following well-known lemma.

Lemma 4.4. *Let V be a separable Banach space, and let $f = f(x)$ be a V-valued function of $x \in \mathbb{R}^d$ such that $f \in L_p(V) = L_p(\mathbb{R}^d, V)$ for some $p \geq 1$. Then*

$$|f^{(\varepsilon)}|_{L_p(V)} \leq |f|_{L_p(V)} \quad \textit{for every } \varepsilon > 0, \quad \textit{and} \quad \lim_{\varepsilon \to 0} |f^{(\varepsilon)} - f|_{L_p(V)} = 0.$$

Proof. By the properties of Bochner integrals, Jensen's inequality and Fubini's theorem

$$|f^{(\varepsilon)}|^p_{L_p(V)} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) k_\varepsilon(x-y) dy \right|_V^p dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)|_V^p k_\varepsilon(x-y)\, dy\, dx = |f|^p_{L_p(V)}.$$

Since V is separable, it has a countable dense subset V_0. Denote by $\mathcal{H} \subset L_p(V)$ the space of functions h of the form

$$h(x) = \sum_{i=1}^k v_i \varphi_i(x)$$

for some integer $k \geq 1$, $v_i \in V_0$ and continuous real functions φ_i on $\mathbb{R}^d$ with compact support. Then for such an h we have

$$|h^{(\varepsilon)} - h|_{L_p(V)} \leq \sum_{i=1}^k |\varphi_i^{(\varepsilon)} - \varphi_i|_{L_p} |v_i|_V \to 0 \quad \text{as } \varepsilon \to 0,$$

where $L_p = L_p(\mathbb{R}^d, \mathbb{R})$. For $f \in L_p(V)$ and $h \in \mathcal{H}$ we have

$$|f - f^{(\varepsilon)}|_{L_p(V)} \leq |f-h|_{L_p(V)} + |h-h^{(\varepsilon)}|_{L_p(V)} + |(f-h)^{(\varepsilon)}|_{L_p(V)} \leq 2|f-h|_{L_p(V)} + |h-h^{(\varepsilon)}|_{L_p(V)}.$$

Letting here $\varepsilon \to 0$ for each $f \in L_p(V)$ we obtain

$$\limsup_{\varepsilon \to 0} |f - f^{(\varepsilon)}|_{L_p(V)} \leq 2|f-h|_{L_p(V)} \quad \text{for all } h \in \mathcal{H}.$$

Since $\mathcal{H}$ is dense in $L_p(V)$, we can choose $h \in \mathcal{H}$ to make $|f-h|_{L_p(V)}$ arbitrarily small, which proves $\lim_{\varepsilon\to 0} |f - f^{(\varepsilon)}|_{L_p(V)} = 0$. □

Proof of Theorem 2.1. By Lemmas 4.1, 4.2 and 4.3 there exist $a = (a^i)$ and $b = (b^i)$ and $m = (m^i)$ in $\mathcal{U}_p$ such that for each $\varphi \in C_0^\infty$ almost surely

$$(a^i_t, \varphi) = \int_0^t (f^i_s, \varphi) ds, \quad (b^i_t, \varphi) = \int_0^t (g^{ir}_s, \varphi) dw^r_s$$

and

$$(m^i_t, \varphi) = \int_0^t \int_Z (h^i_s, \varphi)\, \tilde{\pi}(dz, ds)$$

for all $t \in [0, T]$ and $i = 1, ..., M$. Thus $a + b + m$ is an L_p-valued adapted cadlag process such that for $\bar{u}_t := \psi + a_t + b_t + m_t$ we have $(\bar{u}_t, \varphi) = (u_t, \varphi)$ for each $\varphi \in C_0^\infty$ for $P \otimes dt$ almost every $(\omega, t) \in \Omega \times [0, T]$. Hence, by taking a countable set $\Phi \subset C_0^\infty$ such that Φ is dense in

L_q, we get that $\bar{u} = u$ for $P \otimes dt$ almost everywhere as L_p-valued functions. Moreover, for each $\varphi \in C_0^\infty$

$$(\bar{u}^i_t, \varphi) = (\psi, \varphi) + \int_0^t (f^i_s, \varphi)\, ds + \int_0^t (g^{ir}_s, \varphi)\, dw^r_s + \int_0^t \int_Z (h^i_s(z), \varphi)\, \tilde{\pi}(dz, ds) \tag{4.6}$$

almost surely for all $t \in [0, T]$, $i = 1, 2, ..., M$, since both sides we have cadlag processes. To ease notation we will denote $\bar{u}$ also by u in the sequel.

By the estimates of Lemmas 4.1, 4.2 and 4.3,

$$E \int_{\mathbb{R}^d} \sup_{t \le T} |u_t(x)|^p\, dx$$
$$\le N \left(E|\psi|^p_{L_p} + |f|^p_{\mathbb{L}_p} + |g_s|^p_{\mathbb{L}_p} + |h|^p_{\mathbb{L}_{p,2}} \right) < \infty, \tag{4.7}$$

where $N = N(p, M, T)$ is a constant. Substituting $k_\varepsilon(x - \cdot) = \varepsilon^{-d} k((x - \cdot)/\varepsilon)$ in place of φ in equation (4.6), for $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we have

$$u^{(\varepsilon)i}_t(x) = \psi^{(\varepsilon)i}(x) + \int_0^t f^{(\varepsilon)i}_s(x)\, ds + \int_0^t g^{(\varepsilon)ir}_s(x)\, dw^r_s + \int_0^t \int_Z h^{(\varepsilon)i}_s(x)\, \tilde{\pi}(dz, ds)$$

almost surely for all $t \in [0, T]$ for $i = 1, 2, ..., M$. By virtue of Lemma 4.4 we have

$$\lim_{\varepsilon \to 0} |h^{(\varepsilon)} - h|^p_{\mathbb{L}_{p,2}} = 0 \tag{4.8}$$

and

$$\lim_{\varepsilon \to 0} (|f^{(\varepsilon)} - f|_{\mathbb{L}_p} + |g^{(\varepsilon)} - g|_{\mathbb{L}_p}) = 0. \tag{4.9}$$

Then using (4.8), (4.9) and estimate (4.7) with $u^{(\varepsilon)} - u$ in place of u, we have

$$\lim_{\varepsilon \to 0} E \sup_{t \le T} |u^{(\varepsilon)}_t - u_t|^p_{L_p} = 0. \tag{4.10}$$

By Minkowski's and Hölder's inequalities, for $\varepsilon > 0$ for each $x \in \mathbb{R}^d$, $s \in [0, T]$ and $\omega \in \Omega$

$$|h^{(\varepsilon)}_s(x)|_{\mathcal{L}_2} \le \int_{\mathbb{R}^d} |h_s(y)|_{\mathcal{L}_2} k_\varepsilon(x - y)\, dy \le N_\varepsilon |h_s|_{L_p(\mathcal{L}_2)} \tag{4.11}$$

with $N_\varepsilon = |k_\varepsilon|_{L_{p/(p-1)}} < \infty$. Similarly, for every $\varepsilon > 0$

$$|f^{(\varepsilon)}_s(x)| \le N_\varepsilon |f_s|_{L_p}, \quad |g^{(\varepsilon)}_s(x)|_{\ell_2} \le N_\varepsilon |g_s|_{L_p}.$$

Hence

$$\int_0^T |f^{(\varepsilon)}_s(x)| + |g^{(\varepsilon)}_s(x)|^2_{\ell_2} + |h^{(\varepsilon)}_s(x)|^2_{\mathcal{L}_2} < \infty \text{ (a.s.)}.$$

Thus we can apply Theorem 3.1 on Itô's formula to $|u^{(\varepsilon)}_t(x)|^p$ for each $x \in \mathbb{R}^d$ to get

$$|u^{(\varepsilon)}_t(x)|^p = |\psi^{(\varepsilon)}(x)|^p + \int_0^t p|u^{(\varepsilon)}_{s-}(x)|^{p-2} u^{(\varepsilon)i}_{s-}(x) g^{(\varepsilon)ir}_s(x)\, dw^r_s$$
$$+ \int_0^t p|u^{(\varepsilon)}_{s-}(x)|^{p-2} u^{(\varepsilon)i}_{s-} f^{(\varepsilon)i}_s(x)\, ds$$
$$+ \tfrac{p}{2} \int_0^t \left((p-2)|u^{(\varepsilon)}_{s-}(x)|^{p-4} |u^{(\varepsilon)i}_{s-}(x) g^{(\varepsilon)i\cdot}_s(x)|^2_{l_2} + |u^{(\varepsilon)}_{s-}(x)|^{p-2} |g^{(\varepsilon)}_s(x)|^2_{\ell_2} \right) ds$$

$$+\int_0^t\int_Z p|u^{(\varepsilon)}_{s-}(x)|^{p-2}u^{(\varepsilon)i}_{s-}(x)h^{(\varepsilon)i}_s(x)\,\tilde\pi(dz,ds)+\int_0^t\int_Z J^{h^{(\varepsilon)}_s(x,z)}|u^{(\varepsilon)}_{s-}(x)|^p\,\pi(dz,ds),\quad (4.12)$$

where the notation

$$J^a|v|^p=|v+a|^p-|v|^p-a^iD_i|v|^p=|v+a|^p-|v|^p-pa^i|v|^{p-2}v^i$$

is used for vectors $a=(a^1,...,a^M)$ and $(v^1,...,v^M)\in\mathbb{R}^M$. In order to integrate both sides of (4.12) against dx over $\mathbb{R}^d$ and apply deterministic and stochastic Fubini theorems, we are going to check that almost surely

$$A_1(x):=\int_0^T\int_Z|J^{h^{(\varepsilon)}_s}|u^{(\varepsilon)}_{s-}(x)|^p|\,\mu(dz)\,ds<\infty\quad\text{for all }x\in\mathbb{R}^d,$$

$$B_1:=\int_0^T\int_Z\int_{\mathbb{R}^d}|J^{h^{(\varepsilon)}_s}|u^{(\varepsilon)}_{s-}|^p|\,dx\,\mu(dz)\,ds<\infty,$$

$$A_2(x):=\int_0^T\int_Z|u^{(\varepsilon)}_{s-}(x)|^{2p-2}|h^{(\varepsilon)}_s(x)|^2\,\mu(dz)\,ds<\infty\quad\text{for all }x\in\mathbb{R}^d,$$

$$B_2:=\int_{\mathbb{R}^d}\left(\int_0^T\int_Z|u^{(\varepsilon)}_{s-}|^{2p-2}|h^{(\varepsilon)}_s|^2\,\mu(dz)\,ds\right)^{1/2}dx<\infty,$$

$$A_3(x):=\int_0^T|u^{(\varepsilon)}_{s-}|^{2p-4}|u^{(\varepsilon)i}_{s-}(x)g^{(\varepsilon)i\cdot}_s(x)|^2_{l_2}\,ds<\infty\quad\text{for all }x\in\mathbb{R}^d,$$

$$B_3:=\int_{\mathbb{R}^d}\left(\int_0^T|u^{(\varepsilon)}_{s-}|^{2p-4}|u^{(\varepsilon)i}_{s-}g^{(\varepsilon)i\cdot}_s|^2_{l_2}\,ds\right)^{1/2}dx<\infty$$

and

$$C:=\int_0^T\int_{\mathbb{R}^d}|u^{(\varepsilon)}_s(x)|^{p-1}|f^{(\varepsilon)}_s(x)|\,dx\,ds<\infty.$$

To this end notice first that for $a,v\in\mathbb{R}^M$ by Taylor's formula

$$|J^a|v|^p|\le N(|v|^{p-2}|a|^2+|a|^p)\qquad(4.13)$$

with a constant $N=N(p)$. Using this and Young's inequality, by (4.11) combined with

$$|u^{(\varepsilon)}_s(x)|\le N_\varepsilon|u_s|_{L_p}$$

we get that almost surely

$$A_1(x)\le N\int_0^T\int_Z\left(|u^{(\varepsilon)}_{s-}(x)|^{p-2}|h^{(\varepsilon)}_s(x,z)|^2+|h^{(\varepsilon)}_s(x,z)|^p\right)\,\mu(dz)\,ds$$

$$\le\tfrac{p-2}{p}NN^p_\varepsilon\int_0^T|u_{s-}|^p_{L_p}\,ds+\tfrac{2}{p}NN^p_\varepsilon\int_0^T|h_s|^p_{L_p(\mathcal{L}_2)}\,ds+NN^p_\varepsilon\int_0^T|h_s|^p_{L_p(\mathcal{L}_p)}\,ds<\infty$$

for all $x\in\mathbb{R}^d$. By (4.13), Young's inequality and Lemma 4.4 we have

$$B_1\le N\int_0^T\int_{\mathbb{R}^d}\int_Z\left(|u^{(\varepsilon)}_{s-}|^{p-2}|h^{(\varepsilon)}_s|^2+|h^{(\varepsilon)}_s|^p\right)\,\mu(dz)\,dx\,ds$$

$$\le\tfrac{p-2}{p}N\int_0^T|u_{s-}|^p_{L_p}\,ds+\tfrac{2}{p}N\int_0^T|h_s|^p_{L_p(\mathcal{L}_2)}\,ds+N\int_0^T|h_s|^p_{L_p(\mathcal{L}_p)}\,ds<\infty\text{ (a.s.)}.$$

Using (4.11) and

$$|u_{s-}^{(\varepsilon)}(x)| \le N_\varepsilon |u_{s-}|_{L_p} \quad \text{for } s \in (0,T],\ x \in \mathbb{R}^d,\ \omega \in \Omega,$$

by (4.7) we get that almost surely

$$\begin{aligned} A_2(x) &\le \sup_{t\le T} |u_{t-}^{(\varepsilon)}(x)|^{2p-2} \int_0^T |h_s^{(\varepsilon)}(x)|_{\mathcal{L}_2}^2\,ds \\ &\le N_\varepsilon^{2p} T^{(p-2)/p} \sup_{t\le T} |u_t|_{L_p}^{2p-2} \left(\int_0^T |h_s|_{L_p(\mathcal{L}_2)}^p\,ds \right)^{2/p} < \infty \quad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

By Young's and Hölder's inequalities and by (4.7)

$$\begin{aligned} B_2 &\le q^{-1} \int_{\mathbb{R}^d} \sup_{t\le T} |u_t^{(\varepsilon)}|^p\,dx + p^{-1} \int_{\mathbb{R}^d} \left(\int_0^T |h_s^{(\varepsilon)}|_{\mathcal{L}_2}^2\,ds \right)^{p/2} dx \\ &\le q^{-1} \int_{\mathbb{R}^d} (\sup_{t\le T} |u_t|^p)^{(\varepsilon)}\,dx + T^{(p-2)/2} p^{-1} \int_0^T |h_s^{(\varepsilon)}|_{L_p(\mathcal{L}_2)}^p\,ds \\ &\le q^{-1} \int_{\mathbb{R}^d} \sup_{t\le T} |u_t|^p\,dx + T^{(p-2)/2} p^{-1} \int_0^T |h_s|_{L_p(\mathcal{L}_2)}^p\,ds < \infty \text{ (a.s.)} \end{aligned}$$

with $q = p/(p-1)$. Similarly, for almost every $\omega \in \Omega$

$$A_3(x) \le N_\varepsilon^{2p} T^{(p-2)/p} \sup_{t\le T} |u_t|_{L_p}^{2p-2} \left(\int_0^T |g_s|_{L_p(\ell_2)}^p\,ds \right)^{2/p} < \infty \quad \text{for all } x \in \mathbb{R}^d,$$

$$B_3 \le q^{-1} \int_{\mathbb{R}^d} \sup_{t\le T} |u_t|^p\,dx + T^{(p-2)/2} p^{-1} \int_0^T |g_t|_{L_p(\ell_2)}^p\,dt < \infty,$$

and

$$C \le q^{-1} \int_{\mathbb{R}^d} \sup_{t\le T} |u_t|^p\,dx + T^{p-1} p^{-1} \int_0^T |f_t|_{L_p}^p\,dt < \infty.$$

Note that $u_{t-}^{(\varepsilon)}(x)$ is left continuous in t, it is continuous in x, and it is $\mathcal{F}_t$-measurable for every (t,x). Therefore $u_{t-}(x)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping of $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^d$, and hence it is easy to show that the integrands in (4.12) are also $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions of (ω, t, x).

Thus integrating (4.12) over $\mathbb{R}^d$ we can use the deterministic Fubini theorem together with the stochastic Fubini theorems, Lemma 2.6 from [9] and Theorems 3.4 and 3.5 above, to get

$$\begin{aligned} |u_t^{(\varepsilon)}|_{L_p}^p &= |\psi^{(\varepsilon)}|_{L_p}^p + \int_0^t \int_{\mathbb{R}^d} p |u_s^{(\varepsilon)}|^{p-2} u_s^{(\varepsilon)i} g_s^{(\varepsilon)ir}\,dx\,dw_s^r \\ &+ \tfrac{p}{2} \int_0^t \int_{\mathbb{R}^d} 2|u_s^{(\varepsilon)}|^{p-2} u_s^{(\varepsilon)i} f_s^{(\varepsilon)i} + (p-2)|u_s^{(\varepsilon)}|^{p-4} |u_s^{(\varepsilon)i} g_s^{(\varepsilon)i\cdot}|_{l_2}^2 + |u_s^{(\varepsilon)}|^{p-2} |g_s^{(\varepsilon)}|_{l_2}^2\,dx\,ds \\ &+ \int_0^t \int_Z \int_{\mathbb{R}^d} p |u_{s-}^{(\varepsilon)}|^{p-2} u_{s-}^{(\varepsilon)i} h_s^{(\varepsilon)i}\,dx\,\tilde{\pi}(dz,ds) + \int_0^t \int_Z \int_{\mathbb{R}^d} J^{h_s^{(\varepsilon)}} |u_{s-}^{(\varepsilon)}|^p\,dx\,\pi(dz,ds) \qquad (4.14) \end{aligned}$$

almost surely for all $t \in [0,T]$. In order to take $\varepsilon \to 0$ here, we need to prove

$$A_\varepsilon := \int_0^T \int_Z \left(\int_{\mathbb{R}^d} |u^{(\varepsilon)}_{s-}|^{p-2} u^{(\varepsilon)i}_{s-} h^{(\varepsilon)i}_s - |u_{s-}|^{p-2} u^i_{s-} h^i_s \, dx \right)^2 \mu(dz)\, ds \to 0,$$

$$B_\varepsilon := \int_0^T \int_Z \int_{\mathbb{R}^d} \left| J^{h^{(\varepsilon)}_s} |u^{(\varepsilon)}_{s-}|^p - J^{h_s} |u_{s-}|^p \right| dx\, \pi(dz, ds) \to 0$$

and

$$C_\varepsilon := \int_0^T \left| \int_{\mathbb{R}^d} |u^{(\varepsilon)}_s|^{p-2} u^{(\varepsilon)i}_s g^{(\varepsilon)i\cdot}_s - |u_s|^{p-2} u^i_s g^{i\cdot}_s \, dx \right|^2_{l_2} ds \to 0$$

in probability as $\varepsilon \to 0$. To this end notice that $A_\varepsilon \le A^1_\varepsilon + A^2_\varepsilon$ with

$$A^1_\varepsilon := \int_0^T \int_Z \left(\int_{\mathbb{R}} |u^{(\varepsilon)}_{s-}|^{p-1} |h^{(\varepsilon)}_s - h_s| \, dx \right)^2 \mu(dz)\, ds$$

and

$$A^2_\varepsilon := \int_0^T \int_Z \left(\int_{\mathbb{R}^d} (|u^{(\varepsilon)}_{s-}|^{p-2} u^{(\varepsilon)i}_{s-} - |u_{s-}|^{p-2} u^i_{s-}) h^i_s \, dx \right)^2 \mu(dz)\, ds.$$

By Minkowski's and Hölder inequalities

$$\begin{aligned} A^1_\varepsilon &\le \int_0^T \left(\int_{\mathbb{R}^d} \left(\int_Z |u^{(\varepsilon)}_{s-}|^{2p-2} |h^{(\varepsilon)}_s - h_s|^2 \, \mu(dz) \right)^{1/2} dx \right)^2 ds \\ &\le \int_0^T |u_{s-}|^{2p-2}_{L_p} \left(\int_{\mathbb{R}^d} |h^{(\varepsilon)} - h_s|^p_{\mathcal{L}_2} \, dx \right)^{2/p} ds \\ &\le \sup_{t \le T} |u_t|^{2p-2}_{L_p} \int_0^T |h^{(\varepsilon)}_s - h_s|^p_{L_p(\mathcal{L}_2)} \, ds \to 0 \end{aligned}$$

almost surely as $\varepsilon \to 0$, and

$$\begin{aligned} A^2_\varepsilon &\le \int_0^T \left(\int_{\mathbb{R}^d} \left(\int_Z ||u^{(\varepsilon)}_{s-}|^{p-2} u^{(\varepsilon)}_{s-} - |u_{s-}|^{p-2} u_{s-}|^2 |h_s|^2 \, \mu(dz) \right)^{1/2} dx \right)^2 ds \\ &\le \int_0^T \left(\int_{\mathbb{R}^d} ||u^{(\varepsilon)}_{s-}|^{p-2} u^{(\varepsilon)}_{s-} - |u_{s-}|^{p-2} u_{s-}| |h_s|_{\mathcal{L}_2} \, dx \right)^2 ds \\ &\le \int_0^T ||u^{(\varepsilon)}_{s-}|^{p-2} u^{(\varepsilon)}_{s-} - |u_{s-}|^{p-2} u_{s-}|^2_{L_{p/(p-1)}} |h_s|^2_{L_p(\mathcal{L}_2)} \, ds. \end{aligned} \tag{4.15}$$

Using (4.7), by Lebesgue's theorem on dominated convergence we can see that $u^{(\varepsilon)}_{s-} = (u_{s-})^{(\varepsilon)}$ for every $s \in (0,T]$ and $\omega \in \Omega$. Since $u_{s-} \in L_p(\mathbb{R}^d)$, we have $u^{(\varepsilon)}_{s-} \to u_{s-}$ in $L_p(\mathbb{R}^d)$ as $\varepsilon \to 0$. Hence for fixed ω and $s \in (0,T]$ there is a sequence $\varepsilon_k \to 0$ such that $u^{(\varepsilon_k)}_{s-}(x) \to u_{s-}(x)$ for dx-almost every x, as $k \to \infty$. Applying Lemma 3.8 to the sequence $(u^{(\varepsilon_k)}_{s-})_{k=1}^\infty$ in $V = L_p$ we get a subsequence, for simplicity denoted also by $(u^{(\varepsilon_k)}_{s-})_{k=1}^\infty$, and a function $v \in L_p$ such that $|u^{(\varepsilon_k)}_{s-}(x)| \le v(x)$ for all $x \in \mathbb{R}^d$ and all k. Thus

$$||u^{(\varepsilon_k)}_{s-}|^{p-2} u^{(\varepsilon_k)}_{s-} - |u_{s-}|^{p-2} u_{s-}| \le v^{p-1} + |u(s-)|^{p-1} \in L_{p/p-1},$$

and by Lebesgue's theorem on dominated convergence

$$\lim_{k\to\infty} ||u^{(\varepsilon_k)}_{s-}|^{p-2}u^{(\varepsilon_k)}_{s-} - |u_{s-}|^{p-2}u_{s-}|_{L_{p/p-1}} = 0.$$

Since we have this for a subsequence of any sequence $\varepsilon_k \to 0$, we have

$$\lim_{\varepsilon\to 0} ||u^{(\varepsilon)}_{s-}|^{p-2}u^{(\varepsilon)}_{s-} - |u_{s-}|^{p-2}u_{s-}|_{L_{p/p-1}} = 0 \quad \text{for all } s \in (0,T] \text{ and } \omega \in \Omega.$$

By Hölder's inequality we have

$$||u^{(\varepsilon)}_{s-}|^{p-2}u^{(\varepsilon)}_{s-} - |u_{s-}|^{p-2}u_{s-}|_{L_{p/(p-1)}} \leq 2|u_{s-}|^{p-1}_{L_p} \text{ (a.s.)}.$$

Hence

$$||u^{(\varepsilon)}_{s-}|^{p-2}u^{(\varepsilon)}_{s-} - |u_{s-}|^{p-2}u_{s-}|^2_{L_{p/(p-1)}}|h_s|^2_{L_p(\mathcal{L}_2)} \leq 2|h_s|^2_{L_p(\mathcal{L}_2)}\sup_{s\leq T}|u_s|^{2p-2}_{L_p},$$

and we can use again Lebesgue's theorem on dominated convergence to get that $\lim_{\varepsilon\to 0} A^2_\varepsilon = 0$ almost surely. Consequently, $\lim_{\varepsilon\to 0} A_\varepsilon = 0$ (a.s.), which implies

$$\sup_{t\leq T}\left|\int_0^t\int_Z\int_{\mathbb{R}^d}\Big(|u^{(\varepsilon)}_{s-}|^{p-2}u^{(\varepsilon)i}_{s-}h^{(\varepsilon)i}_s - |u_{s-}|^{p-2}u^i_{s-}h^i_s\Big)\,dx\,\tilde{\pi}(dz,ds)\right| \to 0 \tag{4.16}$$

in probability as $\varepsilon \to 0$. In order to prove $B_\varepsilon \to 0$, we are going to show

$$\lim_{\varepsilon\to 0} E\int_0^T\int_Z\int_{\mathbb{R}^d}\Big|J^{h^{(\varepsilon)}_s}|u^{(\varepsilon)}_{s-}|^p - J^{h_s}|u_{s-}|^p\Big|\,dx\,\mu(dz)\,ds = 0. \tag{4.17}$$

Note that by (4.10), (4.8) and (4.9), for any sequence $\varepsilon_k \to 0$ there is a subsequence, denoted also by ε_k, such that

$$J^{h^{(\varepsilon_k)}_s}|u^{(\varepsilon_k)}_{s-}|^p \to J^h|u_{s-}|^p \quad \text{in } P\otimes dt\otimes dx\otimes \mu(dz) \text{ as } k\to\infty. \tag{4.18}$$

Thus to get (4.17) by virtue of Lebesgue's theorem on dominated convergence we need only show the existence of a function in $L_1(\Omega\times[0,T],L_1(\mathcal{L}_1))$, which dominates the integrand in (4.17) for $\varepsilon = \varepsilon_k$ for all $k \geq 1$. By (4.13)

$$J^{h^{(\varepsilon)}_s}|u^{(\varepsilon)}_{s-}|^p \leq N\big(|u^{(\varepsilon)}_{s-}|^{p-2}|h^{(\varepsilon)}_s|^2 + |h^{(\varepsilon)}_s|^p\big)$$

with a constant $N = N(p)$. Due to (4.8) and (4.10), by Lemma 3.8, there exist a sequence $\varepsilon_k \to 0$ and functions $v \in \mathbb{L}_p$ and $H \in \mathbb{L}_{p,2}$, such that together with (4.18)

$$|u^{(\varepsilon_k)}_{s-}| \leq |v_s| \quad \text{and} \quad |h^{(\varepsilon_k)}_s| \leq |H_s| \quad \text{for all } (\omega,s,z,x) \text{ and } k\geq 1$$

hold. Thus

$$|J^{h^{(\varepsilon_k)}}|u^{(\varepsilon_k)}_{s-}|^p| \leq N(|v_s|^{p-2}|H_s|^2 + |H_s|^p) \quad \text{for } (\omega,s,z,x) \text{ and } k\geq 1.$$

By Hölder's and Young's inequalities,

$$E\int_0^T\int_{\mathbb{R}^d}\int_Z\big(|v_s|^{p-2}|H_s|^2 + |H_s|^p\big)\,\mu(dz)\,dx\,ds$$

$$\leq \tfrac{p-2}{p}|v|^p_{\mathbb{L}_p} + \tfrac{2}{p}E\int_0^T|H_s|^p_{L_p(\mathcal{L}_2)}\,ds + E\int_0^T|H_s|^p_{L_p(\mathcal{L}_2)}\,ds < \infty,$$

which shows that $|v_s|^{p-2}|H_s|^2+|H_s|^p \in L_1(\Omega\times[0,T],L_1(\mathcal{L}_1))$ and finishes the proof of (4.17). Consequently,

$$\begin{aligned}
&E\sup_{t\in[0,T]}\left|\int_0^t\int_Z J^{h_s^{(\varepsilon)}}|u_{s-}^{(\varepsilon)}|^p\,\pi(dz,ds)-\int_0^t\int_Z J^{h_s}|u_{s-}|^p\,\pi(dz,ds)\right|\\
&\qquad\le E\int_0^T\int_Z\left|J^{h_s^{(\varepsilon)}}|u_{s-}^{(\varepsilon)}|^p-J^{h_s}|u_{s-}|^p\right|\,\pi(dz,ds)\\
&=E\int_0^T\int_Z\left|J^{h^{(\varepsilon)}}|u_{s-}^{(\varepsilon)}|^p-J^{h_s}|u_{s-}|^p\right|\,\mu(dz)\,ds\to 0\quad\text{as }\varepsilon\to0.
\end{aligned}\tag{4.19}$$

Now we are going to show that $\lim_{\varepsilon\to0}C_\varepsilon=0$ almost surely, which implies

$$\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}^d}(|u_s^{(\varepsilon)}|^{p-2}u_s^{(\varepsilon)i}g_s^{(\varepsilon)ir}-|u_s|^{p-2}u_s^ig_s^{ir})\,dx\,dw_s^r\right|\to0\tag{4.20}$$

in probability as $\varepsilon\to0$. By Minkowski's inequality

$$C_\varepsilon\le\int_0^T\left(\int_{\mathbb{R}^d}\left||u_t^{(\varepsilon)}|^{p-2}u_t^{(\varepsilon)i}g_t^{(\varepsilon)i\cdot}-|u_t|^{p-2}u_t^ig_t^{i\cdot}\right|_{l_2}dx\right)^2dt.\tag{4.21}$$

Since for fixed $t\in[0,T]$ and $\omega\in\Omega$ we have $|u_t^{(\varepsilon)}-u_t|_{L_p}\to0$ and $|g_t^{(\varepsilon)}-g_t|_{L_p}\to0$ as $\varepsilon\to0$, for any sequence $\varepsilon_k\to0$ there exists a subsequence, denoted also by ε_k, such that almost surely

$$\lim_{\varepsilon_k\to0}u_t^{(\varepsilon_k)}=u_t\quad\text{and}\quad\lim_{\varepsilon_k\to0}g_t^{(\varepsilon_k)}=g_t\quad dx\text{-almost everywhere},$$

and, by virtue of Lemma 3.8 we have functions $v\in L_p$ and $G\in L_p$ such that

$$|u_t^{(\varepsilon_k)}|\le v\quad\text{and}\quad|g_t^{(\varepsilon_k)}|_{\ell_2}\le G\quad\text{for all }x\in\mathbb{R}^d\text{ for all }k\ge1$$

for the fixed $t\in[0,T]$ and $\omega\in\Omega$. Thus

$$\begin{aligned}
&\left||u_t^{(\varepsilon_k)}|^{p-2}|u_t^{(\varepsilon_k)i}g_t^{(\varepsilon_k)i\cdot}-|u_t|^{p-2}u_t^ig_t^{i\cdot}\right|_{l_2}\le v^{p-1}G+|u_t|^{p-1}|g_t|_{\ell_2}\\
&\le\tfrac{p-1}{p}v^p+\tfrac1pG^p+\tfrac{p-1}{p}|u_t|^p+\tfrac1p|g_t|_{\ell_2}^p\in L_1(\mathbb{R}^d,\mathbb{R})\quad\text{for all }k\ge1,
\end{aligned}$$

and by Lebesgue's theorem on dominated convergence

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\left||u_t^{(\varepsilon_k)}|^{p-2}u_t^{(\varepsilon_k)i}g_t^{(\varepsilon_k)i\cdot}-|u_t|^{p-2}u_t^ig_t^{i\cdot}\right|_{l_2}dx=0.$$

Consequently,

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^d}\left||u_t^{(\varepsilon)}|^{p-2}u_t^{(\varepsilon)i}g_t^{(\varepsilon)i\cdot}-|u_t|^{p-2}u_t^ig_t^{i\cdot}\right|_{l_2}dx=0.$$

Notice that by Hölder's inequality

$$\left||u^{(\varepsilon)}|^{p-2}u^{(\varepsilon)i}g^{(\varepsilon)i\cdot}-|u|^{p-2}u^ig^{i\cdot}\right|_{L_1}^2\le4|g|_{L_p}^2\sup_{t\in[0,T]}|u_t|_{L_p}^{2p-2}\in L_1([0,T],\mathbb{R}),$$

where $L_1=L_1(\mathbb{R}^d,\mathbb{R})$ and $L_p=L_p(\mathbb{R}^d,\mathbb{R})$. Thus letting $\varepsilon\to0$ in (4.21) by Lebesgue's theorem on dominated convergence we get $\lim_{\varepsilon\to0}C_\varepsilon=0$. Finally we show

$$E\int_0^T\int_{\mathbb{R}^d}\left||u_s^{(\varepsilon)}|^{p-2}u_s^{(\varepsilon)i}f_s^{(\varepsilon)i}-|u_s|^{p-2}u_s^if_s^i\right|dx\,ds\to0,$$

$$E\int_0^T\int_{\mathbb{R}^d}\Big||u_s^{(\varepsilon)}|^{p-4}|u_s^{(\varepsilon)i}g_s^{(\varepsilon)i\cdot}|_{l_2}^2-|u_s|^{p-4}|u_s^ig_s^{i\cdot}|_{l_2}^2\Big|\,dx\,ds\to0,$$

$$E\int_0^T\int_{\mathbb{R}^d}\Big||u_s^{(\varepsilon)}|^{p-2}|g_s^{(\varepsilon)}|_{\ell_2}^2-|u_s|^{p-2}|g_s|_{\ell_2}^2\Big|\,dx\,ds\to0. \tag{4.22}$$

as $\varepsilon\to0$. Since $|u^{(\varepsilon)}-u|_{\mathbb{L}_p}\to0$, $|f^{(\varepsilon)}-f|_{\mathbb{L}_p}\to0$ and $|g^{(\varepsilon)}-g|_{\mathbb{L}_p}\to0$ as $\varepsilon\to0$, for any sequence $\varepsilon_k\to0$ there exist a subsequence, denoted also by ε_k such that

$$\lim_{k\to\infty}(|u^{(\varepsilon_k)}-u|+|f^{(\varepsilon_k)}-f|+|g^{(\varepsilon_k)}-g|_{\ell_2})=0\quad P\otimes dt\otimes dx\,(a.e.),$$

and by virtue of Lemma 3.8 there are functions $\mathbf{v}\in\mathbb{L}_p$, $\mathbf{f}\in\mathbb{L}_p$ and $\mathbf{g}\in\mathbb{L}_p$ such that

$$|u^{(\varepsilon_k)}|\le\mathbf{v},\quad|f^{(\varepsilon_k)}|\le\mathbf{f},\quad|g^{(\varepsilon_k)}|_{\ell_2}\le\mathbf{g}\quad\text{for all }(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^d\text{ and }k\ge1.$$

Thus by Hölder's inequality

$$||u^{(\varepsilon_k)}|^{p-2}u^{(\varepsilon_k)i}f^{(\varepsilon_k)i}-|u|^{p-2}u^if^i|\le\mathbf{v}^{p-1}\mathbf{f}+|u|^{p-1}|f|\in\mathbb{L}_1,$$

$$||u^{(\varepsilon_k)}|^{p-4}|u^{(\varepsilon_k)i}g^{(\varepsilon_k)i\cdot}|_{l_2}^2-|u|^{p-4}|u^ig^{i\cdot}|_{l_2}^2|\le\mathbf{v}^{p-2}\mathbf{g}^2+|u|^{p-2}|g|_{\ell_2}^2\in\mathbb{L}_1,$$

$$||u^{(\varepsilon_k)}|^{p-2}|g^{(\varepsilon_k)}|_{\ell_2}^2-|u|^{p-2}|g|_{\ell_2}^2|\le\mathbf{v}^{p-2}\mathbf{g}^2+|u|^{p-2}|g|_{\ell_2}^2\in\mathbb{L}_1,$$

and by Lebesgue's theorem on dominated convergence we get (4.22) for $\varepsilon_k\to0$, and hence for $\varepsilon\to0$ as well. Using this together with (4.16), (4.19) and (4.20), we obtain (2.2) by letting $\varepsilon\to0$ in (4.14). □

Proof of Theorem 2.2. By taking $\varphi^{(\varepsilon)}$ in place of φ in equation (2.4), we get for each $\varphi\in C_0^\infty$

$$(u_t^{(\varepsilon)},\varphi)=(\psi^{(\varepsilon)},\varphi)+\int_0^t(f_s^{(\varepsilon)},\varphi)\,ds+\int_0^t(g_s^{(\varepsilon)r},\varphi)\,dw_s^r+\int_0^t\int_Z(h_s^{(\varepsilon)},\varphi)\,\tilde{\pi}(dz,ds) \tag{4.23}$$

$P\otimes dt$ almost every $(\omega,t)\in\Omega\times[0,T]$, where

$$f_s^{(\varepsilon)}:=f_s^{i(\varepsilon)}+f_s^{0(\varepsilon)},$$

where i runs through $\{1,2,...,d\}$. Hence by Theorem 2.2 we have an L_p-valued adapted cadlag process $\bar{u}^\varepsilon$ such that for each $\varphi\in C_0^\infty$ almost surely (4.23) holds with $\bar{u}^\varepsilon$ in place of $u^{(\varepsilon)}$ for all $t\in[0,T]$. In particular, for each $\varphi\in C_0^\infty$ we have $(u^{(\varepsilon)},\varphi)=(\bar{u}^\varepsilon,\varphi)$ for $P\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,T]$. Thus $u^{(\varepsilon)}=\bar{u}^\varepsilon$, as L_p-valued functions, for $P\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,T]$, and almost surely

$$|\bar{u}_t^\varepsilon|_{L_p}^p=|\psi^{(\varepsilon)}|_{L_p}^p+p\int_0^t\int_{\mathbb{R}^d}|\bar{u}_s^\varepsilon|^{p-2}\bar{u}_s^\varepsilon g_s^{(\varepsilon)r}\,dx\,dw_s^r$$

$$+p\int_0^t\int_{\mathbb{R}^d}\Big(|\bar{u}_s^\varepsilon|^{p-2}\bar{u}_s^\varepsilon f_s^{0(\varepsilon)}+|u_s^{(\varepsilon)}|^{p-2}u_s^{(\varepsilon)}D_if_s^{i(\varepsilon)}+\tfrac12(p-1)|\bar{u}_s^\varepsilon|^{p-2}|g_s^{(\varepsilon)}|_{l_2}^2\Big)\,dx\,ds$$

$$+p\int_0^t\int_Z\int_{\mathbb{R}^d}|\bar{u}_{s-}^\varepsilon|^{p-2}u_{s-}^{(\varepsilon)}h_s^{(\varepsilon)}\,dx\,\tilde{\pi}(dz,ds)+\int_0^t\int_Z\int_{\mathbb{R}^d}J^{h^{(\varepsilon)}}|\bar{u}_{s-}^\varepsilon|^p\,dx\,\pi(dz,ds)$$

for all $t\in[0,T]$. Hence, using that by integration by parts

$$\int_{\mathbb{R}^d}|u_s^{(\varepsilon)}|^{p-2}u_s^{(\varepsilon)}D_if_s^{i(\varepsilon)}\,dx=-\int_{\mathbb{R}^d}(p-1)|\bar{u}_s^\varepsilon|^{p-2}f_s^{(\varepsilon)i}D_iu_s^{(\varepsilon)}\,dx,$$

for $P\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,T]$ we get

$$|\bar u^\varepsilon_t|^p_{L_p} = |\psi^{(\varepsilon)}|^p_{L_p} + p\int_0^t\int_{\mathbb{R}^d} |\bar u^\varepsilon_s|^{p-2}\bar u^\varepsilon_s g^{(\varepsilon)r}_s\,dx\,dw^r_s$$

$$+p\int_0^t\int_{\mathbb{R}^d}\Big(|\bar u^\varepsilon_s|^{p-2}\bar u^\varepsilon_s f^{0(\varepsilon)}_s - (p-1)|\bar u^\varepsilon_s|^{p-2} f^{i(\varepsilon)}_s D_i u^{(\varepsilon)}_s + \tfrac12 (p-1)|\bar u^\varepsilon_s|^{p-2}|g^{(\varepsilon)}_s|^2_{l_2}\Big)\,dx\,ds$$

$$+\,p\int_0^t\int_Z\int_{\mathbb{R}^d} |\bar u^\varepsilon_{s-}|^{p-2}\bar u^\varepsilon_{s-}h^{(\varepsilon)}_s\,dx\,\tilde\pi(dz,ds) + \int_0^t\int_Z\int_{\mathbb{R}^d} J^{h^{(\varepsilon)}}|\bar u^\varepsilon_{s-}|^p\,dx\,\pi(dz,ds). \qquad (4.24)$$

almost surely for all $t\in[0,T]$. By Davis', Minkowski and Hölder inequalities we have

$$E\sup_{t\le T}\left|\int_0^t\int_Z\int_{\mathbb{R}^d} p|\bar u^\varepsilon_{s-}|^{p-2}\bar u^\varepsilon_{s-}h^{(\varepsilon)}_s\,dx\,\tilde\pi(dz,ds)\right|$$

$$\le 3pE\left(\int_0^T\int_Z\left(\int_{\mathbb{R}^d}|\bar u^\varepsilon_{s-}|^{p-2}\bar u^\varepsilon_{s-}h^{(\varepsilon)}_s\,dx\right)^2\mu(dz)\,ds\right)^{1/2}$$

$$\le 3pE\left(\int_0^T\left(\int_{\mathbb{R}^d}|\bar u^\varepsilon_s|^{p-1}|h^{(\varepsilon)}_s|_{\mathcal{L}_2}\,dx\right)^2 ds\right)^{1/2}$$

$$\le 3pE\left(\int_0^T |\bar u^\varepsilon_s|^{2p-2}_{L_p}|h^{(\varepsilon)}_s|^2_{L_p(\mathcal{L}_2)}\,ds\right)^{1/2} \le \frac{1}{12}\,E\sup_{t\le T}|\bar u^\varepsilon_t|^p_{L_p} + NT^{(p-2)/2}|h^{(\varepsilon)}|^p_{\mathbb{L}_{p,2}}, \qquad (4.25)$$

with a constant $N=N(p,d)$. Similarly,

$$E\sup_{t\le T}\left|\int_0^t\int_{\mathbb{R}^d} p|\bar u^\varepsilon_s|^{p-2}\bar u^\varepsilon_s g^{(\varepsilon)r}_s\,dx\,dw^r_s\right| \le \frac{1}{12}\,E\sup_{t\le T}|\bar u^\varepsilon_t|^p_{L_p} + NT^{(p-2)/2}|g^{(\varepsilon)}|^p_{\mathbb{L}_p}. \qquad (4.26)$$

By (4.13) and Hölder inequality we have

$$E\int_0^T\int_Z\int_{\mathbb{R}^d} J^{h^{(\varepsilon)}}|\bar u^\varepsilon_{s-}|^p\,dx\,\pi(dz,ds)$$

$$\le N\,E\int_0^T\int_{\mathbb{R}^d}\int_Z\big(|\bar u^\varepsilon_{s-}|^{p-2}|h^{(\varepsilon)}_s|^2 + |h^{(\varepsilon)}_s|^p\big)\,\mu(dz)\,dx\,ds$$

$$\le \frac{1}{12}\,E\sup_{t\le T}|\bar u^\varepsilon_t|^p_{L_p} + N'T^{(p-2)/2}E\int_0^T|h^{(\varepsilon)}_t|^p_{L_p(\mathcal{L}_2)}\,dt + NE\int_0^T|h^{(\varepsilon)}_t|^p_{L_p(\mathcal{L}_p)}\,dt, \qquad (4.27)$$

with constants N and N' depending only on p and d. By Hölder's and Young's inequalities

$$pE\int_0^T\int_{\mathbb{R}^d}|\bar u^\varepsilon_s|^{p-2}|\bar u^\varepsilon_s f^{0(\varepsilon)}_s|\,dx\,ds \le \frac{1}{12}\,E\sup_{t\le T}|\bar u^\varepsilon|^p_{L_p} + NT^{p-1}|f^{0(\varepsilon)}_s|^p_{\mathbb{L}_p}$$

$$p(p-1)E\int_0^T\int_{\mathbb{R}^d}|\bar u^\varepsilon_s|^{p-2}f^{i(\varepsilon)}_s D_i u^{(\varepsilon)}_s\,dx\,ds$$

$$\le \frac{1}{12}\,E\sup_{t\le T}|\bar u^\varepsilon|^p_{L_p} + NT^{(p-2)/2}\Big(\sum_{i=1}^d|f^{i(\varepsilon)}|^p_{\mathbb{L}_p} + |Du^{(\varepsilon)}|^p_{\mathbb{L}_p}\Big)$$

$$\tfrac{1}{2}p(p-1)E\int_0^T\int_{\mathbb{R}^d}|\bar{u}^{\varepsilon}_s|^{p-2}|g^{(\varepsilon)}_s|^2_{l_2}\,dx\,ds\leq\frac{1}{12}\,E\sup_{t\leq T}|\bar{u}^{\varepsilon}_t|^p_{L_p}+NT^{(p-2)/2}|g^{(\varepsilon)}|^p_{\mathbb{L}_p}$$

Using these inequalities together with (4.25), (4.26) and (4.27), from (4.24) we obtain

$$\begin{aligned}E\sup_{t\leq T}|\bar{u}^{\varepsilon}_t|^p_{L_p}&\leq 2\,E|\psi^{(\varepsilon)}|^p_{L_p}+NE\int_0^T|h^{(\varepsilon)}_t|^p_{L_p(\mathcal{L}_p)}\,dt+NT^{p-1}|f^{0(\varepsilon)}|^p_{\mathbb{L}_p}\\&+NT^{(p-2)/2}\big(|g^{(\varepsilon)}|^p_{\mathbb{L}_p}+E\int_0^T|h^{(\varepsilon)}_t|^p_{L_p(\mathcal{L}_2)}+\sum_{i=1}^d|f^{i(\varepsilon)}|^p_{\mathbb{L}_p}+|Du^{(\varepsilon)}|^p_{\mathbb{L}_p}\big)\end{aligned}\tag{4.28}$$

with a constant $N=N(p,d)$. Hence

$$E\sup_{t\leq T}|\bar{u}^{\varepsilon}_t-\bar{u}^{\varepsilon'}_t|^p_{L_p}\to 0\quad\text{as }\varepsilon,\,\varepsilon'\to 0.$$

Consequently, there is an L_p-valued adapted cadlag process $\bar{u}=(\bar{u}_t)_{t\in[0,T]}$ such that

$$\lim_{\varepsilon\to 0}E\sup_{t\leq T}|\bar{u}^{\varepsilon}_t-\bar{u}|^p_{L_p}=0.$$

Thus for each $\varphi\in C_0^{\infty}(\mathbb{R}^d)$ we can take $\varepsilon\to 0$ in

$$\begin{aligned}(\bar{u}^{\varepsilon}_t,\varphi)&=(\psi^{(\varepsilon)},\varphi)+\int_0^t(f^{(\varepsilon)}_s,\varphi)\,ds+\int_0^t(g^{(\varepsilon)r}_s,\varphi)\,dw^r_s+\int_0^t\int_Z(h^{(\varepsilon)}_s,\varphi)\,\tilde{\pi}(dz,ds)\\&=(\psi^{(\varepsilon)},\varphi)+\int_0^t(f^{0(\varepsilon)}_s,\varphi)\,ds-\int_0^t(f^{i(\varepsilon)}_s,D_i\varphi)\,ds+\int_0^t(g^{(\varepsilon)r}_s,\varphi)\,dw^r_s\\&\quad+\int_0^t\int_Z(h^{(\varepsilon)}_s,\varphi)\,\tilde{\pi}(dz,ds)\end{aligned}$$

and it is easy to see we get

$$(\bar{u}_t,\varphi)=(\psi,\varphi)+\int_0^t(f^{\alpha}_s,D^*_{\alpha}\varphi)\,ds+\int_0^t(g^r_s,\varphi)\,dw^r_s+\int_0^t\int_Z(h_s,\varphi)\,\tilde{\pi}(dz,ds)$$

almost surely for all $t\in[0,T]$. Hence $\bar{u}=u$ for $P\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,T]$. Letting $\varepsilon\to 0$ in (4.28), we get estimate (2.6). Finally letting $\varepsilon\to 0$ in (4.14), by analogous arguments as in the proof of Theorem 2.2, we obtain (2.5). □

Acknowledgement. The main results of the paper were presented at the conference on "New Directions in Stochastic Analysis: Rough Paths, SPDEs and Related Topics", on the occasion of Professor Terry Lyons' 65th Birthday, in Zuse Institute Berlin, 8-22 March, 2019. The authors are grateful to the organisers for the invitation.

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