Introductory Physics I

Elementary Mechanics

by

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Notice

This is a "lecture note" style textbook, designed to support my personal teaching activities at Duke University, in particular teaching its Physics 41/42, 53/54, or 61/62 series (Introductory Physics for potential physics majors, life science majors, or engineers respectively). It is freely available in its entirety in a downloadable PDF form or to be read online at:

http://www.phy.duke.edu/~rgb/Class/intro_physics_2.php

and will be made available in an inexpensive print version via Lulu press as soon as it is in a sufficiently polished and complete state.

In this way the text can be used by students all over the world, where each student can pay (or not) according to their means. Nevertheless, I am hoping that students who truly find this work useful will purchase a copy through Lulu or Amazon when that option becomes available, if only to help subsidize me while I continue to write inexpensive textbooks in physics or other subjects.

Although I no longer use notes to lecture from (having taught the class for decades now, they are hardly necessary) these are 'real' lecture notes and are organized for ease of presentation and ease of learning. They do not try to say every single thing that can be said about each and every topic covered, and are hierarchically organized in a way that directly supports efficient learning.

As a "live" document, these notes have errors great and small, missing figures (that I usually draw from memory in class and will add to the notes themselves as I have time or energy to draw them in a publishable form), and they cover and omit topics according to *my own* view of what is or isn't important to cover in a one-semester course. Expect them to change without warning as I add content or correct errors. Purchasers of any eventual paper version should be aware of its probable imperfection and be prepared to either live with it or mark up their *own* copies with corrections or additions as need be (in the lecture note spirit) as I do mine. The text has generous margins, is widely spaced, and contains scattered blank pages for students' or instructors' own use to facilitate this.

I cherish good-hearted communication from students or other instructors pointing out errors or suggesting new content (and have in the past done my best to implement many such corrections or suggestions).

Books by Robert G. Brown

Physics Textbooks

- *Introductory Physics I and II*

  A lecture note style textbook series intended to support the teaching of introductory physics, with calculus, at a level suitable for Duke undergraduates.

- *Classical Electrodynamics*

  A lecture note style textbook intended to support the second semester (primarily the dynamical portion, little statics covered) of a two semester course of graduate Classical Electrodynamics.

Computing Books

- *How to Engineer a Beowulf Cluster*

  An online classic for years, this is the print version of the famous free online book on cluster engineering. It too is being actively rewritten and developed, no guarantees, but it is probably still useful in its current incarnation.

Fiction

- *The Book of Lilith*

  ISBN: 978-1-4303-2245-0
  Web: http://www.phy.duke.edu/∼rgb/Lilith/Lilith.php

  Lilith is the *first* person to be given a soul by God, and is given the job of giving all the things in the world souls by loving them, beginning with Adam. Adam is given the job of making up rules and the definitions of sin so that humans may one day live in an ethical society. Unfortunately Adam is weak, jealous, and greedy, and insists on being on *top* during sex to "be closer to God".

  Lilith, however, refuses to be second to Adam or anyone else. *The Book of Lilith* is a funny, sad, satirical, uplifting tale of her spiritual journey through the ancient world soulgiving and judging to find at the end of that journey – herself.

Poetry

- *Who Shall Sing, When Man is Gone*

  Original poetry, including the epic-length poem about an imagined end of the world brought about by a nuclear war that gives the collection its name. Includes many long and short works on love and life, pain and death.

  > Ocean roaring, whipped by storm
  > in damned defiance, hating hell

with every wave and every swell,
every shark and every shell
and shoreline.

- *Hot Tea!*

  More original poetry with a distinctly Zen cast to it. Works range from funny and satirical to inspiring and uplifting, with a few erotic poems thrown in.

  Chop water, carry
  wood. Ice all around,
  fire is dying. Winter Zen?

All of these books can be found on the online Lulu store here:

http://stores.lulu.com/store.php?fAcctID=877977

The Book of Lilith is available on Amazon, Barnes and Noble and other online bookseller websites.

Contents

Preface

This introductory electromagnetism and optics text is intended to be used in the first semester of a two-semester series of courses teaching *introductory physics* at the college level, followed by a second semester course in electricity and magnetism and optics. The text is intended to support teaching the material at a rapid, but *advanced* level – it was developed to support teaching introductory calculus-based physics to potential physics majors, engineers, and other natural science majors at Duke University over a period of more than twenty-five years.

Students who hope to succeed in learning physics from this text will need, as a minimum prerequisite, a solid grasp of mathematics. It is strongly recommended that all students have mastered mathematics at least through single-variable differential calculus (typified by the AB advanced placement test or a first-semester college calculus course). Students should also be *taking* (or have completed) single variable integral calculus (typified by the BC advanced placement test or a second-semester college calculus course). In the text it is presumed that students are competent in geometry, trigonometry, algebra, and single variable calculus; more advanced multivariate calculus is used in a number of places but it is taught in context as it is needed and is always "separable" into two or three independent one-dimensional integrals.

Many students are, unfortunately *weak* in their mastery of mathematics at the time they take physics. This enormously complicates the process of learning for them, especially if they are years removed from when they took their algebra, trig, and calculus classes as is frequently the case for pre-medical students. For that reason, a separate supplementary text intended *specifically to help students of introductory physics quickly and efficiently review the required math* is being prepared as a companion volume to all semesters of introductory physics. Indeed, it should really be quite useful for any course being taught with any textbook series and not just this one.

This book is located here:

http://www.phy.duke.edu/~rgb/Class/math_for_intro_physics.php

and I *strongly suggest* that all students who are reading these words preparing to begin studying physics pause for a moment, visit this site, and either download the pdf or bookmark the site.

Note that *Week 0: How to Learn Physics* is not part of the course *per se*, but I usually do a quick review of this material (as well as the course structure, grading scheme, and so on) in my first lecture of any given semester, the one where students are still finding the room, dropping and adding courses, and one cannot present real content in good conscience unless you plan to do it again in the second lecture as well. Students *greatly benefit* from guidance on how to study, as most enter physics thinking that they can master it with nothing but the memorization and rote learning skills that have served them so well for their many other fact-based classes. Of course this is completely false – physics is *reason* based and *conceptual* and it requires a very different pattern of study than simply staring at and trying to memorize lists of formulae or examples.

Students, however, should not count on their instructor doing this – they need to be self-actualized in their study from the beginning. It is therefore *strongly suggested* that all students read this preliminary chapter right away as their first "assignment" whether or not it is covered in the first lecture or assigned. In fact, (if you're just such a student reading these words) you can always decide to read it *right now* (as soon as you finish this Preface). It won't take you an hour, and might make as much as a full letter difference (to the good) in your final grade. What do you have to lose?

Even if you think that you are an excellent student and learn things totally effortlessly, I strongly suggest reading it. It describes a new perspective on the teaching and learning process supported by very recent research in neuroscience and psychology, and makes very specific suggestions as to the best way to proceed to learn physics.

Finally, the *Introduction* is a rapid summary of *the entire course!* If you read it and look at the pictures *before* beginning the course proper you can get a good conceptual overview of everything you're going to learn. If you *begin* by learning in a *quick* pass the broad strokes for the whole course, when you go through each chapter in all of its detail, all those facts and ideas have a place to live in your mind.

That's the primary idea behind this textbook – in order to be easy to remember, ideas need a house, a place to live. Most courses try to build you that house by giving you one nail and piece of wood at a time, and force you to build it in complete detail from the ground up.

Real houses aren't built that way at all! First a foundation is established, then the *frame of the whole house* is erected, and then, slowly but surely, the frame is wired and plumbed and drywalled and finished with all of those picky little details. It works better that way. So it is with learning.

Textbook Layout and Design

This textbook has a design that is just about perfectly backwards compared to most textbooks that currently cover the subject. Here are its primary design features:

- All mathematics required by the student is reviewed in a standalone, cross-referenced (free) work at the *beginning* of the book rather than in an appendix that many students never find.

- There are only *twelve chapters*. The book is organized so that it can be sanely taught in a *single college semester* with at *most* a chapter a week.

- It *begins* each chapter with an "abstract" and chapter summary. Detail, especially lecture-note style mathematical detail, follows the summary rather than the other way around.

- This text does *not* spend page after page trying to explain in English how physics works (prose which to my experience nobody reads anyway). Instead, a terse "lecture note" style presentation outlines the main points and presents considerable mathematical detail to support solving problems.

- Verbal and conceptual understanding *is*, of course, very important. It is expected to come from verbal instruction and discussion in the classroom and recitation and lab. This textbook *relies* on having a committed and competent instructor and a sensible learning process.

- Each chapter ends with a *short* (by modern standards) selection of *challenging* homework problems. A good student might well get through all of the problems in the book, rather than at most 10% of them as is the general rule for other texts.

- The problems are weakly sorted out by level, as this text is intended to support non-physics science and pre-health profession students, engineers, and physics majors all three. The *material* covered is of course the same for all three, but the level of detail and difficulty of the math used and required is a bit different.

- The textbook is entirely algebraic in its presentation and problem solving requirements – with *very few exceptions* no calculators should be required to solve problems. The author assumes that any student taking physics is capable of punching numbers into a calculator, but it is *algebra* that ultimately determines the formula that they should be computing. Numbers are used in problems only to illustrate what "reasonable" numbers might be for a given real-world physical situation or where the problems cannot reasonably be solved algebraically (e.g. resistance networks).

This layout provides considerable benefits to both instructor and student. This textbook supports a *top-down* style of learning, where one learns each distinct chapter topic by quickly getting the main points onboard via the summary, then derives them or explores them in detail, then applies them to example problems. Finally one uses what one has started to learn working in groups and with direct mentoring and support from the instructors, to solve highly challenging problems that *cannot* be solved without acquiring the deeper level of understanding that is, or should be, the goal one is striving for.

It's without doubt a lot of work. Nobody said learning physics would be *easy*, and this book certainly doesn't claim to make it so. However, this approach will (for most students) *work*.

The reward, in the end, is the ability to see the entire world around you through new eyes, understanding much of the "magic" of the causal chain of physical forces that makes all things unfold in time. Natural Law is a strange, beautiful sort of magic; one that is utterly impersonal and mechanical and yet filled with structure and mathematics and light. It *makes sense*, both in and of itself and of the physical world you observe.

Enjoy.

I: Getting Ready to Learn Physics

Preliminaries

See, Do, *Teach*

If you are reading this, I assume that you are either taking a course in physics or wish to learn physics on your own. If this is the case, I want to begin by teaching you the importance of your personal *engagement* in the learning process. If it comes right down to it, how well you learn physics, how good a grade you get, and how much *fun* you have all depend on how enthusiastically you tackle the learning process. If you remain disengaged, detatched from the learning process, you almost certainly will do poorly and be miserable while doing it. If you can find *any degree* of engagement – or open enthusiasm – with the learning process you will very likely do well, or at least as well as possible.

Note that I use the term *learning*, not *teaching* – this is to emphasize from the beginning that learning is a choice and that *you* are in control. Learning is active; being taught is passive. It is up to you to *seize control* of your own educational process and *fully participate*, not sit back and wait for knowledge to be forcibly injected into your brain.

You may find yourself stuck in a course that is taught in a traditional way, by an instructor that lectures, assigns some readings, and maybe on a good day puts on a little dog-and-pony show in the classroom with some audiovisual aids or some demonstrations. The standard expectation in this class is to sit in your chair and watch, passive, taking notes. No real engagement is "required" by the instructor, and lacking activities or a structure that encourages it, you lapse into becoming a lecture transcription machine, recording all kinds of things that make no immediate sense to you and telling yourself that you'll sort it all out later.

You may find yourself floundering in such a class – for good reason. The instructor presents an ocean of material in each lecture, and you're going to actually retain at most a few cupfuls of it functioning as a scribe and passively copying his pictures and symbols without first extracting their sense. And the lecture *makes* little sense, at least at first, and reading (if you do any reading at all) does little to help. Demonstrations can sometimes make one or two ideas come clear, but only at the expense of twenty other things that the instructor now has no time to cover and expects you to get from the readings alone. You continually postpone going over the lectures and readings to understand the material any more than is strictly required to do the homework, until one day a *big test* draws nigh and you realize that you really don't understand anything and have forgotten most of what you did, briefly, understand. Doom and destruction loom.

Sound familiar?

On the other hand, you may be in a course where the instructor has structured the course with a balanced mix of *open* lecture (held as a freeform discussion where questions aren't just encouraged but required) and group interactive learning situations such as a carefully structured recitation and lab where discussion and doing blend together, where students teach each other and use what they have learned in many ways and contexts. If so, you're lucky, but luck only goes so far.

Even in a course like this you may *still* be floundering because you may not understand *why* it is important for you to participate with your whole spirit in the quest to learn anything you ever choose to study. In a word, you simply may not give a rodent's furry behind about learning the material so that studying is always a fight with yourself to "make" yourself do it – so that no matter what happens, *you lose.* This too may sound very familiar to some.

The importance of engagement and participation in "active learning" (as opposed to passively being taught) is not really a new idea. Medical schools were four year programs in the year 1900. They are four year programs today, where the amount of information that a physician must now master in those four years is probably *ten times greater* today than it was back then. Medical students are necessarily among the most efficient learners on earth, or they simply cannot survive.

In medical schools, the optimal learning strategy is compressed to a three-step adage: See one, do one, teach one.

See a procedure (done by a trained expert).

Do the procedure yourself, with the direct supervision and guidance of a trained expert.

Teach a student to do the procedure.

See, do, teach. Now you *are* a trained expert (of sorts), or at least so we devoutly hope, because that's all the training you are likely to get until you start doing the procedure over and over again with real humans and with limited oversight from an attending physician with too many other things to do. So you practice and study on your own until you achieve real mastery, because a mistake can *kill* somebody.

This recipe is quite general, and can be used to increase *your own* learning in almost *any* class. In fact, lifelong success in learning with or without the guidance of a good teacher is a matter of discovering the importance of *active engagement and participation* that this recipe (non-uniquely) encodes. Let us rank learning methodologies in terms of "probable degree of active engagement of the student". By probable I mean the degree of active engagement that I as an instructor have observed in students over many years and which is significantly reinforced by research in teaching methodology, especially in physics and mathematics.

Listening to a lecture as a transcription machine with your brain in "copy machine" mode is almost entirely passive and is for *most* students *probably* a nearly complete waste of time. That's not to say that "lecture" in the form of an organized presentation and review of the material to be learned isn't important or is completely useless! It serves one *very important purpose* in the grand scheme of learning, but by being passive *during* lecture *you* cause it to fail in its purpose. Its purpose is *not* to give you a complete, line by line transcription of the words of your instructor to ponder later and alone. It is to convey, for a brief shining moment, the *sense* of the *concepts* so that you *understand them*.

It is difficult to sufficiently emphasize this point. If lecture doesn't make sense *to you* when the instructor presents it, you will have to work much harder to achieve the sense of the material "later", if later ever comes at all. If you fail to identify the important concepts during the presentation and see the lecture as a string of disconnected facts, you will have to remember *each* fact as if it were an abstract string of symbols, placing impossible demands on your memory even if you are extraordinarily bright. If you fail to achieve some degree of understanding (or *synthesis* of the material, if you prefer) in lecture by asking questions and getting expert explanations on the spot, you will have to build it later out of your notes on a set of abstract symbols that made no sense to you at the time. You might as well be trying to translate Egyptian Hieroglyphs without a Rosetta Stone, and the best of luck to you with *that*.

Reading is a bit more active – at the very least your brain is more likely to be somewhat engaged if you aren't "just" transcribing the book onto a piece of paper or letting the words and symbols happen in your mind – but is still pretty passive. Even watching nifty movies or cool-ee-oh demonstrations is basically sedentary – you're still just sitting there while somebody or something *else* makes it all happen in your brain while you aren't *doing* much of anything. At best it grabs your attention a bit better (on average) than lecture, but *you* are mentally *passive*.

In all of these forms of learning, the single active thing you are likely to be doing is taking notes or moving an eye muscle from time to time. For better or worse, the human brain isn't designed to learn well in passive mode. Parts of your brain are likely to take charge and pull your eyes irresistably to the window to look outside where *active* things are going on, things that might not be so damn *boring!*

With your active engagement, with your taking charge of and participating in the learning process, things change dramatically. Instead of passively listening in lecture, you can at least *try* to ask questions and initiate discussions whenever an idea is presented that makes no intial sense to you. Discussion is an *active* process even if you aren't the one talking at the time. *You participate!* Even a tiny bit of participation in a classroom setting where students are constantly asking questions, where the instructor is constantly answering them and asking the students questions in turn makes a huge difference. Humans being social creatures, it also makes the class a lot more fun!

In summary, sitting on your ass[1] and writing meaningless (to you, so far) things down

[1] I mean, of course, your donkey. What did you think I meant?

as somebody says them in the hopes of being able to "study" them and discover their meaning on your own later is *boring* and for most students, later never comes because you are busy with *many* classes, because you haven't discovered anything beautiful or exciting (which is the *reward* for figuring it all out – if you ever get there) and then there is partying and hanging out with friends and having *fun*. Even if you do find the time and really want to succeed, in a complicated subject like physics you are less likely to be *able* to discover the meaning on your own (unless you are *so bright* that learning methodology is irrelevant and you learn in a single pass no matter what). Most introductory students are swamped by the details, and have small chance of discovering the *patterns* within those details that constitute "making sense" and make the detailed information *much, much easier to learn* by enabling a compression of the detail into a much smaller set of connected ideas.

Articulation of ideas, whether it is to yourself or to others in a discussion setting, *requires* you to create tentative patterns that might describe and organize all the details you are being presented with. Using those patterns and applying them to the details as they are presented, you naturally encounter places where your tentative patterns are wrong, or don't quite work, where something "doesn't make sense". In an "active" lecture students participate in the process, and can ask questions and kick ideas around until they *do* make sense. Participation is also *fun* and helps you pay far more attention to what's going on than when you are in passive mode. It may be that this increased attention, this consideration of many alternatives and rejecting some while retaining others with social reinforcement, is what makes all the difference. To learn optimally, even "seeing" must be an active process, one where you are not a vessel waiting to be filled through your eyes but rather part of a team studying a puzzle and looking for the patterns *together* that will help you eventually solve it.

Learning is increased still further by *doing*, the very essence of activity and engagement. "Doing" varies from course to course, depending on just what there is for you to do, but it always is the *application* of what you are learning to some sort of activity, exercise, problem. It is *not* just a recapitulation of symbols: "looking over your notes" or "(re)reading the text". The symbols for any given course of study (in a physics class, they very likely will *be* algebraic symbols for real although I'm speaking more generally here) do not, initially, mean a lot to you. If I write $\vec{F} = q(\vec{v} \times \vec{B})$ on the board, it means a great deal to *me*, but if you are taking this course for the first time it probably means zilch to *you*, and yet I pop it up there, draw some pictures, make some noises that hopefully make sense to you at the time, and blow on by. Later you read it in your notes to try to recreate that sense, but you've *forgotten* most of it. Am I describing the income I expect to make selling $\vec{B}$ tons of barley with a market value of $\vec{v}$ and a profit margin of q?

To *learn* this expression (for yes, this is a force law of nature and one that we very much must learn this semester) we have to learn what the symbols stand for – q is the charge of a point-like object in motion at velocity $\vec{v}$ in a magnetic field $\vec{B}$, and $\vec{F}$ is the resulting force acting on the particle. We have to learn that the $\times$ symbol is the *cross product of evil* (to most students at any rate, at least at first). In order to get a *gut feeling* for what this equation represents, for the directions associated with the cross product, for the trajectories it implies for charged particles moving in a magnetic field in a variety of contexts one has to *use* this expression to solve problems, *see* this expression in action in laboratory experiments that let you prove to yourself that it isn't bullshit and that the world really does have cross product force laws in it. You have to do your homework that involves this law, and be fully engaged.

The learning process isn't exactly linear, so if you participate fully in the discussion and the doing while going to even the most traditional of lectures, you have an excellent chance of getting to the point where you can score anywhere from a 75% to an 85% in the course. In most schools, say a C+ to B+ performance. Not bad, but not really excellent. A few students will still get A's – they either work extra hard, or really like the subject, or they have some sort of secret, some way of getting over that barrier at the 90's that is only crossed by those that really do understand the material quite well.

Here is the secret for getting *yourself* over that 90% hump, even in a physics class (arguably one of the most difficult courses you can take in college), even if you're *not* a super-genius (or have never managed in the past to learn like one, a glance and you're done): *Work in groups!*

That's it. Nothing really complex or horrible, just get together with your friends who are also taking the course and do your homework *together*. In a well designed physics course (and many courses in mathematics, economics, and other subjects these days) you'll have *some* aspects of the class, such as a recitation or lab, where you are *required* to work in groups, and the groups and group activities may be highly structured or freeform. "Studio" methods for teaching physics have even wrapped the lecture itself into a group-structured setting, so *everything* is done in groups, and (probably by making it nearly impossible to be disengaged and sit passively in class waiting for learning to "happen") this approach yields measureable improvements (all things being equal) on at least some objective instruments for measurement of learning.

If you take charge of your own learning, though, you will quickly see that in *any* course, however taught, *you can study in a group!* This is true even in a course where "the homework" is to be done alone by fiat of the (unfortunately ignorant and misguided) instructor. Just study "around" the actual assignment – assign *yourselves* problems "like" the actual assignment – most textbooks have plenty of extra problems and then there is the Internet and other textbooks – and do them in a group, then (afterwards!) break up and do your actual assignment alone. Note that if you use a completely different textbook to pick your group problems from and do them together before *looking* at your assignment in *your* textbook, you can't even be blamed if some of the ones you pick turn out to be ones your instructor happened to assign.

Oh, and not-so-subtly – give the instructor a PDF copy of this book (it's free for instructors, after all, and a click away on the Internet) and point to this page and paragraph containing the following little message from me to them:

> Yo! Teacher! Let's wake up and smell the coffee! Don't prevent your students from doing homework in groups – require it! Make the homework correspond-

> ingly more difficult! Give them quite a lot of course credit for doing it well! Construct a recitation or review session where students – in groups – who still cannot get the most difficult problems can get socratic tutorial help *after* working hard on the problems on their own! Integrate discussion and deliberately teach to increase *active engagement* (instead of passive wandering attention) in lecture[2]. Then watch as student performance and engagement spirals into the stratosphere compared to what it was before...

Then pray. Some instructors have their egos tied up in things to the point where *they* cannot learn, and then what can you do? If an instructor lets ego or politics obstruct their search for functional methodology, you're screwed anyway, and you might as well just tackle the material on your own. Or heck, maybe their expertise and teaching experience vastly exceeds my own so that their naked words *are* sufficiently golden that any student should be able to learn by just hearing them and doing homework all alone in isolation from any peer-interaction process that might be of use to help them make sense of it all – all data to the contrary.

My own words and lecture – in spite of my 29 years of experience in the classroom, in spite of the fact that it has been over twenty years since I actually used lecture notes to teach the course, in spite of the fact I never, ever prepare for recitation because solving the homework problems with the students "cold" *as* a peer member of their groups is useful where copying my privately worked out solutions onto a blackboard for them to passively copy on their papers in turn is useless, in spite of the fact that I wrote this book *similarly* without the use of any outside resource – my words and lecture are *not*. On the other hand, students who work effectively in groups and learn to use this book (and other resources) and do all of the homework "to perfection" might well learn physics quite well without my involvement at all!

Let's understand *why* working in groups has such a dramatic effect on learning. What happens in a group? Well, a lot of *discussion* happens, because humans working on a common problem like to talk. There is plenty of *doing* going on, presuming that the group has a common task list to work through, like a small mountain of really difficult problems that nobody can possibly solve working on their own and are *barely* within their abilities working as a group backed up by the course instructor! Finally, in a group everybody has the opportunity to *teach!*

The importance of teaching – not only seeing the lecture presentation with your whole brain actively engaged and participating in an ongoing discussion so that it makes sense at the time, not only doing lots of homework problems and exercises that apply the material in some way, but *articulating* what you have discovered in this process and *answering*

[2] Perhaps by using Studio methods, but I've found that in mid-sized classes and smaller (less than around fifty students) one can get very good results without a specially designed classroom by the Chocolate Method – I lecture without notes and offer a piece of chocolate or cheap toy or nifty pencil to any student who catches me making a mistake on the board before I catch it myself, who asks a particularly good question, who looks like they are nodding off to sleep (seriously, chocolate works wonders here, especially when ceremoniously offered). Anything that keeps students *focussed* during lecture by making it into a game, by allowing/encouraging them to speak out without raising their hands, by praising them and rewarding them for engagement makes a huge difference.

questions that force you to consider and reject alternative solutions or pathways (or not) cannot be overemphasized. Teaching each other in a peer setting (ideally with mentorship and oversight to keep you from teaching each other *mistakes*) is *essential!*

This problem you "get", and teach *others* (and actually learn it better from teaching it than they do from your presentation – never begrudge the effort required to teach your group peers even if some of them are very slow to understand). The next problem you don't get but some *other* group member does – they get to teach *you*. In the end you all learn *far more* about every problem as a consequence of the struggle, the exploration of false paths, the discovery and articulation of the correct path, the process of discussion, resolution and agreement in teaching whereby *everybody* in the group reaches full understanding.

I would assert that it is all but *impossible* for someone to become a (halfway decent) teacher of *anything* without learning along the way that the absolute best way to learn *any* set of material deeply is to *teach* it – it is the very foundation of Academe and has been for two or three thousand years. It is, as we have noted, built right into the intensive learning process of medical school and graduate school in general. For some reason, however, we don't incorporate a teaching component in most *undergraduate* classes, which is a shame, and it is basically nonexistent in nearly all K-12 schools, which is an open tragedy.

As an engaged student *you don't have to live with that!* Put it there yourself, by incorporating group study and mutual teaching into your learning process *with or without the help or permission of your teachers!* A really smart and effective group soon learns to *iterate* the teaching – I teach you, and to make sure you got it you *immediately* use the material I taught you and try to articulate it back to me. Eventually everybody in the group understands, everybody in the group benefits, *everybody in the group gets the best possible grade on the material.* This process will actually make you (quite literally) more intelligent. You may or may not become smart enough to lock down an A, but you will get the best grade you are capable of getting, for your given investment of effort.

This is close to the ultimate in engagement – highly active learning, with all cylinders of your brain firing away on the process. You can *see* why learning is enhanced. It is simply a bonus, a sign of a just and caring God, that it is also a lot more *fun* to work in a group, especially in a relaxed context with food and drink present. Yes, I'm encouraging you to have "physics study parties" (or history study parties, or psychology study parties). Hold contests. Give silly prizes. See. Do. Teach.

Other Conditions for Learning

Learning isn't *only* dependent on the engagement pattern implicit in the See, Do, Teach rule. Let's absorb a few more True Facts about learning, in particular let's come up with a handful of things that can act as "switches" and turn your ability to learn on and off quite independent of how your instructor structures your courses. Most of these things aren't *binary* switches – they are more like dimmer switches that can be slid up between dim (but not off) and bright (but not fully on). Some of these switches, or environmental parameters, act together more powerfully than they act alone. We'll start with the most important pair, a pair that research has shown work together to potentiate or block learning.

Instead of just telling you what they are, arguing that they are important for a paragraph or six, and moving on, I'm going to give you an early opportunity to *practice* active learning in the context of reading a chapter on active learning. That is, I want you to participate in a tiny mini-experiment. It works a little bit better if it is done verbally in a one-on-one meeting, but it should still work well enough even if it is done in this text that you are reading.

I going to give you a string of ten or so digits and ask you to glance at it one time for a count of three and then look away. No fair peeking once your three seconds are up! Then I want you to do something else for at least a minute – anything else that uses your whole attention and interrupts your ability to rehearse the numbers in your mind in the way that you've doubtless learned permits you to learn other strings of digits, such as holding your mind blank, thinking of the phone numbers of friends or your social security number. Even rereading this paragraph will do.

At the end of the minute, try to recall the number I gave you and write down what you remember. Then turn back to right here and compare what you wrote down with the actual number.

Ready? (No peeking yet...) Set? Go!

Ok, here it is, in a footnote at the bottom of the page to keep your eye from naturally reading ahead to catch a glimpse of it while reading the instructions above[3].

How did you do?

If you are like most people, this string of numbers is a bit too long to get into your immediate memory or visual memory in only three seconds. There was very little time for rehearsal, and then you went and did something else for a bit right away that was supposed to *keep* you from rehearsing whatever of the string you *did* manage to verbalize in three seconds. Most people will get anywhere from the first three to as many as seven or eight of the digits right, but probably not in the correct order, unless...

...they are particularly smart or lucky and in that brief three second glance have time to notice that the number consists of all the digits used exactly once! Folks that happened to "see" this at a glance probably did better than average, getting all of the correct digits but maybe in not quite the correct order.

People who are downright *brilliant* (and equally lucky) realized in only three seconds (without cheating an extra second or three, you know who you are) that it consisted of the string of odd digits in ascending order followed by the even digits in descending order. Those people probably got it *all perfectly right* even without time to rehearse and "memorize" the string! Look again at the string, see the pattern now?

The moral of this little mini-demonstration is that it is *easy* to overwhelm the mind's

[3]1357986420 (one, two, three, quit and do something else for one minute...)

capacity for processing and remembering "meaningless" or "random" information. A string of ten measely (apparently) random digits is too much to remember for one lousy minute, especially if you aren't given time to do rehearsal and all of the other things we have to make ourselves do to "memorize" meaningless information.

Of course things *changed radically* the instant I pointed out the pattern! At this point you could very likely go away and come back to this point in the text *tomorrow* or even *a year from now* and have an *excellent* chance of remembering this particular digit string, because it *makes sense* of a sort, and there are plenty of cues in the text to trigger recall of the particular pattern that "compresses and encodes" the actual string. You don't have to remember *ten* random things at all – only two and a half – odd ascending digits followed by the opposite (of both). Patterns rock!

This example has obvious connections to lecture and class time, and is one reason retention from lecture is so lousy. For *most* students, lecture in any nontrivial college-level course is a long-running litany of stuff they don't know yet. Since it is all new to them, it might as well be random digits as far as their cognitive abilities are concerned, at least at first. Sure, there is pattern there, but you have to *discover* the pattern, which requires *time* and a certain amount of *meditation* on all of the information. Basically, you have to have a chance for the pattern to jump out of the stream of information and punch the switch of the damn light bulb we all carry around inside our heads, the one that is endlessly portrayed in cartoons. That light bulb is *real* – it actually exists, in more than just a metaphorical sense – and if you study long enough and hard enough to obtain a sudden, epiphinaic realization in any topic you are studying, however trivial or complex (like the pattern exposed above) it is quite likely to be accompanied by a purely mental flash of "light". You'll know it when it happens to you, in other words, and it feels *great*.

Unfortunately, the instructor doesn't usually give students a *chance* to experience this in lecture. No sooner is one seemingly random factoid laid out on the table than along comes a new, apparently disconnected one that pushes it out of place long before we can either memorize it the hard way or make sense out of it so we can remember it with a lot less work. This isn't really anybody's fault, of course; the light bulb is quite unlikely to go off in lecture *just* from lecture no matter *what* you or the lecturer do – it is something that happens to the prepared mind at the end of a process, not something that just fires away every time you hear a new idea.

The humble and unsurprising conclusion I want you to draw from this silly little mini-experiment is that *things are easier to learn when they make sense!* A *lot* easier. In fact, things that don't make sense to you are never "learned" – they are at best memorized. Information can almost always be *compressed* when you discover the patterns that run through it, especially when the patterns all fit together into the marvelously complex and beautiful and mysterious process we call "deep understanding" of some subject.

There is one more example I like to use to illustrate how important this information compression is to memory and intelligence. I play chess, badly. That is, I know the legal moves of the game, and have no idea at all how to use them effectively to improve my position and eventually win. Ten moves into a typical chess game I can't recall how I got myself into the mess I'm typically in, and at the end of the game I probably can't remember *any* of what went on except that I got trounced, again.

A chess *master*, on the other hand, can play umpty games at once, blindfolded, against pitiful fools like myself and when they've finished winning them all they can go back and recontruct *each one* move by move, criticizing each move as they go. Often they can remember the games in their entirety days or even years later.

This isn't just because they are *smarter* – *they* might be completely unable to derive the Lorentz group from first principles, and I can, and this doesn't automatically make me smarter than them either. It is because chess makes *sense* to them – they've achieved a deep understanding of the game, as it were – and they've built a complex meta-structure memory in their brains into which they can poke chess moves so that they can be retrieved extremely efficiently. This gives them the *attendant* capability of searching vast portions of the game tree at a glance, where I have to tediously work through each branch, one step at a time, usually omitting some really important possibility because I don't realize that that knight on the far side of the board can affect things on this side where we are both moving pieces.

This sort of "deep" (synthetic) understanding of physics is very much the goal of *this* course (the one in the textbook you are reading, since I use this intro in many textbooks), and to achieve it you must *not* memorize things as if they are random factoids, you must work to abstract the beautiful intertwining of patterns that compress all of those apparently random factoids into things that you can easily remember offhand, that you can easily reconstruct from the pattern even if you forget the details, and that you can search through at a glance. But the process I describe can be applied to learning pretty much anything, as patterns and structure exist in abundance in *all* subjects of interest. There are even sensible rules that govern or describe the anti-pattern of *pure randomness!*

There's one more important thing you can learn from thinking over the digit experiment. *Some* of you reading this very likely didn't do what I asked, you didn't play along with the game. Perhaps it was too much of a bother – you didn't want to waste a *whole minute* learning something by actually *doing* it, just wanted to read the damn chapter and get it over with so you could do, well, whatever the hell else it is you were planning to do today that's more important to you than physics or learning in other courses.

If you're one of these people, you probably don't remember *any* of the digit string at this point from actually seeing it – you never even *tried* to memorize it. A very few of you may actually be so terribly jaded that you don't even remember the little mnemonic *formula* I gave above for the digit string (although frankly, people that are *that* disengaged are probably not about to do things like actually read a textbook in the first place, so possibly not). After all, either way the string is pretty damn meaningless, pattern or not.

Pattern and meaning aren't exactly the same thing. There are all sorts of patterns one can find in random number strings, they just aren't "real" (where we could wax poetic at this point about information entropy and randomness and monkeys typing Shakespeare if this were a different course). So why bother wasting brain energy on even the *easy* way to remember this string when doing so is utterly unimportant to you in the grand scheme of all things?

From this we can learn the *second* humble and unsurprising conclusion I want you to draw from this one elementary thought experiment. *Things are easier to learn when you care about learning them!* In fact, they are damn near impossible to learn if you really *don't* care about learning them.

Let's put the two observations together and plot them as a graph, just for fun (and because graphs help one learn for reasons we will explore just a bit in a minute). If you care about learning what you are studying, and the information you are trying to learn makes sense (if only for a moment, perhaps during lecture), the chances of your learning it are quite good. This alone isn't *enough* to guarantee that you'll learn it, but it they are basically both necessary conditions, and one of them is directly connected to degree of engagement.

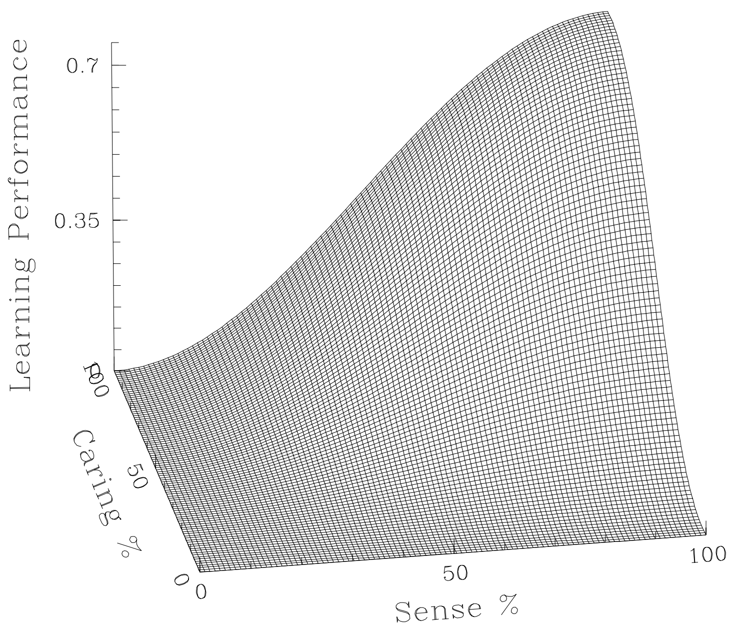

Figure 1: Relation between sense, care and learning

On the other hand, if you care but the information you want to learn makes no sense, or if it makes sense but you hate the subject, the instructor, your school, your life and just don't care, your chances of learning it aren't so good, probably a bit better in the first case than in the second as if you care you have a *chance* of finding someone or some way that will help you make sense of whatever it is you wish to learn, where the person who doesn't cares, well, they don't care. Why should they remember it?

If you don't give a rat's ass about the material *and* it makes no sense to you, go home. Leave school. Do something else. You basically have almost no chance of learning the material unless you are gifted with a transcendent intelligence (wasted on a dilettante who lives in a state of perpetual ennui) and are miraculously gifted with the ability learn things effortlessly even when they make no sense to you and you don't really care about them. All the learning tricks and study patterns in the world won't help a student who doesn't try, doesn't care, and for whom the material never makes sense.

If we worked at it, we could probably find other "logistic" controlling parameters to associate with learning – things that increase your probability of learning monotonically as they vary. Some of them are already apparent from the discussion above. Let's list a few more of them with explanations just so that you can see how *easy* it is to sit down to study and try to learn and have "something wrong" that decreases your ability to learn in that particular place and time.

Learning is actual work and involves a fair bit of biological stress, just like working out. Your brain needs *food* – it burns a whopping *20-30% of your daily calorie intake all by itself* just living day to day, even more when you are really using it or are somewhat sedentary in your physical habits. Note that your brain runs on pure, energy-rich glucose, so when your blood sugar drops your brain activity drops right along with it. This can happen (paradoxically) because you *just ate a carbohydrate rich meal*. A balanced diet containing foods with a lower glycemic index[4] tends to be harder to digest and provides a longer period of sustained energy for your brain. A daily multivitamin (and various antioxidant supplements such as alpha lipoic acid) can also help maintain your body's energy release mechanisms at the cellular level.

Blood sugar is typically lowest first thing in the morning, so this is a lousy time to actively study. On the other hand, a good hearty breakfast, eaten at least an hour before plunging in to your studies, is a great idea and is a far better habit to develop for a lifetime than eating no breakfast and instead eating a huge meal right before bed.

Learning requires adequate *sleep*. Sure this is tough to manage at college – there are no parents to tell you to go to bed, lots of things to do, and of course you're in *class* during the day and then you study, so late night is when you have fun. Unfortunately, learning is clearly correlated with engagement, activity, and mental alertness, and all of these tend to shut down when you're tired. Furthermore, the formation of *long term memory of any kind* from a day's experiences has been shown in both animal and human studies to *depend* on the brain undergoing at least a few natural sleep cycles of deep sleep alternating with REM (Rapid Eye Movement) sleep, dreaming sleep. Rats taught a maze and then deprived of REM sleep cannot run the maze well the next day; rats that are taught the *same* maze but that get a good night's of rat sleep with plenty of rat dreaming

[4]Wikipedia: http://www.wikipedia.org/wiki/glycemic_index.

can run the maze well the next day. People conked on the head who remain unconscious for hours and are thereby deprived of normal sleep often have permanent amnesia of the previous day – it never gets turned into long term memory.

This is hardly surprising. Pure common sense and experience tell you that your brain won't work too well if it is hungry and tired. Common sense (and yes, experience) will rapidly convince you that learning generally works better if you're not stoned or drunk when you study. Learning works *much* better when you have *time* to learn and haven't put everything off to the last minute. In fact, all of Maslow's hierarchy of needs[5] are important parameters that contribute to the probability of success in learning.

There is one more set of very important variables that strongly affect our ability to learn, and they are in some ways the least well understood. These are variables that describe you as an *individual*, that describe your *particular* brain and how it works. Pretty much everybody will learn better if they are self-actualized and fully and actively engaged, if the material they are trying to learn is available in a form that makes sense and clearly communicates the implicit patterns that enable efficient information compression and storage, and above all if they *care* about what they are studying and learning, if it has *value* to them.

But everybody is not the same, and the *optimal* learning strategy for one person is not going to be what works well, or even at all, for another. This is one of the things that confounds "simple" empirical research that attempts to find benefit in one teaching/learning methodology over another. Some students *do* improve, even dramatically improve – when this or that teaching/learning methodology is introduced. In others there is no change. Still others actually do worse. In the end, the beneficial effect to a selected subgroup of the students may be lost in the statistical noise of the study and the fact that no attempt is made to identify commonalities among students that succeed or fail.

The point is that finding an optimal teaching and learning strategy is *technically* an *optimization problem on a high dimensional space.* We've discussed *some* of the important dimensions above, isolating a few that appear to have a monotonic effect on the desired outcome in at least some range (relying on common sense to cut off that range or suggest trade-offs – one cannot learn better by simply discussing one idea for weeks at the expense of participating in lecture or discussing many other ideas of equal and coordinated importance; sleeping for twenty hours a day leaves little time for experience to fix into long term memory with all of that sleep). We've omitted one that is crucial, however. That is *your brain!*

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[5]Wikipedia: http://www.wikipedia.org/wiki/Maslow's_hierarchy_of_needs. In a nutshell, in order to become *self-actualized* and realize your full potential in activities such as learning you need to have your physiological needs met, you need to be safe, you need to be loved and secure in the world, you need to have good self-esteem and the esteem of others. Only then is it particularly likely that you can become self-actualized and become a great learner and problem solver.

Your Brain and Learning

Your brain is more than just a unique instrument. In some sense it is you. You could imagine having your brain removed from your body and being hooked up to machinary that provided it with sight, sound, and touch in such a way that "you" remain[6]. It is difficult to imagine that you still exist in any meaningful sense if your brain is taken out of your body and destroyed while your body is artificially kept alive.

Your brain, however, *is* an instrument. It has internal structure. It uses energy. It does "work". It is, in fact, a biological machine of sublime complexity and subtlety, one of the true wonders of the world! Note that this statement can be made quite independent of whether "you" are your brain per se or a spiritual being who happens to be using it (a debate that need not concern us at this time, however much fun it might be to get into it) – either way the brain itself is quite marvelous.

For all of that, few indeed are the people who bother to learn to actually *use* their brain effectively *as* an instrument. It just works, after all, whether or not we do this. Which is fine. If you want to get the most mileage out of it, however, it helps to read the manual.

So here's at least *one* user manual for your brain. It is by no means complete or authoritative, but it should be enough to get you started, to help you discover that you are actually a lot smarter than you think, or that you've been in the past, once you realize that you can *change* the way you think and learn and experience life and gradually *improve* it.

In the spirit of the learning methodology that we eventually hope to adopt, let's simply itemize in no particular order the various features of the brain[7] that bear on the process of learning. Bear in mind that such a minimal presentation is more of a *metaphor* than anything else because simple (and extremely common) generalizations such as "creativity is a right-brain function" are not strictly true as the brain is far more complex than that.

- The brain is *bicameral*: it has two *cerebral hemispheres*[8] , right and left, with brain functions *asymmetrically* split up between them.

- The brain's hemispheres are connected by a networked membrane called the *corpus callosum* that is how the two halves talk to each other.

- The human brain consists of *layers* with a structure that recapitulates evolutionary phylogeny; that is, the core structures are found in very primitive animals and common to nearly all vertebrate animals, with new layers (apparently) added by evolution on top of this core as the various phyla differentiated, fish, amphibian, reptile, mammal, primate, human. The outermost layer where most actual thinking occurs (in animals that think) is known as the *cerebral cortex*.

[6]Imagine very easily if you've ever seen *The Matrix* movie trilogy...
[7]Wikipedia: http://www.wikipedia.org/wiki/brain.
[8]Wikipedia: http://www.wikipedia.org/wiki/cerebral_hemisphere.

- The *cerebral cortex*[9] – especially the outermost layer of *it* called the *neocortex* – is where "higher thought" activities associated with learning and problem solving take place, although the brain is a very complex instrument with functions spread out over many regions.

- An important brain model is a *neural network*[10] . Computer simulated neural networks provide us with insight into how the brain can remember past events and process new information.

- The fundamental operational units of the brain's information processing functionality are called *neurons*[11] . Neurons receive electrochemical signals from other neurons that are transmitted through long fibers called *axons*[12] *Neurotransmitters*[13] are the actual chemicals responsible for the triggered functioning of neurons and hence the neural network in the cortex that spans the halves of the brain.

- Parts of the cortex are devoted to the senses. These parts often contain a *map* of sorts of the world as seen by the associated sense mechanism. For example, there exists a topographic map in the brain that roughly corresponds to points in the retina, which in turn are stimulated by an image of the outside world that is projected onto the retina by your eye's lens in a way we will learn about later in this course! There is thus a *representation of your visual field* laid out inside your brain!

- Similar maps exist for the other senses, although sensations from the right side of your body are generally processed in a laterally inverted way by the *opposite* hemisphere of the brain. What your right eye sees, what your right hand touches, is ultimately transmitted to a sensory area in your left brain hemisphere and vice versa, and volitional muscle control flows from these brain halves the other way.

- Neurotransmitters require biological resources to produce and consume bioenergy (provided as glucose) in their operation. You can *exhaust* the resources, and *saturate* the receptors for the various neurotransmitters on the neurons by overstimulation.

- You can also block neurotransmitters by chemical means, put neurotransmitter analogues into your system, and alter the chemical trigger potentials of your neurons by taking various drugs, poisons, or hormones. The *biochemistry of your brain* is extremely important to its function, and (unfortunately) is not infrequently a bit "out of whack" for many individuals, resulting in e.g. attention deficit or mood disorders that can greatly affect one's ability to easily learn while leaving one otherwise highly functional.

- Intelligence[14] , learning ability, and problem solving capabilities are not fixed; they can vary (often improving) over your whole lifetime! Your brain is highly *plastic* and

[9] Wikipedia: http://www.wikipedia.org/wiki/Cerebral_cortex.
[10] Wikipedia: http://www.wikipedia.org/wiki/Neural_network.
[11] Wikipedia: http://www.wikipedia.org/wiki/Neurons.
[12] Wikipedia: http://www.wikipedia.org/wiki/axon. .
[13] Wikipedia: http://www.wikipedia.org/wiki/neurotransmitters.
[14] Wikipedia: http://www.wikipedia.org/wiki/intelligence.

can sometimes even reprogram itself to full functionality when it is e.g. damaged by a stroke or accident. On the other hand neither is it infinitely plastic – any given brain has a range of accessible capabilities and can be improved only to a certain point. However, for people of supposedly "normal" intelligence and above, it is by no means clear what that point is! Note well that *intelligence is an extremely controversial subject* and you should not take things like your own measured "IQ" too seriously.

- Intelligence is not even fixed within a population over time. A phenomenon known as "the Flynn effect"[15] (after its discoverer) suggests that IQ tests have increased almost six points a decade, on average, over a timescale of tens of years, with most of the increases coming from the lower half of the distribution of intelligence. This is an active area of research (as one might well imagine) and some of that research has demonstrated fairly conclusively that individual intelligences can be improved by five to ten points (a significant amount) by environmentally correlated factors such as nutrition, education, complexity of environment.

- The best time for the brain to learn is right before sleep. The process of sleep appears to "fix" long term memories in the brain and things one studies right before going to bed are retained much better than things studied first thing in the morning. Note that this conflicts directly with the party/entertainment schedule of many students, who tend to study early in the evening and then amuse themselves until bedtime. It works much better the other way around.

- Sensory memory[16] corresponds to the roughly 0.5 second (for most people) that a sensory impression remains in the brain's "active sensory register", the sensory cortex. It can typically hold less than 12 "objects" that can be retrieved. It quickly decays and cannot be improved by rehearsal, although there is some evidence that its object capacity can be improved over a longer term by practice.

- Short term memory is where *some* of the information that comes into sensory memory is transferred. Just which information is transferred depends on where one's "attention" is, and the mechanics of the attention process are not well understood and are an area of active research. Attention acts like a filtering process, as there is a *wealth* of parallel information in our sensory memory at any given instant in time but the thread of our awareness and experience of time is serial. We tend to "pay attention" to one thing at a time. Short term memory lasts from a few seconds to as long as a minute without rehearsal, and for nearly all people it holds $4-5$ objects[17]. However, its capacity can be increased by a process called "chunking" that is basically the information compression mechanism demonstrated in the earlier example with numbers – grouping of the data to be recalled into "objects" that permit a larger set to still fit in short term memory.

[15] Wikipedia: http://www.wikipedia.org/wiki/flynn_effect.

[16] Wikipedia: http://www.wikipedia.org/wiki/memory. Several items in a row are connected to this page.

[17] From this you can see why I used ten digits, gave you only a few seconds to look, and blocked rehearsal in our earlier exercise.

- Studies of chunking show that the ideal size for data chunking is three. That is, if you try to remember the string of letters:

  FBINSACIAIBMATTMSN

  with the usual three second look you'll almost certainly find it impossible. If, however, I insert the following spaces:

  FBI NSA CIA IBM ATT MSN

  It is suddenly much easier to get at least the first four. If I parenthesize:

  (FBI NSA CIA) (IBM ATT MSN)

  so that you can recognize the first three are all government agencies in the general category of "intelligence and law enforcement" and the last three are all market symbols for information technology mega-corporations, you can once again recall the information a day later with only the most cursory of rehearsals. You've taken eighteen "random" objects that were meaningless and could hence be recalled only through the most arduous of rehearsal processes, converted them to six "chunks" of three that can be easily tagged by the brain's existing long term memory (note that you are *not learning* the string FBI, you are building an *association* to the already existing memory of what the string FBI *means*, which is *much easier* for the brain to do), and chunking the chunks into *two* objects.

  Eighteen objects without meaning – difficult indeed! Those *same* eighteen objects *with* meaning – umm, looks pretty easy, doesn't it...

  Short term memory is still that – short term. It typically decays on a time scale that ranges from minutes for nearly everything to order of a day for a few things unless the information can be transferred to *long* term memory. Long term memory is the big payoff – *learning* is associated with formation of long term memory.

- Now we get to the really good stuff. Long term is memory that you form that lasts a long time in human terms. A "long time" can be days, weeks, months, years, or a lifetime. Long term memory is encoded *completely differently* from short term or sensory/immediate memory – it appears to be encoded *semantically*[18] , that is to say, *associatively* in terms of its *meaning*. There is considerable evidence for this, and it is one reason we focus so much on the importance of meaning in the previous sections.

  To miraculously transform things we try to remember from "difficult" to learn random factoids that have to be brute-force stuffed into disconnected semantic storage units created as it were one at a time for the task at hand into "easy" to learn factoids, all we have to do is *discover* meaning associations with things we already know, or *create* a strong memory of the global meaning or *conceptualization* of a subject that serves as an associative home for all those little factoids.

  A characteristic of this as a successful process is that when one works systematically to learn by means of the latter process, learning gets *easier* as time goes on. Every factoid you add to the semantic structure of the global conceptualization strengthens

[18] Wikipedia: http://www.wikipedia.org/wiki/semantics.

it, and makes it even easier to add new factoids. In fact, the mind's extraordinary rational capacity permits it to interpolate and extrapolate, to *fill in* parts of the structure on its own *without effort* and in many cases without even being exposed to the information that needs to be "learned"!

- One area where this extrapolation is particularly evident and powerful is in *mathematics*. Any time we can learn, or discover from experience a *formula* for some phenomenon, a mathematical *pattern*, we don't have to actually see something to be able to "remember" it. Once again, it is easy to find examples. If I give you data from sales figures over a year such as January = $1000, October = $10,000, December = $12,000, March=$3000, May = $5000, February = $2000, September = $9000, June = $6000, November = $11,000, July = $7000, August = $8000, April = $4000, at first glance they look quite difficult to remember. If you organize them temporally by month and look at them for a moment, you recognize that sales increased *linearly* by month, starting at $1000 in January, and suddenly you can reduce the whole series to a simple mental formula (straight line) and a couple pieces of initial data (slope and starting point). One amazing thing about this is that if I asked you to "remember" something that you *have not seen*, such as sales in February in the *next* year, you could make a very plausible guess that they will be $14,000!

  Note that this isn't a memory, it is a guess. Guessing is what the mind is designed to do, as it is part of the process by which it "predicts the future" even in the most mundane of ways. When I put ten dollars in my pocket and reach in my pocket for it later, I'm basically guessing, on the basis of my memory and experience, that I'll find ten dollars there. Maybe my guess is wrong – my pocket could have been picked[19], maybe it fell out through a hole. My *concept* of object permanence plus my *memory* of an initial state permit me to make a *predictive guess* about the Universe!

  This is, in fact, physics! This is what physics is all about – coming up with a set of rules (like conservation of matter) that encode observations of object permanence, more rules (equations of motion) that dictate how objects move around, and allow me to conclude that "I put a ten dollar bill, at rest, into my pocket, and objects at rest remain at rest. The matter the bill is made of cannot be created or destroyed and is bound together in a way that is unlikely to come apart over a period of days. Therefore the ten dollar bill is still there!" Nearly anything that you do or that happens in your everyday life can be formulated as a predictive physics problem.

- The *hippocampus*[20] appears to be partly responsible for both forming spatial maps or visualizations of your environment and also for forming the *cognitive map* that organizes what you know and transforms short term memory into long term memory, and it appears to do its job (as noted above) *in your sleep*. Sleep deprivation *prevents the formation of long term memory*. Being rendered unconscious for a long

[19] With three sons constantly looking for funds to attend movies and the like, it isn't as unlikely as you might think!

[20] Wikipedia: http://www.wikipedia.org/wiki/hippocampus.

period often produces *short term amnesia* as the brain loses short term memory before it gets put into long term memory. The hippocampus shows evidence of plasticity – taxi drivers who have to learn to navigate large cities actually have larger than normal hippocampi, with a size proportional to the length of time they've been driving. This suggests (once again) that it is possible to *deliberately increase the capacity* of your *own* hippocampus through the exercise of its functions, and consequently *increase your ability to store and retrieve information*, which is an important component (although not the only component) of intelligence!

- Memory is improved by *increasing the supply of oxygen to the brain*, which is best accomplished by *exercise*. Unsurprisingly. Indeed, as noted above, having good general health, good nutrition, good oxygenation and perfusion – having all the biomechanism in tip-top running order – is perfectly reasonably linked to being able to perform at your best in anything, mental activity included.

- Finally, the *amygdala*[21] is a brain organ in our *limbic system* (part of our "old", reptile brain). The amygdala is an important part of our *emotional* system. It is associated with primitive survival responses, with sexual response, and appears to play a *key role* in modulating (filtering) the process of turning short term memory into long term memory. Basically, any sort term memory associated with a powerful emotion is much more likely to make it into long term memory.

  There are clear evolutionary advantages to this. If you narrowly escape being killed by a saber-toothed tiger at a particular pool in the forest, and then forget that this happened by the next day and return again to drink there, chances are decent that the saber-tooth is still there and you'll get eaten. On the other hand, if you come upon a particular fruit tree in that same forest and get a free meal of high quality food and forget about the tree a day later, you might starve.

  We see that both negative and positive emotional experiences are strongly correlated with learning! *Powerful* experiences, especially, are correlated with learning. This translates into learning strategies in two ways, one for the instructor and one for the student. For the instructor, there are two general strategies open to helping students learn. One is to create an atmosphere of *fear, hatred, disgust, anger* – powerful negative emotions. The other is to create an atmosphere of *love, security, humor, joy* – powerful positive emotions. In between there is a great wasteland of bo-ring, bo-ring, bo-ring where students plod along, struggling to form memories because there is nothing "exciting" about the course in either a positive or negative way and so their amygdala degrades the memory formation process in favor of other more "interesting" experiences.

  Now, in my opinion, negative experiences in the classroom do indeed promote the formation of long term memories, but they aren't the memories the instructor intended. The student is likely to remember, and loath, the instructor for the rest of their life but is *not* more likely to remember the material except sporadically in association with particularly traumatic episodes. They may well be *less* likely, as

[21] Wikipedia: http://www.wikipedia.org/wiki/amygdala.

we naturally avoid negative experiences and will study less and work less hard on things we can't stand doing.

For the instructor, then, positive is the way to go. Creating a warm, nurturing classroom environment, ensuring that the students know that you *care* about their learning and about them as individuals helps to promote learning. Making your lectures and teaching processes *fun* – and *funny* – helps as well. Many successful lecturers make a powerful *positive* impression on the students, creating an atmosphere of amazement or surprise. A classroom experience should really be a *joy* in order to optimize learning in so many ways.

For the student, be aware that *your attitude matters!* As noted in previous sections, *caring* is an essential component of successful learning because you have to attach *value* to the process in order to get your amygdala to do its job. However, you can do *much more*. You can see how *many* aspects of learning can be enhanced through the simple expedient of making it a positive experience! Working in groups is *fun*, and you learn more when you're having fun (or quavering in abject fear, or in an interesting mix of the two). Attending an interesting lecture is fun, and you'll retain more than average. Participation is fun, especially if you are "rewarded" in some way that makes a moment or two special to you, and you'll remember more of what goes on.

From all of these little factoids (presented in a way that I'm hoping helps you to build at least the beginnings of a working conceptual model of your own brain) I'm hoping that you are coming to realize that *all of this is at least partially under your control!* Even if your instructor is scary or boring, the material at first glance seems dry and meaningless, and so on – all the negative-neutral things that make learning difficult, *you* can decide to make it fun and exciting, *you* can ferret out the meaning, *you* can adopt study strategies that focus on the formation of cognitive maps and organizing structures *first* and *then* on applications, rehearsal, factoids, and so on, *you* can learn to study right before bed, get enough sleep, become aware of your brain's learning biorhythms.

Finally, you can learn to *increase your functional learning capabilities* by a *significant* amount. Solving puzzles, playing mental games, doing crossword puzzles or sudoku, working homework problems, writing papers, arguing and discussing, just plain *thinking* about difficult subjects and problems even when you don't *have* to all increase your active intelligence in initially small but cumulative ways. You too can increase the size of your hippocampus, learn to engage your amygdala by *choosing* in a self-actualized way what you value and learning to discipline your emotions accordingly, and create more conceptual maps within your brain that can be shared as components across the various things you wish to learn. The more you know about *anything*, the easier it is to learn *everything* – this is the pure biology underlying the value of the liberal arts education.

Use your whole brain, exercise it often, don't think that you "just" need math and not spatial relations, visualization, verbal skills, a knowledge of history, a memory of performing experiments with your hands or mind or both – you need it all! Remember, just as is the case with physical exercise (which you should get plenty of), *mental* exercise gradually makes you mentally stronger, so that you can eventually do easily things that at first appear insurmountably difficult. You can learn to learn *three to ten times as fast* as you did in high school, to have more fun while doing it, and to gain tremendous reasoning capabilities along the way just by *trying* to learn to learn more efficiently instead of continuing to use learning strategies that worked (possibly indifferently) back in elementary and high school.

The next section, at long last, will make a very specific set of suggestions for *one* very good way to study physics (or nearly anything else) in a way that maximally takes advantage of your own volitional biology to make learning as efficient and pleasant as it is possible to be.

How to Do Your Homework Effectively

By now in your academic career (and given the information above) it should be very apparent just where homework exists in the grand scheme of (learning) things. Ideally, you attend a class where a warm and attentive professor clearly explains some abstruse concept and a whole raft of facts in some moderately interactive way that encourages engagement and "being earnest". Alas, there are *too many* facts to fit in short term/immediate memory and *too little time* to move most of them through into long term/working memory before finishing with one and moving on to the next one. The material may appear to be boring and random so that it is difficult to pay full attention to the *patterns* being communicated and remain emotionally enthusiastic all the while to help the process along. As a consequence, by the end of lecture you've already *forgotten* many if not most of the facts, but if you were paying attention, asked questions as needed, and really cared about learning the material you *would* remember a handful of the most important ones, the ones that made your brief understanding of the material hang (for a brief shining moment) together.

This conceptual overview, however initially tenuous, is the skeleton you will eventually clothe with facts and experiences to transform it into an entire system of associative memory and reasoning where you can work intellectually at a high level with little effort and usually with a great deal of pleasure associated with the very act of thinking. But you aren't there yet.

You now know that you are not terribly likely to retain a lot of what you are shown in lecture without engagement. In order to actually learn it, you must *stop* being a passive recipient of facts. You must *actively* develop your understanding, by means of *discussing* the material and kicking it around with others, by *using* the material in some way, by *teaching* the material to peers as you come to understand it.

To help facilitate this process, associated with lecture your professor almost certainly gave you an *assignment*. Amazingly enough, its purpose is not to torment you or to be the basis of your grade (although it may well do both). It is to give you some concrete stuff to *do* while thinking about the material to be learned, while discussing the material to be learned, while using the material to be learned to accomplish specific goals, while teaching some of what you figure out to others who are sharing this whole experience while being taught by them in turn. The assignment is *much more important* than lecture, as it is entirely participatory, where real learning is *far more likely to occur.* You could, once you learn the trick of it, blow off lecture and do fine in a course in all other respects. If you fail to do the assignments *with your entire spirit engaged*, you are doomed.

In other words, to learn you must *do your homework*, ideally at least partly in a *group* setting. The only question is: *how* should you do it to both finish learning all that stuff you sort-of-got in lecture and to re-attain the moment(s) of clarity that you then experienced, until eventually it becomes a permanent characteristic of your awareness and you *know* and *fully understand* it all on your own?

There are two general steps that need to be *iterated* to finish learning anything at all. They are a lot of work. In fact, they are far *more* work than (passively) attending lecture, and are *more important* than attending lecture. You can learn the material with these steps without *ever* attending lecture, as long as you have access to what you need to learn in some media or human form. You in all probability will *never* learn it, lecture or not, without making a few passes through these steps. They are:

a) Review the whole (typically textbooks and/or notes)

b) Work on the parts (do homework, use it for something)

(iterate until you thoroughly understand whatever it is you are trying to learn).

Let's examine these steps.

The first is pretty obvious. You didn't "get it" from one lecture. There was too much material. If you were *lucky* and well prepared and blessed with a good instructor, perhaps you grasped *some* of it for a *moment* (and if your instructor was poor or you were particularly poorly prepared you may not have managed even that) but what you did momentarily understand is fading, flitting further and further away with every moment that passes. You need to review the entire topic, as a whole, as well as all its parts. A set of good summary notes might contain all the relative factoids, but there are *relations* between those factoids – a temporal sequencing, mathematical derivations connecting them to other things you know, a topical association with other things that you know. They tell a *story*, or part of a story, and you need to know that story in *broad* terms, not try to memorize it word for word.

Reviewing the material should be done in layers, skimming the textbook and your notes, creating a *new* set of notes out of the text in combination with your lecture notes, maybe reading in more detail to understand some particular point that puzzles you, reworking a few of the examples presented. Lots of increasingly deep passes through it (starting with the merest skim-reading or reading a summary of the whole thing) are *much* better than trying to work through the whole text one line at a time and not moving on until you understand it. Many things you might want to understand will only come clear from things you are exposed to *later*, as it is not the case that all knowledge is ordinal, hierarchical, and derivatory.

You especially do *not* have to work on *memorizing* the content. In fact, it is *not* desireable to try to memorize content at this point – you want the big picture *first* so that facts have a place to live in your brain. If you build them a house, they'll move right in without a fuss, where if you try to grasp them one at a time with no place to put them, they'll (metaphorically) slip away again as fast as you try to take up the next one. Let's understand this a bit.

As we've seen, your brain is fabulously efficient at storing information in a *compressed associative* form. It also tends to remember things that are *important* – whatever that means – and forget things that aren't important to make room for more important stuff, as your brain structures work together in understandable ways on the process. Building the cognitive map, the "house", is what it's all about. But as it turns out, building this house *takes time.*

This is the goal of your iterated review process. At first you are memorizing things the hard way, trying to connect what you learn to very simple hierarchical concepts such as this step comes before that step. As you do this over and over again, though, you find that absorbing new information takes you less and less time, and you remember it much more easily and for a longer time without additional rehearsal. Sometimes your brain even *outruns* the learning process and "discovers" a missing part of the structure before you even read about it! By reviewing the whole, well-organized structure over and over again, you gradually build a greatly compressed representation of it in your brain and tremendously reduce the amount of work required to flesh out that structure with increasing levels of detail *and remember them and be able to work with them* for a long, long time.

Now let's understand the second part of doing homework – working problems. As you can probably guess on your own at this point, there are good ways and bad ways to do homework problems. The worst way to do homework (aside from not doing it at all, which is *far too common* a practice and a *bad idea* if you have any intention of learning the material) is to do it all in one sitting, right before it is due, and to never again look at it.

Doing your homework in a single sitting, working on it just one time *fails to repeat and rehearse the material* (essential for turning short term memory into long term in nearly all cases). It *exhausts the neurons in your brain* (quite literally – there is metabolic energy consumed in thinking) as one often ends up working on a problem far too long in one sitting just to get done. It *fails to incrementally build up* in your brain's long term memory the *structures* upon which the more complex solutions are based, so you have to constantly go back to the book to get them into short term memory long enough to get through a problem. Even this simple bit of repetition does *initiate* a learning process. Unfortunately, by not repeating them after this one sitting they soon fade, often without a discernable trace in long term memory.

Just as was the case in our experiment with memorizing the number above, the problems almost invariably are *not* going to be a matter of random noise. They have certain key facts and ideas that are the basis of their solution, and those ideas are used over and over again. There is plenty of pattern and meaning there for your brain to exploit in information compression, and it may well be *very cool stuff to know* and hence *important* to you once learned, but it takes time and repetition and a certain amount of meditation for the "gestalt" of it to spring into your awareness and burn itself into your conceptual memory as "high order understanding".

You have to *give* it this time, and perform the repetitions, while maintaining an optimistic, philosophical attitude towards the process. You have to do your best to have *fun* with it. You don't get strong by lifting light weights a single time. You get strong lifting weights repeatedly, starting with light weights to be sure, but then working up to the *heaviest weights you can manage.* When you *do* build up to where you're lifting hundreds of pounds, the fifty pounds you started with seems light as a feather to you.

As with the body, so with the brain. Repeat broad strokes for the big picture with increasingly deep and "heavy" excursions into the material to explore it in detail as the overall picture emerges. Intersperse this with sessions where you *work on problems* and try to *use* the material you've figured out so far. Be sure to *discuss* it and *teach it to others* as you go as much as possible, as articulating what you've figured out to others both uses a different part of your brain than taking it in (and hence solidifies the memory) and it helps you articulate the ideas to *yourself!* This process will help you learn more, better, faster than you ever have before, and to have fun doing it!

Your brain is more complicated than you think. You are very likely used to *working hard* to try to *make* it figure things out, but you've probably observed that this doesn't work very well. A lot of times you simply *cannot* "figure things out" because your brain doesn't yet know the key things required to do this, or doesn't "see" how those parts you do know fit together. Learning and discovery is not, alas, "intentional" – it is more like trying to get a bird to light on your hand that flits away the moment you try to grasp it.

People who do really hard crossword puzzles (one form of great brain exercise) have learned the following. After making a pass through the puzzle and filling in all the words they can "get", and maybe making a couple of extra passes through thinking hard about ones they can't get right away, looking for patterns, trying partial guesses, they arrive at an impasse. If they continue working hard on it, they are unlikely to make further progress, no matter how long they stare at it.

On the other hand, if they *put the puzzle down* and *do something else for a while* – especially if the something else is go to bed and sleep – when they come back to the puzzle they often can *immediately see* a dozen or more words that the day before were absolutely invisible to them. Sometimes one of the *long theme answers* (perhaps 25 characters long) where they have no more than *two letters* just "gives up" – they can simply "see" what the answer must be.

Where do these answers come from? The person has not "figured them out", they have "recognized" them. They come all at once, and they don't come about as the result of a logical sequential process.

Often they come from the person's *right brain*[22]. The left brain tries to use logic

[22]Note that this description is at least partly metaphor, for while there is some hemispherical specialization of some of these functions, it isn't always sharp. I'm retaining them here (oh you brain specialists

and simple memory when it works on crosswork puzzles. This is usually good for some words, but for many of the words there are *many possible answers* and without any insight one can't even recall *one* of the possibilities. The clues don't suffice to connect you up to a word. Even as letters get filled in this continues to be the case, not because you don't *know* the word (although in really hard puzzles this can sometimes be the case) but because you don't know how to *recognize* the word "all at once" from a cleverly nonlinear clue and a few letters in this context.

The right brain is (to some extent) responsible for *insight* and *non-linear thinking.* It sees *patterns*, and *wholes*, not sequential relations between the parts. It isn't intentional – we can't "make" our right brains figure something out, it is often the other way around! Working hard on a problem, then "sleeping on it" (to get that all important hippocampal involvement going) is actually a *great* way to develop "insight" that lets you solve it *without really working terribly hard* after a few tries. It also utilizes more of your brain – left and right brain, sequential reasoning and insight, and if you articulate it, or use it, or make something with your hands, then it exercieses these parts of your brain as well, strengthening the memory and your understanding still more. The learning that is associated with this process, and the problem solving power of the method, is *much greater* than just working on a problem linearly the night before it is due until you hack your way through it using information assembled a part at a time from the book.

The following "Method of Three Passes" is a *specific* strategy that implements many of the tricks discussed above. It is known to be effective for learning by means of doing homework (or in a generalized way, learning anything at all). It is ideal for "problem oriented homework", and will pay off big in learning dividends should you adopt it, especially when supported by a *group oriented recitation* with *strong tutorial support* and *many opportunities for peer discussion and teaching.*

The Method of Three Passes

Pass 1 Three or more nights before recitation (or when the homework is due), make a *fast* pass through all problems. Plan to spend 1-1.5 hours on this pass. With roughly 10-12 problems, this gives you around 6-8 minutes per problem. Spend *no more* than this much time *per problem* and if you can solve them in this much time fine, otherwise move on to the next. Try to do this the last thing before bed at night (seriously) and *then go to sleep.*

Pass 2 After at least one night's sleep, make a *medium speed* pass through all problems. Plan to spend 1-1.5 hours on this pass as well. Some of the problems will already be solved from the first pass or nearly so. *Quickly* review their solution and then move on to concentrate on the still unsolved problems. If you solved 1/4 to 1/3 of the problems in the first pass, you should be able to spend 10 minutes or so per problem in the second pass. Again, do this right before bed if possible and then go immediately to sleep.

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who might be reading this) because they are a *valuable* metaphor.

Pass 3 After at least one night's sleep, make a *final* pass through all the problems. Begin as before by quickly reviewing all the problems you solved in the previous two passes. Then spend fifteen minutes or more (as needed) to solve the remaining unsolved problems. Leave any "impossible" problems for recitation – there should be no more than three from any given assignment, as a general rule. Go immediately to bed.

This is an *extremely powerful* prescription for deeply learning nearly *anything*. Here is the motivation. Memory is formed by repetition, and this obviously contains a lot of that. Permanent (long term) memory is actually formed in your sleep, and studies have shown that whatever you study right before sleep is most likely to be retained. Physics is actually a "whole brain" subject – it requires a synthesis of both right brain visualization and conceptualization and left brain verbal/analytical processing – both geometry and algebra, if you like, and you'll often find that problems that stumped you the night before just solve themselves "like magic" on the second or third pass if you work hard on them for a short, intense, session and then sleep on it. This is your right (nonverbal) brain participating as it develops intuition to guide your left brain algebraic engine.

Other suggestions to improve learning include working in a study group for that third pass (the first one or two are best done alone to "prepare" for the third pass). Teaching is one of the best ways to learn, and by working in a group you'll have opportunities to both teach and learn more deeply than you would otherwise as you have to articulate your solutions.

Make the learning *fun* – the *right* brain is the key to forming long term memory and it is the seat of your *emotions*. If you are happy studying and make it a positive experience, you will increase retention, it is that simple. Order pizza, play music, make it a "physics homework party night".

Use your whole brain on the problems – draw lots of pictures and figures (right brain) to go with the algebra (left brain). Listen to quiet music (right brain) while thinking through the sequences of events in the problem (left brain). Build little "demos" of problems where possible – even using your hands in this way helps strengthen memory.

Avoid memorization. You will learn physics far better if you learn to *solve* problems and *understand* the concepts rather than attempt to *memorize* the umpty-zillion formulas, factoids, and specific problems or examples covered at one time or another in the class. That isn't to say that you shouldn't learn the important formulas, Laws of Nature, and all of that – it's just that the learning should generally *not* consist of putting them on a big sheet of paper all jumbled together and then trying to memorize them as abstract collections of symbols out of context.

Be sure to review the problems *one last time* when you get your graded homework back. Learn from your mistakes or you will, as they say, be doomed to repeat them.

If you follow this prescription, you will have seen *every assigned homework problem* a minimum of five or six times – three original passes, recitation itself, a final write up pass after recitation, and a review pass when you get it back. At least three of these should occur after you have solved *all* of the problems correctly, since recitation is devoted to ensuring this. When the time comes to study for exams, it should really be (for once) a *review* process, not a cram. Every problem will be like an old friend, and a very brief review will form a *seventh* pass or *eighth* pass through the assigned homework.

With this methodology (enhanced as required by the physics resource rooms, tutors, and help from your instructors) there is no reason for you do poorly in the course and every reason to expect that you will do well, perhaps very well indeed! And you'll still be spending only the 3 to 6 hours per week on homework that is expected of you in any college course of this level of difficulty!

This ends our discussion of course preliminaries (for nearly *any* serious course you might take, not just physics courses) and it is time to get on with the actual material for *this* course.

Mathematics

Physics, as was noted in the preface, requires a solid knowledge of all mathematics through calculus. That's right, the whole nine yards: number theory, algebra, geometry, trigonometry, vectors, differential calculus, integral calculus, even a smattering of differential equations. Somebody may have told you that you can go ahead and take physics having gotten C's in introductory calculus, perhaps in a remedial course that you took because you had such a hard time with precalc or because you failed straight up calculus when you took it.

They lied.

Sorry to be blunt, but that's the simple truth. If you are not competent enough in math *right this minute* to know, or be able to find without help the answers *instantly* to the following selection of questions:

- What are the two values of α that solve $\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$?
- What is $Q(r) = \int_0^r \frac{\rho}{R} r'^3 dr'$?
- What is $\frac{d\cos(\omega t + \delta)}{dt}$?
- What are the x and y components of a vector of length A that makes an angle of θ with the positive x axis (proceeding, as usual, counterclockwise for positive θ)?
- What is the cross product of the two vectors $\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{x}}$ and $\vec{\boldsymbol{F}} = F_y\hat{\boldsymbol{y}}$ (magnitude and direction)?
- What is the inner/dot product of the two vectors $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}}$ and $\vec{\boldsymbol{B}} = B_x\hat{\boldsymbol{y}} + B_y\hat{\boldsymbol{y}}$?

then you are going to have to *add* to the burden of learning physics per se the burden of learning, or re-learning, all of the basic mathematics that would have permitted you to answer them easily. These are not idly selected examples, either; they are all things you will have to in fact do early and ***often*** in this course!

My strong *advice* to you, if you are now feeling the cold icy grip of panic because in fact you are signed up for physics using this book and you couldn't answer *any* of these questions, is to drop the course and go study math until you really master it, *then* come back and take physics. Seriously dude (or dudess). Or resign yourself to a life of misery and enormously hard work just to pass.

Don't go blaming the course, the teacher, this textbook, or anybody but yourself if you proceed unprepared and then fail or suffer – You Have Been Warned.

So, what if you could do *almost* all of these short problems (and can at the very least remember the tools, like the Quadratic Formula, that you were *supposed* to use to solve them if you could remember them? What if you have no choice but to take physics now, and are just going to have to do your best and relearn the math as required along the way? What if you did in fact understand math pretty well once upon a time and are sure it won't be *much* of an obstacle, but you really would like a review, a summary, a listing of the things you need to know someplace handy so you can instantly look them up as you struggle with the problems that uses the math it contains? What if you are (or were) *really good* at math, but want to be able to look at derivations or reread explanations to bring stuff you learned right back to your fingertips once again?

For all of these latter cases, for students of the course in general, I provide the following online (free!) book: ***Mathematics for Introductory Physics***. It is located here:

http://www.phy.duke.edu/~rgb/Class/math_for_intro_physics.php

It is a work in progress, and is quite possibly still somewhat incomplete, but it should help you with a lot of what you are missing, and if you let *me* know what you are missing that you didn't find there, I can work to add it!

I would strongly advise all students of introductory physics (any semester) to visit this site *right now* and bookmark it or download the PDF, and to visit the site from time to time to see if I've posted an update. It is on my back burner, so to speak, until I finish the actual physics texts themselves that I'm working on, but I will still add things to them as motivated by my own needs teaching courses using this series of books.

Summary

That's enough preliminary stuff. At this point, if you've read all of this "week"'s material and vowed to adopt the method of three passes in all of your homework efforts, if you've bookmarked the math help or downloaded it to your personal ebook viewer or computer, if you've realized that your brain is actually something that you can *help and enhance* in various ways as you try to learn things, then my purpose is well-served and you are as well-prepared as you can be to tackle physics.

Homework for Week 0

Problem 1.

Skim read this entire section (Week 0: How to Learn Physics), then read it like a novel, front to back. Think about the connection between engagement and learning and how important it is to try to have *fun* in a physics course. Write a short essay (say, three paragraphs) describing at least one time in the past where you were extremely engaged in a course you were taking, had lots of fun in the class, and had a really great learning experience.

Problem 2.

Skim-read the entire content of ***Mathematics for Introductory Physics*** (linked above). Identify things that it covers that you don't remember or don't understand. Pick one and learn it.

Problem 3.

Apply the *Method of Three Passes* to *this* homework assignment. You can either write three short essays or revise your one essay three times, trying to improve it and enhance it each time for the first problem, and review both the original topic and any additional topics you don't remember in the math review problem. On the *last* pass, write a short (two paragraph) essay on whether or not you found multiple passes to be effective in helping you remember the content.

Note well: You may well have found the content *boring* on the third pass because it was so familiar to you, but that's not a bad thing. If you learn physics so thoroughly that its laws become *boring*, not because they confuse you and you'd rather play World of Warcraft but because you know them so well that reviewing them isn't adding anything to your understanding, well *damn* you'll do well on the exams testing the concept, won't you?

II: Elementary Mechanics

The chapters in the first part, as one can easily see, are not actual physics. The first is a *very important* review of *how to efficiently learn* physics (or anything else), intended for you (the student) to *read like a novel, one time*, at the very beginning of the course and then refer back to as needed.

The second is a general review of pretty much all of the actual mathematics required for the course – I *assume* that if you're taking physics in college *or* high school you've have had algebra, geometry, trigonometry, and differential and integral calculus (all *required prerequisites* for any sensible calculus-based physics course) but (being no fool) I *also* assume that most students, including ones that got A's in those classes and/or high scores on the relevant advanced placement tests have *forgotten* a lot of it, at least to the point where they are very shaky when it comes to knowing what things mean, how to derive them, or how to apply them to even quite simple problems where the variables all *means* something. If nothing else, putting all the math you might need here in one impossible-to-miss place means you don't have to keep four or five math books handy to get through the problems if you *do* forget (or never learned) something that turns out to be important.

We now depart this "general review" layout (which is obviously not intended to be lectured on, although I usually review the content of the *Preliminaries* on the first day of class at the same time I review the syllabus and set down the class rules and grading scheme that I will use) and embark upon the actual week by week, day by day progress through the course material. For maximal ease of use for you the student and (one hopes) your instructor, the course is designed to cover *one chapter per week-equivalent*, whether or not the chapter is broken up into a day and a half of lecture (summer school), an hour a day (MWF), or an hour and a half a day (TTh) in a semester based scheme. To emphasize this preferred rhythm, each chapter will be referred to by the *week* it would normally be covered in my own semester-long course.

A week's work in all cases covers just about exactly one "topic" in the course. A very few are spread out over two weeks; one or two compress two related topics into one week, but in all cases the *homework* is assigned on a weekly rhythm to give you ample opportunity to use the *method of three passes* described in the first part of the book, culminating in an expected 2-3 hour *recitation* where you should go over the assigned homework *in a group* of three to six students, with a mentor handy to help you where you get stuck, with a goal of *getting all of the homework perfectly correct by the end of recitation*. That is, at the end of a week plus its recitation, you *should* be able to do *all* of the week's homework, *perfectly*, and *without looking*. You will usually *need* all three passes, the last one working in a group, *plus* the mentored recitation to achieve this degree of competence!

However, *if* you do this, you are almost certain to do well on a quiz that terminates the recitation period, and you will be very likely to *retain* the material and not have to "cram" it in again for the hour exams and/or final exam later in the course. Once you achieve *understanding* and reinforce it with a fair bit of repetition and practice, most students will naturally transform this experience into remarkably deep and permanent learning.

Note well that each week is organized for *maximal ease of learning* with the week/chapter review *first*. Try to *always look at this review before lecture* even if you skip reading the chapter itself until later, when you start your homework. Skimming the whole thing before lecture is, of course, better still. It is a "first pass" that can often make lecture much easier to follow and help free you from the tyranny of note-taking as you only need to note *differences* in the presentation from this text and perhaps the answers to *questions* that helped you understand something during the discussion. Then read or skim it again right before each homework pass.

Week 1: Newton's Laws

Dynamics Summary

- Physics is a *language* – in particular the language of a certain kind of *worldview*. For philosophically inclined students who wish to read more deeply on this, I include links to terms that provide background for this point of view.
  - Wikipedia: http://www.wikipedia.org/wiki/Worldview
  - Wikipedia: http://www.wikipedia.org/wiki/Semantics
  - Wikipedia: http://www.wikipedia.org/wiki/Ontology

  Mathematics is the natural language and logical language of physics, not for any particularly deep reason but because it *works*. The components of the semantic language of physics are thus generally expressed as mathematical or logical *coordinates*, and the semantic expressions themselves are generally mathematical/algebraic *laws*.

- **Coordinates** are the *fundamental adjectival modifiers* corresponding to the differentiating properties of "things" (nouns) in the real Universe, where the term fundamental can also be thought of as meaning *irreducible* – adjectival properties that cannot be readily be expressed in terms of or derived from other adjectival properties of a given object/thing. See:
  - Wikipedia: http://www.wikipedia.org/wiki/Coordinate System

- **Units.** Physical coordinates are basically mathematical numbers with units (or can be so considered even when they are discrete non-ordinal sets). In this class we will consistently and universally use *Standard International* (SI) units unless otherwise noted. Initially, the irreducible units we will need are:

  a) *meters* – the SI units of *length*
  b) *seconds* – the SI units of *time*
  c) *kilograms* – the SI units of *mass*

  All other units for at least a short while will be expressed in terms of these three, for example units of *velocity* will be meters per second, units of *force* will be kilogram-meters per second squared. We will often give names to some of these combinations, such as the SI units of force:

$$1 \text{ Newton} = \frac{\text{kg}-\text{m}}{\text{sec}^2}$$

Later you may learn of other irreducible coordinates that describe elementary particles or extended macroscopic objects in physics such as electrical charge, as well as derivative quantities such as electrical current (charge per unit time), spin, angular momentum, momentum, and more.

- **Laws of Nature** are essentially *mathematical postulates* that permit us to understand natural phenomena and *make predictions* concerning the time evolution or static relations between the coordinates associated with objects in nature that are consistent mathematical theorems of the postulates. These predictions can then be compared to *experimental observation* and, if they are consistent (uniformly successful) we *increase our degree of belief in them.* If they are inconsistent with observations, we *decrease* our degree of belief in them, and seek alternative or modified postulates that work better.

  The entire body of human scientific knowledge is the more or less successful outcome of this process. This body is not fixed – indeed it is *constantly changing* because it is an *adaptive* process, one that self-corrects whenever observation and prediction fail to correspond or when new evidence spurs insight (sometimes revolutionary insight!)

  Newton's Laws built on top of the analytic geometry of Descartes (as the basis for at least the abstract spatial coordinates of things) are the *dynamical principle* that proved *successful* at predicting the outcome of many, many everyday experiences and experiments as well as cosmological observations in the late 1600's and early 1700's; when combined with associated empirical *force laws* they form the basis of the physics you will learn in this course.

- **Newton's Laws**:

  a) **Law of Inertia:** Objects at rest or in uniform motion (at a constant velocity) in an *inertial reference frame* remain so unless acted upon by an unbalanced (net) force. We can write this algebraically as:

  $$\sum_i \vec{F}_i = 0 = m\vec{a} = m\frac{d\vec{v}}{dt} \quad \Rightarrow \quad \vec{v} = \text{constant vector} \tag{1}$$

  b) **Law of Dynamics:** The net force applied to an object is directly proportional to its acceleration in an *inertial reference frame.* The constant of proportionality is called the **mass** of the object. We write this algebraically as:

  $$\vec{F} = \sum_i \vec{F}_i = m\vec{a} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt} \tag{2}$$

  where we introduce the *momentum* of a particle, $\vec{p} = m\vec{v}$, in the last way of writing it.

  c) **Law of Reaction:** If object A exerts a force $\vec{F}_{AB}$ on object B *along a line connecting the two objects*, then object B exerts an equal and opposite reaction force of $\vec{\boldsymbol{F}}_{BA} = -\vec{\boldsymbol{F}}_{AB}$ on object A. We write this algebraically as:

$$\vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji} \tag{3}$$

$$\text{(or)} \quad \sum_{i,j} \vec{\boldsymbol{F}}_{ij} = 0 \tag{4}$$

(the latter form means that the sum of all *internal* forces between particles in any closed system of particles cancel).

- An **inertial reference frame** is a *spatial coordinate system* (or *frame*) that is either at rest or moving at a constant speed, a *non-accelerating* set of coordinates that can be used to describe the locations of real objects. In physics one has considerable leeway when it comes to choosing the (inertial) coordinate frame to be used to solve a problem – some lead to much simpler solutions than others!

- **Forces of Nature** (weakest to strongest):

  a) Gravity
  b) Weak Nuclear
  c) Electromagnetic
  d) Strong Nuclear

  It is possible that there are more forces of nature waiting to be discovered. Because physics is not a *dogma*, this presents no real problem. If they are, we'll simply give the discoverer a Nobel Prize and add their name to the "pantheon" of great physicists, add the force to the list above, and move on. Science, as noted, is self-correcting.

- **Force** is a *vector*. For each force rule below we therefore need *both* a formula or rule for the magnitude of the force (which we may have to compute in the solution to a problem – in the case of forces of constraint such as the normal force we will *usually* have to do so) and a way of determining or specifying the *direction* of the force. Often this direction will be obvious and in corresponence with experience and mere common sense – strings pull, solid surfaces push, gravity points down and not up. Other times it will be more complicated or geometric and (along with the magnitude) may vary with position and time.

- *Force Rules* The following set of force rules will be used both in this chapter and throughout this course. All of these rules can be derived or understood (with some effort) from the forces of nature, that is to say from "elementary" natural laws.

  a) **Gravity** (near the surface of the earth):

  $$F_g = mg \tag{5}$$

  The direction of this force is *down*, so one could write this in vector form as $\vec{\boldsymbol{F}}_g = -mg\hat{\boldsymbol{y}}$ in a coordinate system such that up is the $+y$ direction. This rule follows from Newton's Law of Gravitation, the elementary law of nature in the list above, evaluated "near" the surface of the earth where it is approximately constant.

b) **The Spring** (Hooke's Law) in one dimension:

$$F_x = -k\Delta x \tag{6}$$

This force is directed back to the equilibrium point of unstretched spring, in the *opposite direction* to Δx, the displacement of the mass on the spring from equilibrium. This rule arises from the primarily electrostatic forces holding the atoms or molecules of the spring material together, which tend to linearly oppose *small* forces that pull them apart or push them together (for reasons we will understand in some detail later).

c) The **Normal Force**:

$$F_\perp = N \tag{7}$$

This points perpendicular and away from solid surface, magnitude sufficient to oppose the force of contact *whatever it might be!* This is an example of a *force of constraint* – a force whose magnitude is determined by the *constraint* that one solid object cannot generally interpenetrate another solid object, so that the solid surfaces exert whatever force is needed to prevent it (up to the point where the "solid" property itself fails). The physical basis is once again the electrostatic molecular forces holding the solid object together, and *microscopically* the surface deforms, however slightly, more or less like a spring.

d) **Tension** in an Acme (massless, unstretchable, unbreakable) string:

$$F_s = T \tag{8}$$

This force simply transmits an *attractive* force between two objects on opposite ends of the string, in the directions of the taut string at the points of contact. It is another constraint force with no fixed value. Physically, the string is like a spring once again – it microscopically is made of bound atoms or molecules that pull ever so slightly apart when the string is stretched until the restoring force balances the applied force.

e) **Static Friction**

$$f_s \le \mu_s N \tag{9}$$

(directed opposite towards net force parallel to surface to contact). This is another force of constraint, as large as it needs to be to keep the object in question travelling at the same speed as the surface it is in contact with, up to the *maximum* value static friction can exert before the object starts to slide. This force arises from mechanical interlocking at the microscopic level plus the electrostatic molecular forces that hold the surfaces themselves together.

f) **Kinetic Friction**

$$f_k = \mu_k N \tag{10}$$

(opposite to direction of relative sliding motion of surfaces and parallel to surface of contact). This force *does* have a fixed value when the right conditions (sliding) hold. This force arises from the forming and breaking of microscopic adhesive bonds between atoms on the surfaces plus some mechanical linkage between the small irregularities on the surfaces.

g) **Fluid Forces, Pressure**: A fluid in contact with a solid surface (or anything else) in general exerts a force on that surface that is related to the *pressure* of the fluid:

$$F_P = PA \tag{11}$$

which you should read as "the force exerted by the fluid on the surface is the pressure in the fluid times the area of the surface". If the pressure varies or the surface is curved one may have to use calculus to add up a total force. In general the direction of the force exerted is *perpendicular* to the surface. An object at rest in a fluid often has balanced forces due to pressure. The force arises from the molecules in the fluid literally bouncing off of the surface of the object, transferring momentum (and exerting an average force) as they do so. We will study this in some detail and will even derive a kinetic model for a gas that is in good agreement with real gases.

h) **Drag Forces**:

$$F_d = -bv^n \tag{12}$$

(directed opposite to relative velocity of motion through fluid, n usually between 1 (low velocity) and 2 (high velocity). This force also has a determined value, although one that depends in detail on the motion of the object. It arises first because the surface of an object moving through a fluid is literally bouncing fluid particles off in the leading direction while moving away from particles in the trailing direction, so that there is a differential pressure on the two surfaces.

The first week summary would not be incomplete without some sort of reference to *methodologies of problem solving* using Newton's Laws and the force laws or rules above. The following rubric should be of great use to you as you go about solving *any* of the problems you will encounter in this course, although we will modify it slightly when we treat e.g. energy instead of force, torque instead of force, and so on.

Dynamical Recipe for Newton's Second Law

a) *Draw* a good picture of what is going on. In general you should probably do this even if one has been provided for you – visualization is key to success in physics.

b) On your drawing (or on a second one) decorate the objects with all of the *forces* that act on them, creating a *free body diagram* for the forces on each object.

c) *Write* e.g. Newton's Second Law for each object (summing the forces and setting the result to $m_i\vec{\boldsymbol{a}}_i$ for each – ith – object) and algebraically rearrange it into (vector) differential equations of motion (practically speaking, this means solving for or isolating the *acceleration* $\vec{\boldsymbol{a}}_i = \frac{d^2\vec{\boldsymbol{x}}_i}{dt^2}$ of the particles in the equations of motion).

d) *Decompose* the 1, 2 or 3 dimensional equations of motion for each object into a *set of independent* 1 dimensional equations of motion for each of the orthogonal coordinates by choosing a suitable coordinate system (which may *not* be cartesian, for some problems) and using trig/geometry. Note that a "coordinate" here may even wrap around a corner following a string, for example – or we can use a different coordinate system for each particle, as long as we have a known relation between the coordinate systems.

e) *Solve* the independent 1 dimensional systems for each of the independent orthogonal coordinates chosen, plus any coordinate system constraints or relations. In many problems the *constraints* will eliminate one or more degrees of freedom from consideration. Note that in most nontrivial cases, these solutions will have to be *simultaneous* solutions, obtained by e.g. algebraic substitution or elimination.

f) *Reconstruct* the multidimensional trajectory by adding the vectors components thus obtained back up (for a common independent variable, time).

g) *Answer* algebraically any questions requested concerning the resultant trajectory.

Some parts of this rubric will require *experience and common sense* to implement correctly for any given kind of problem. That is why homework is so critically important! We want to make solving the problems *easy*, and they will only get to be easy with practice followed by reflection, practice followed by reflection.

Introduction: A Bit of History and Philosophy

It has been remarked by at least one of my colleagues that one reason we have such a hard time teaching Newtonian physics to college students is that we have to first *unteach* them their already prevailing world"natural" view of physics, which dates all the way back to Aristotle.

In a nutshell and in very general terms (skipping all the "nature is a source or cause of being moved and of being at rest" as primary attributes, see Aristotle's *Physica*, book II) Aristotle considered motion or the lack thereof of an object to be innate properties of materials, according to their proportion of the big four basic elements: Earth, Air, Fire and Water. He extended the idea of the moving and the immovable to cosmology and to his Metaphysics as well.

In this primitive view of things, the observation that (most) physical objects (being "Earth") set in motion slow down is translated into the notion that their natural state is to be at rest, and that one has to add *something* from one of the other essences to produce a state of uniform motion. This was not an unreasonable hypothesis; a great deal of a person's everyday experience (then and now) is consistent with this. When we pick something up and throw it, it moves for a time and bounces, rolls, slides to a halt. We need to press down on the accelerator of a car to keep it going, adding something from the "fire" burning in the motor to the "earth" of the body of the car. Even our selves seem to run on "something" that goes away when we die.

Unfortunately, it is completely and totally wrong. Indeed, it is almost precisely *Newton's first law* stated backwards. It is very likely that the *reason* Newton wrote down his first law (which is otherwise a trivial consequence of his second law) is to directly confront the fallacy of Aristotle, to force the philosophers of his day to confront the fact that his theory of physics was *irreconcilable* with that of Aristotle, and that (since his actually worked to make precise predictions of nearly any kind of classical motion) Aristotle's physics was *wrong*.

This was a core component of the Enlightment, a period of a few hundred years in which Europe went from a state of almost slavish, church-endorsed belief in the infallibility and correctness of the Aristotelian worldview to a state where humans, for the first time in history, let nature *speak for itself* by using a *consistent framework* to listen to what nature had to say[23]. Aristotle lost, but his *ideas* are slow to die *because* they closely parallel everyday experience. The *first* chore for any serious student of physics is thus to unlearn this Aristotelian view of things[24].

This is not, unfortunately, an abstract problem. It is very concrete and very current. Because I have an online physics textbook, and because physics is in some very fundamental sense the "magic" that makes the world work, I not infrequently am contacted by individuals who do *not* understand the material covered in this textbook, who do *not* want to do the very hard work required to master it, but who still want to be "magicians". So they invent their *own* version of the magic, usually altering the *mathematically precise* meanings of things like "force", "work", "energy" to be something else altogether that *they* think that they understand but that, when examined closely, usually are dimensionally or conceptually inconsistent and mean nothing at all.

Usually their "new" physics is in fact a reversion to the physics of Aristotle. They recreate the magic of earth and air, fire and water, a magic where things slow down unless fire (energy) is added to sustain their motion or where it can be extracted from invisible an inexhaustible resources, a world where the *mathematical* relations between work and energy and force and acceleration do not hold. A world, in short, that violates a huge, vast, truly stupdendous body of accumulated experimental evidence including the very evidence that you yourselves will very likely observe in person in the physics labs associated with this course. A world in which things like perpetual motion machines are possible, where free lunches abound, and where humble dilettantes can be crowned "the next Einstein" without having a solid understanding of algebra, geometry, advanced calculus, or the physics that everybody *else* seems to understand perfectly.

This world does not exist. Seriously, it is a fantasy, and a *very dangerous one*, one that threatens modern civilization itself. One of the *most important* reasons you are taking this course, *whatever* your long term dreams and aspirations are professionally,

[23]Students who like to read historical fiction will doubtless enjoy Neal Stephenson's *Baroque Cycle*, a set of novels that spans the Enlightenment and in which Newton, Liebnitz, Hooke and other luminaries to whom we owe our modern conceptualization of physics all play active parts.

[24]This is not the last chore, by the way. Physicists have long since turned time into a coordinate just like space so that how long things take depends on one's point of view, eliminated the assumption that one can measure any set of measureable quantities to arbitrary precision in an arbitrary order, replaced the determinism of mathematically precise trajectories with a funny kind of stochastic quasi-determinism, made (some) forces into just an aspect of geometry, and demonstrated a degree of mathematical structure (still incomplete, we're working on it) beyond the wildest dreams of Aristotle or his mathematical-mystic buddies, the Pythagoreans.

is to come to fully and deeply understand this. You will come to understand the magic of science, at the same time you learn to reject the notion that science *is* magic or vice versa. I personally find it very comforting that the individuals that take care of my body (physicians) and who design things like jet airplanes and automobiles (engineers) share a common and consistent *Newtonian*[25] view of just how things work, and would find it very disturbing if any of them believed in magic, in fairies, in earth, air, fire and water as constituent elements, in "crystal energies", in the power of a drawn pentagram or ritually chanted words in any context whatsoever. These all represent a sort of willful *wishful thinking* on the part of the believer, a desire for things to *not* follow the rigorous and mathematically precise rules that they appear to follow as they evolve in time.

Let me be therefore be precise. In the physics we will study week by week below, the natural state of "things" (e.g. objects made of matter) will be to move uniformly. We will learn *non*-Aristotelian physics, *Newtonian* physics. It is only when things are acted on from outside by *unbalanced forces* that the motion becomes non-uniform; they will speed up or slow down. By the time we are done, you will understand how this can still lead to the damping of motion observed in everyday life, why things *do* generally slow down. In the meantime, be aware of the problem and resist applying the Aristotelian view to real physics problems, and consider, *based on the evidence and your experiences taking this course* rejecting "magic" as an acceptable component of your personal worldview unless and until it too has some sort of objective empirical support (which would, of course, just make it part of physics!)

Dynamics

Physics is the study of *dynamics*. Dynamics is the description of the actual forces of nature that, we believe, underlie the causal structure of the Universe and are responsible for its *evolution in time*. We are about to embark upon the intensive study of a simple description of nature that introduces the concept of a *force*, due to Isaac Newton. A force is considered to be the *causal agent* that produces the *effect of acceleration* in any massive object, altering its dynamic state of motion.

Newton was not the first person to attempt to describe the underlying nature of causality. Many, many others, including my favorite 'dumb philosopher', Aristotle, had attempted this. The major difference between Newton's attempt and previous ones is that Newton did not frame his as a philosophical postulate per se. Instead he formulated it as a *mathematical theory* and proposed a set of *laws* that (he hoped) precisely described the regularities of motion in nature.

In physics a law is the equivalent of a *postulated axiom* in mathematics. That is, a physical law is, like an axiom, an *assumption* about how nature operates that *not* formally provable by any means, including experience, within the theory. A physical law is thus

[25]Or better. Of course actual modern physics is non-Newtonian quantum mechanics, but this is just as non-magical and concrete and it reduces to Newtonian physics in the macroscopic world of our quotidian experience.

not considered "correct" – rather we ascribe to it a "degree of belief" based on how well and consistently it describes nature in experiments designed to verify or falsify its correspondance. It must successfully withstand the test of repeated experiments in order to survive and become plausible.

If a set of laws survive all the experimental tests we can think up and subject it to, we consider it *likely* that it is a good approximation to the true laws of nature; if it passes many tests but then fails others (often failing consistently at some length or time scale) then we may continue to call the postulates laws (applicable within the appropriate milieu) but recognize that they are only approximately true and that they are superceded by some more fundamental laws that are closer (at least) to being the "true laws of nature".

Newton's Laws, as it happens, are in this latter category – early postulates of physics that worked remarkably well up to a point (in a certain "classical" regime) and then failed. They are "exact" (for all practical purposes) for massive, large objects and long times such as those we encounter in the everyday world of human experience (as described by SI scale units). They fail badly (as a basis for prediction) for microscopic phenomena involving short distances, small times and masses, for very strong forces, and for the laboratory description of phenomena occurring at relativistic velocities. Nevertheless, even here they survive in a distorted but still recognizable form, and the constructs they introduce to help us study dynamics still survive.

Interestingly, Newton's laws lead us to second order differential equations, and even quantum mechanics appears to be based on differential equations of second order or less. Third order and higher systems of differential equations seem to have potential problems with temporal causality (where effects always follow, or are at worst simultaneous with, the causes in time); it is part of the genius of Newton's description that it precisely and sufficiently allows for a full description of causal phenomena, even where the details of that causality turn out to be incorrect.

Incidentally, one of the other interesting features of Newton's Laws is that *Newton invented calculus* to enable him to solve the problems they described. Now you know why calculus is so essential to physics: physics was the original motivation behind the invention of calculus itself. Calculus was also (more or less simultaneously) invented in the more useful and recognizable form that we still use today by other mathematical-philosophers such as Leibnitz, and further developed by many, many people such as Gauss, Poincare, Poisson, Laplace and others. In the overwhelming majority of cases, especially in the early days, solving one or more problems in the physics that was still being invented was the motivation behind the most significant developments in calculus and differential equation theory. This trend continues today, with physics providing an underlying structure and motivation for the development of the most advanced mathematics.

Coordinates

Think about any *thing*, any entity that objectively exists in the real, visible, Universe. What defines the object and differentiates it from all of the *other* things that make up the Universe? Before we can talk about how the Universe and its contents change in time, we have to talk about how to describe its contents *at all*.

As I type this I'm looking over at just such a thing, an object that I truly believe exists in the real Universe. To help you understand this object, I have to use *language*. I might tell you how large it is, what is weight is, what it looks like, where it is, how long it has been there, what it is for, and – of course – I could substitute a single *word* for much of this, add a few adjectival modifiers, and speak of an "empty beer glass sitting on a table in my den just to my side", where now I have only to tell you just where my den is, where the tale is in the den, and perhaps differentiate this particular beer glass from other beer glasses you might have seen. Eventually, if I use enough words, construct a detailed map, make careful measurements, perhaps include a picture, I can convey to you a very precise *mental picture* of the beer glass, one sufficiently accurate that you can *visualize* just where, when and what it is (or was).

Of course this prose description is *not the glass itself!* If you like, the *map is not the territory*[26]! That is, it is an *informational representation* of the glass, a collection of symbols with an agreed upon meaning (where meaning in this context is a correspondence between the symbols and the presumed general sensory experience of the glass that one would have if one *looked* at the glass from my current point of view.

Constructing just such a map is the whole point of physics, only the map is not just of mundane objects such as a glass; it is the map of the whole world, the whole *universe*. To the extent that this *worldview* is successful, especially in a predictive sense and not just hindsight, the physical map in your mind works well to predict the universe you perceive through your sensory apparatus. A perfect understanding of physics (and a knowledge of certain data) is equivalent to a possessing a perfect map, one that precisely locates every thing within the universe for all possible times.

Maps require several things. It is convenient, but not necessary, to have a set of single term descriptors, symbols for various "things" in the world the map is supposed to describe. So this symbol might stand for a chuch, that one for a bridge, still another one for a street or railroad crossing. Another *absolutely essential* part of a map is the actual *coordinates* within it. The coordinate representation within the map is supposed to exist in an accurate one-to-one but *abstract* correspondence with the *concrete* territory in the real world where things represented by the symbols actually exist and move about.

Of course the symbols such as the term "beer glass" can *themselves* be abstractly modelled as points in some sort of space; Complex or composite objects with "simple" coordinates can be represented as a collection of far more coordinates for the smaller

[26]This is an adage of a field called *General Semantics* and is something to keep carefully in mind while studying physics. Not even my direct perception of the glass is the glass itself; it is just a more complex and dynamics kind of map.

objects that make up the composite object. Ultimately one arrives at *elementary* objects, objects that are not made up of other objects at all. The various kinds of elementary objects, the list of their variable properties, and their spatial and temporal coordinates are in some deep sense *all coordinates*, and every object in the universe can be thought of as a *point in this enormous and highly complex space.*

In this sense **coordinates** are the *fundamental adjectival modifiers* corresponding to the differentiating properties of "things" (nouns) in the real Universe, where the term fundamental can also be thought of as meaning elementary or *irreducible* – adjectival properties that cannot be readily be expressed in terms of or derived from other adjectival properties of a given object/thing.

Physical coordinates are, then, basically mathematical numbers with units (and can be so considered even when they are discrete non-ordinal sets). At first we will omit most of the details from the objects we study to keep things simple. In fact, we will spend most of the first part of the course interested in only three quantities that set the scale for the coordinate system we need to describe the classical classical physics of a rather generic "particle": space (where it is), time (when it is there), and mass (an intrinsic property).

We need *units* to describe intervals or values in all three coordinates so that we can talk or think about those objects in ways that don't depend on the listener. In this class we will consistently and universally use *Standard International* (SI) units unless otherwise noted. Initially, the irreducible units we will need are:

a) *meters* – the SI units of *length*

b) *seconds* – the SI units of *time*

c) *kilograms* – the SI units of *mass*

All other units for at least a short while will be expressed in terms of these three.

For example units of *velocity* will be meters per second, units of *force* will be kilogram-meters per second squared. We will often give names to some of these combinations, such as the SI units of force:

$$1 \text{ Newton} = \frac{\text{kg}-\text{m}}{\text{sec}^2}$$

Later you may learn of other irreducible coordinates that describe elementary particles or extended macroscopic objects in physics such as electrical charge, as well as derivative quantities such as electrical current (charge per unit time), spin, angular momentum, momentum, and more.

Coordinates are enormously powerful ideas, the very essence of mapmaking. To assist us in working with them, we introduce the notion of *coordinate frame* – a *system* of all of the relevant coordinates that describe at least the position of a particle (in one, two or three dimensions, usually). Below is a picture of a simple single particle that might represent my car on a set of *coordinates* that describes at least part of the actual space where my care is sitting. The solid line on this figure represents the *trajectory* of my car as it moves about. Armed with a watch, an apparatus for measuring mass, meter sticks and some imagination, one can imagine a virtual car rolling up or down along its virtual trajectory and compare its motion *in our internal maps* with the correspondent happenings in the world outside of our minds where the real car moves along a real track.

Note well that we *idealize* my car by treating the whole thing as a *single object* with a *single position* (located somewhere "in the middle") when we know that it is *really* made up of many, many tinier objects (elementary particles assembled into atoms, molecules, and eventually things like steering wheels and bucket seats that are "objects" in their own right that are further assembled into a "car"). Later in this semester we will *formally justify* our ability to do this, and we will improve on our description of things like cars and wheels and solid composite objects by learning how they can translate as if they are a "single object" and also rotate about a certain center.

In the meantime, we will continue with this idealization and treat solid objects as *particles* – masses that are at a single point in space at a single time. So we will treat planets, baseballs, people, cars, blocks and more as if they have a single spatial location at any instant in time, a single mass – as a particle.

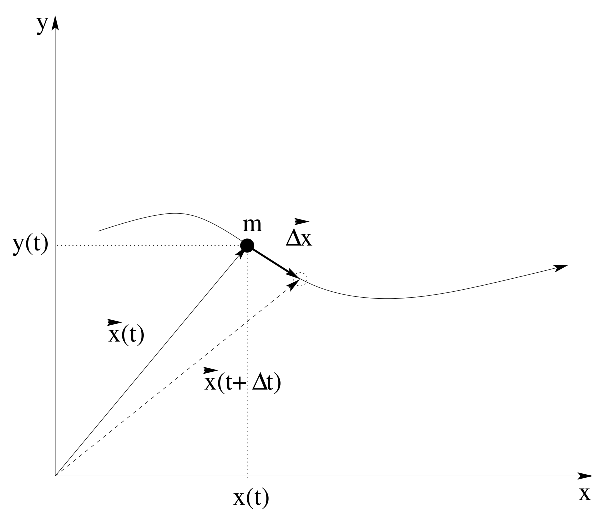


Figure 2: Coordinatized visualization of the motion of a particle of mass m along a trajectory $\vec{\boldsymbol{x}}(t)$. Note that in a short time Δt the particle's position changes from $\vec{\boldsymbol{x}}(t)$ to $\vec{\boldsymbol{x}}(t + \Delta t)$.

In physical dynamics, we will be concerned with finding the *trajectory* of particles or systems – the position of each particle in the system as a function of time. We can write the trajectory as a *vector function* on a spatial coordinate system (e.g. cartesian coordinates in 2 dimensions):

$$\vec{\boldsymbol{x}}(t) = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}} \tag{13}$$

Note that $vx(t)$ stands for a vector from the origin to the particle, where $x(t)$ by itself (no boldface or vector sign) stands for the x-component of this vector. This trajectory is *visualized* in figure 2. In all of the problems we work on throughout the semester, *visualization* will be a key component of the solution.

The human brain doesn't, actually, excel at being able to keep all of these details onboard in your "mind's eye", the virtual visual field of your imagination. Consequently, you must *always* draw figures, usually with coordinates, forces, and other "decorations", when you solve a physics problem. The paper (or blackboard or whiteboard) then becomes an extension of your brain – a kind of "scratch space" that augments your visualization capabilities and your sequential symbolic reasoning capabilities. To put it bluntly, you are *more intelligent* when you reason with a piece of paper and pen than you are when you are forced to rely on your brain alone. To succeed in physics, you need all of the intelligence you can get, and you need to *synthesize* solutions using both halves of your brain, visualization and sequential reason. Paper and pen facilitate this process and one of the most important lessons of this course is how to *use* them to attain the full benefit of the added intelligence they enable not just in physics problems, but everywhere else as well.

If we know the trajectory function of a particle, we know a lot of other things too. Since we know *where* it is at any given time, we can easily tell *how fast it is going* in *what direction*. This combination of the speed of the particle and its direction forms a vector called the *velocity* of the particle.

The *average* velocity of the particle is by definition the vector change in its position $\Delta\vec{x}$ in some time Δt divided by that time:

$$\vec{v}_{\rm av} = \frac{\Delta\vec{x}}{\Delta t} \tag{14}$$

Sometimes average velocity is useful, but usually it is not. It isn't because it is often a poor measure for how fast a particle is moving at any given time. For example, I might get in my car and drive around a racetrack at speed of 50 meters per second – really booking it, tires screeching, smoke coming out of my engine (at least if I tried this in *my* car, as it would likely explode if I tried to go 112 mph for any extended time), and screech to a halt where I began. My *average* velocity is then *zero* – I'm back where I started! That zero is a poor explanation for the heat waves pulsing out from under my hood.

More often we will be interested in the *instantaneous velocity* of the particle. This is basically the average velocity, averaged over as small a time interval as possible – one so short that it is just long enough for the car to move at all. Calculus permits us to take this limit, and indeed uses just this limit as the *definition* of the *derivative*. We thus define the instantaneous velocity vector as the time-derivative of the position vector:

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\vec{x}(t+\Delta t)-\vec{x}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\vec{x}}{\Delta t} = \frac{d\vec{x}}{dt} \tag{15}$$

In turn, our cars speed up and slow down. To see how the *velocity* changes in time, we will need to consider the *acceleration* of a particle, or the rate at which the *velocity* changes. As before, we can easily define an average acceleration over a possibly long time interval Δt as:

$$\vec{a}_{\rm av} = \frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t} \tag{16}$$

Also as before, this average is usually a poor measure of the actual acceleration a particle (or car) experiences. If I am on a straight track at rest and stamp on the accelerator, burning rubber until I reach 50 meters per second (112 miles per hour) and then stamp on the break to screech to a halt, my *average* acceleration is once again zero, but at no time during the trip was my *actual* acceleration anything close to zero. In fact, my average acceleration around the track loop was zero as well, in spite of the fact that particles moving in a circular curve (as we will shortly see) are *always* accelerating.

The only acceleration that really matters is thus the limit of the average over very short times; the time derivative of the velocity. This limit is thus defined to be the *instantaneous* accleration:

$$\vec{\boldsymbol{a}}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{\boldsymbol{v}}}{\Delta t} = \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d^2\vec{\boldsymbol{x}}}{dt^2} \tag{17}$$

Obviously we *could* continue this process and define the time derivative of the acceleration and higher order derivatives. However, we generally do not have to in physics – we will not need to continue this process. As it turns out, the dynamic principle that appears sufficient to describe pretty much all classical motion will involve *force* and *acceleration*, and pretty much all of the math we do (at least at first) will involve moving *backwards* from a knowledge of the acceleration to a knowledge of the velocity and position vectors via *integration* or more generally (later) *solving the differential equation of motion*.

We are now prepared to formulate this dynamical principle – Newton's Second Law.

Newton's Laws

The following are Newton's Laws as you will need to know them to both solve problems and answer conceptual questions in this course. Note well that they are framed in terms of the *spatial coordinates* defined in the previous section.

a) **Law of Inertia:** Objects at rest or in uniform motion (at a constant velocity) in an *inertial reference frame* remain so unless acted upon by an unbalanced (net) force. We can write this algebraically as:

$$\sum_i \vec{\boldsymbol{F}}_i = 0 = m\vec{\boldsymbol{a}} = m\frac{d\vec{\boldsymbol{v}}}{dt} \quad \Rightarrow \quad \vec{\boldsymbol{v}} = \text{constant vector} \tag{18}$$

b) **Law of Dynamics:** The net force applied to an object is directly proportional to its acceleration in an *inertial reference frame*. The constant of proportionality is called the **mass** of the object. We write this algebraically as:

$$\vec{\boldsymbol{F}} = \sum_i \vec{\boldsymbol{F}}_i = m\vec{\boldsymbol{a}} = \frac{d(m\vec{\boldsymbol{v}})}{dt} = \frac{d\vec{\boldsymbol{p}}}{dt} \tag{19}$$

where we introduce the *momentum* of a particle, $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$, in the last way of writing it.

c) **Law of Reaction:** If object A exerts a force $\vec{\boldsymbol{F}}_{AB}$ on object B *along a line connecting the two objects*, then object B exerts an equal and opposite reaction force of $\vec{\boldsymbol{F}}_{BA} = -\vec{\boldsymbol{F}}_{AB}$ on object A. We write this algebraically as:

$$\vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji} \tag{20}$$

$$\text{(or)} \quad \sum_{i,j} \vec{\boldsymbol{F}}_{ij} = 0 \tag{21}$$

(the latter form means that the sum of all *internal* forces between particles in any closed system of particles cancel).

Note that these laws are *not all independent* as mathematics goes. The first law is a clear and direct consequence of the second. The third is not – it is an independent statement. The first law *historically*, however, had an important purpose. It *rejected* the dynamics of Aristotle, introducing the new idea of *intertia* where an object in motion continues in that motion unless acted upon by some external agency. This is directly opposed to the Aristotelian view that things only moved when acted on by an external agency and that they naturally came to *rest* when that agency was removed.

The second law is our basic dynamical principle. It tells one how to go from a problem description (in words) plus a knowledge of the force laws of nature to an "equation of motion" (typically a statement of Newton's second law). The equation of motion, generally solved for the acceleration, becomes a *kinematical* equation from which we can develop a full knowledge of the solution using mathematics guided by our experience and physical intuition.

The third law leads (as we shall see) to the surprising result that *systems of particles behave collectively like a particle!* This is indeed fortunate! We know that something like a baseball is really made up of a *lot* of teensy particles itself, and yet it obeys Newton's Second law as if it *is* a particle. We will use the third law to derive this and the closely related *Law of Conservation of Momentum* in a later week of the course.

An inertial reference frame is a coordinate system (or frame) that is either at rest or moving at a constant speed, a *non-accelerating* frame of reference. For example, the ground, or lab frame, is a coordinate system at rest relative to the approximately non-accelerating ground or lab and is considered to be an inertial frame to a good approximation. A car travelling at a constant speed *relative* to the ground, a spaceship coasting in a region free from fields, a railroad car going down straight tracks at constant speed are also inertial frames. A car that is accelerating (say by going around a curve), a spaceship that is accelerating, a freight car that is speeding up or slowing down – these are all examples of *non*-inertial frames. All of Newton's laws suppose an inertial reference frame (yes, the third law too) and are generally *false* for accelerations evaluated in an *accelerating* frame as we will prove next.

Inertial Reference Frames – the Galilean Transformation

We have already spoken about coordinate systems, or "frames", that we need to imagine when we create the mental map between a physics *problem* in the abstract and the supposed *reality* that it describes. One immediate problem we face is that there are many frames we might choose to solve a problem in, but that our choice of frames isn't *completely* arbitrary. We need to reason out how much freedom we have, so that we can use that freedom to make a "good choice" and select a frame that makes the problem relatively simple.

Students that go on in physics will learn that there is more to this process than meets the eye – the symmetries of frames that preserve certain quantities actually leads us to an understanding of conserved quantities and restricts acceptable *physical theories* in certain key ways. But even students with no particular interest in relativity theory or quantum theory or advanced classical mechanics (where all of this is developed) have to understand the ideas developed in this section, simply to be able to solve problems efficiently.

Time

Let us start by thinking about time. Suppose that you wish to time a race (as physicists). The first thing one has to do is *understand* what the conditions are for the start of the race and the end of the race. The start of the race is the instant in time that the gun goes off and the racers (as particles) start accelerating towards the finish line. This time is concrete, an actual event that you can "instantly" observe[27]. Similarly, the end of the race is the instant in time that the racers (as particles) cross the finish line.

Consider three observers timing the same racer. One uses a "perfect" stop watch, one that is triggered by the gun and stopped by the racer crossing the finish line. The race starts at time $t = 0$ on the stop watch, and stops at time t_f, the time it took the racer to complete the race.

The second doesn't have a stop watch – she has to use their watch set to local time. When the gun goes off she records t_0, the time her watch reads at the start of the race. When the racer crosses the finish line, she records t_1, the finish time *in local time coordinates*. To find the time of the race, she converts her watch's time to seconds and subtracts to obtain $t_f = t_1 - t_0$, which must non-relativistically[28] *agree* with the first observer.

The third has just arrived from India, and hasn't had time to reset his watch. He records t'_0 for the start, t'_1 for the finish, and subtracts to once again obtain $t_f = t'_1 - t'_0$.

All three of these times must agree because clearly the time required for the racer to cross the finish line has nothing to do with the observers. We could use any clock we wished, set to any time zone or started so that "t = 0" occurs at any time you like to

[27]For the purpose of this example we will ignore things like the speed of sound or the speed of light and assume that our observation of the gun going off is instantaneous.

[28]Students not going on in physics should just ignore this adverb. Students *going* on in physics should be aware that the real, relativistic Universe those times might *not* agree.

time the race as long as it records times in seconds accurately. In physics we express this invariance by stating that we can change clocks at will when considering a particular problem by means of the transformation:

$$t' = t - t_0 \tag{22}$$

where t_0 is the time in our *old* time-coordinate frame that we wish to be *zero* in our new, primed frame. This is basically a linear change of variables, a so-called "u-substitution" in calculus, but because we shift the "zero" of our clock in all cases by a *constant*, it is true that:

$$dt' = dt \tag{23}$$

so differentiation by t' is identical to differentiation by t and:

$$\vec{\boldsymbol{F}} = m\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = m\frac{d^2\vec{\boldsymbol{x}}}{dt'^2} \tag{24}$$

That is, *Newton's second law* is *invariant* under uniform translations of time, so we can start our clocks whenever we wish and still accurately describe all motion relative to that time.

Space

We can reason the same way about space. If we want to measure the distance between two points on a line, we can do so by putting the zero on our meter stick at the first and reading off the distance of the second, or we can put the first at an arbitrary point, record the position of the second, and subtract to get the same distance. In fact, we can place the origin of our coordinate system anywhere we like and measure all of our locations *relative* to this origin, just as we can choose to start our clock at any time and measure all times relative to that time.

Displacing the origin is described by:

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}} + \vec{\boldsymbol{x}}_0 \tag{25}$$

and as above,

$$d\vec{\boldsymbol{x}}' = d\vec{\boldsymbol{x}} \tag{26}$$

so differentiating by $\vec{\boldsymbol{x}}$ is the same as differentiating by a displaced $\vec{\boldsymbol{x}}'$.

However, there is another freedom we have in coordinate transformations. Suppose you are driving a car at a uniform speed down the highway. Inside the car is a fly, flying from the back of the car to the front of the car. To *you*, the fly is moving slowly – in fact, if you place a coordinate frame inside the car, you can describe the fly's position and velocity relative to that coordinate frame very easily. To an observer on the *ground*, however, the fly is flying by at the speed of the car *plus* the speed of the car. The observer on the ground can use a coordinate frame on the ground and can *also* describe the position of the fly perfectly well.

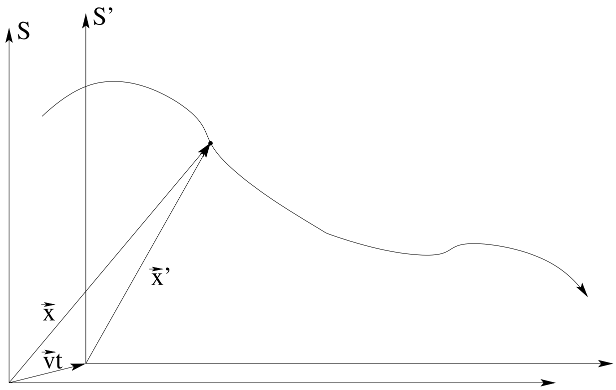


Figure 3: Frame S is the ground, or lab frame at rest. Frame S' is the car (or an abstract frame) moving at a constant velocity $\vec{\boldsymbol{v}}$ relative to the ground. The position of a fly relative to the ground frame is the position of the car relative to the ground frame plus the position of the fly relative to the car. This is an easy mentalo model to use to understand frame transformations.

In figure 3 one can consider the frame S to be the "ground" frame. $\vec{\boldsymbol{x}}$ is the position of the fly relative to the ground. The frame S' is the car, moving at a *constant* velocity $\vec{\boldsymbol{v}}$ relative to the ground, and $\vec{\boldsymbol{x}}'$ is the position of the fly relative to the car. In this way we can describe the position of the fly *in time* in either one of the two frames, where (looking at the triangle of vectors in 3):

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{x}}'(t) + \vec{\boldsymbol{v}}_{\rm frame} t \tag{27}$$

So much for description; what about dynamics? If we differentiate this equation twice, we get:

$$\frac{d\vec{\boldsymbol{x}}}{dt} = \frac{d\vec{\boldsymbol{x}}'}{dt} + \vec{\boldsymbol{v}}_{\rm frame} \tag{28}$$

$$\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = \frac{d^2\vec{\boldsymbol{x}}'}{dt^2} \tag{29}$$

(where we use the fact that the velocity $\vec{\boldsymbol{v}}_{\rm frame}$ is a *constant* so that it disappears from the second derivative) so that if we multiply both sides by m we prove:

$$\vec{\boldsymbol{F}} = m\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = m\frac{d^2\vec{\boldsymbol{x}}'}{dt^2} \tag{30}$$

or *Newton's second law is invariant* under the *Galilean transformation* represented by equation 27.

Any coordinate frame travelling at a *constant velocity* is called an *inertial reference frame*, and since our law of dynamics is *invariant* with respect to changes of inertial frame (as long as the force law itself is), we have complete freedom to choose the one that is the most convenient. The physics of the fly in the car are identical to the physics of the fly relative to the ground.

Equation 28 can be written as:

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}' + \vec{\boldsymbol{v}}_{\text{frame}} \tag{31}$$

which you can think of as the velocity relative to the ground is the velocity *in* the frame plus the velocity *of* the frame. This is the conceptual rule for velocity transformations: The fly may be moving only at 1 meter per second in the car, but if the car is travelling at 19 meters per second relative to the ground, the fly is travelling at 20 meters per second relative to the ground!

The Galilean transformation isn't the only possible way to relate frames, and in fact it doesn't correctly describe nature. A different transformation called the *Lorentz* transformation from the *theory of relativity* works much better, where both length intervals and time intervals *change* when changing inertial reference frames. However, describing and deriving relativistic transformations (and the postulates that lead us to consider them in the first place) is beyond the scope of this course, and they are not terribly important in the classical regime where things move at speeds much less than that of light.

Accelerating (non-inertial) Frames

Note that if the frame S' is *not* travelling at a constant velocity, then:

$$\frac{d\vec{\boldsymbol{x}}}{dt} = \frac{d\vec{\boldsymbol{x}}'}{dt} + \vec{\boldsymbol{v}}(t) \tag{32}$$

$$\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = \frac{d^2\vec{\boldsymbol{x}}'}{dt^2} + \vec{\boldsymbol{a}}_{\text{frame}} \tag{33}$$

where $\vec{\boldsymbol{a}}_{\text{frame}}$ is the *acceleration of the frame* S'. Newton's second law is then *not* invariant:

$$\vec{\boldsymbol{F}} = m\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = m\frac{d^2\vec{\boldsymbol{x}}'}{dt^2} + m\vec{\boldsymbol{a}}_{\text{frame}} \tag{34}$$

Note that *inside* the accelerated frame, the acceleration relative to that frame is given by:

$$\vec{\boldsymbol{F}} + \vec{\boldsymbol{F}}' = \vec{\boldsymbol{F}} - m\vec{\boldsymbol{a}}_{\text{frame}} = m\frac{d^2\vec{\boldsymbol{x}}'}{dt^2} \tag{35}$$

In other words, a *variant* of Newton's second law is satisfied, with an additional *pseudo-force*

$$\vec{\boldsymbol{F}}' = -m\vec{\boldsymbol{a}}_{\text{frame}} \tag{36}$$

that does not correspond to any law of nature that due to the acceleration of the frame. We will discuss frame-related pseudoforces later when we consider accelerating elevators and rotating frames.

Forces

Classical dynamics at this level, in a nutshell, is very simple. Find the total force on an object. Use Newton's second law to obtain its acceleration (as a differential equation of

Particle	Location	Size
Up or Down Quark	Nucleon (Proton or Neutron)	pointlike
Proton	Nucleus	10^{-15} meters
Neutron	Nucleus	10^{-15} meters
Nucleus	Atom	10^{-15} meters
Electron	Atom	pointlike
Atom	Molecules or Objects	10^{-10} meters

Table 1: Basic building blocks of normal matter.

motion). Solve the equation of motion by direct integration or otherwise for the position and velocity.

That's it!

Well, except for answering those pesky *questions* that we typically ask in a physics problem, but we'll get to that later. For the moment, the next most important problem is: how do we evaluate the total force?

To answer it, we need a knowledge of the forces at our disposal, the force laws and rules that we are likely to encounter in our everyday experience of the world. Some of these forces are *fundamental* forces – *elementary* forces that we call "laws of nature" because the forces themselves aren't caused by some *other* force, they are themselves the actual causes of dynamical action in the visible Universe. Other force laws aren't quite so fundamental – they are more like "approximate rules" and aren't exactly correct. They are also usually derivable from (or at least understandable from) the elementary natural laws, although it may be quite a lot of work to do so.

We quickly review the forces we will be working with in the first part of the course, both the forces of nature and the force *rules* that apply to our everyday existence in approximate form.

The Forces of Nature

At this point in your life, you almost certainly know that all normal matter of your everyday experience is made up of *atoms*. Most of you also know that an atom itself is made up of a positively charged *atomic nucleus* that is very tiny indeed surrounded by a cloud of negatively charged *electrons* that are much lighter. Atoms in turn bond together to make molecules, molecules in turn form gases, liquids, solids – things, including those macroscopic things that we are treating as particles.

The *actual* elementary particles from which they are made are much tinier than atoms. It is worth providing a *greatly simplified table* of the "stuff" from which normal atoms (and hence molecules, and hence we ourselves) are made:

In this table, up and down quarks and electrons are so-called *elementary particles* – things that are not made up of something else but are fundamental components of nature. Quarks bond together three at a time to form *nucleons* – a proton is made up of "up-up-down" quarks and has a charge of $+e$, where e is the elementary electric charge. A neutron is made up of "up-down-down" and has no charge.

Neutrons and protons, in turn, bond together to make an atomic nucleus. The simplest atomic nucleus is the hydrogen nucleus, which can be as small as a single proton, or can be a proton bound to one neutron (deuterium) or two neutrons (tritium). No matter how many protons and neutrons are bound together, the atomic nucleus is *small* – order of 10^{-15} meters in diameter. The quarks, protons and neutrons are bound together by means of a force of nature called the *strong nuclear interaction*, which is the strongest force we know of relative to the mass of the interacting particles.

The positive nucleus combines with electrons (which are negatively charged and around 2000 times lighter than a proton) to form an *atom*. The force responsible for this binding is the *electromagnetic* force, another force of nature (although in truth nearly all of the interaction is *electrostatic* in nature, just one part of the overall electromagnetic interaction).

The light electrons repel one another electrostatically almost as strongly as they are attracted to the nucleus that anchors them. They also obey the *Pauli exclusion principle* which causes them to avoid one another's company. These things together cause atoms to be much larger than a nucleus, and to have interesting "structure" that gives rise to chemistry and molecular bonding and (eventually) life.

Inside the nucleus (and its nucleons) there is another force that acts at very short range. This force can cause e.g. neutrons to give off an electron and turn into a proton or other strange things like that. This kind of event changes the atomic number of the atom in question and is usually accompanied by *nuclear radiation*. We call this force the *weak nuclear force*. The two nuclear forces thus both exist only at very short length scales, basically in the quantum regive inside an atomic nucleus, where we cannot easily see them in the classical regime. For our purposes it is enough that they exist and bind stable nuclei together so that those nuclei in turn can form atoms, molecules, objects, us.

Our picture of normal matter, then, is that it is made up of *atoms* that may or may not be bonded together into molecules, with three forces all significantly contributing to what goes on inside a nucleus and only one that is relevant to the electronic structure of the atoms themselves.

There is, however, a fourth force (that we know of – there may be more still, but four is all that we've been able to positively identify and understand). That force is gravity. Gravity is a bit "odd". It is a very long range, but *very weak* force – by far the weakest force of the four forces of nature. It only is signficant when one builds *planet* or *star* sized objects, where it can do anything from simply bind the surface of the planet and its atmosphere together to bring about the catastrophic collapse of a dying star. Since we *live* on the surface of a planet, to *us* gravity will be an important force, but the forces we experience every day are primarily electromagnetic

Let's summarize this in a short table of forces of nature, strongest to weakest:

a) Strong Nuclear

b) Electromagnetic

c) Weak Nuclear

d) Gravity

Note well: It is possible that there are more forces of nature waiting to be discovered. Because physics is not a *dogma*, this presents no real problem. If a new force of nature (or radically different way to view the ones we've got), we'll simply give the discoverer a Nobel Prize and add their name to the "pantheon" of great physicists, add the force to the list above, and move on. Science, as noted above, is self-correcting.

Force Rules

The following set of force rules will be used both in this chapter and throughout this course. All of these rules can be derived or understood (with some effort) from the forces of nature, that is to say from "elementary" natural laws.

a) **Gravity** (near the surface of the earth):

$$F_g = mg \tag{37}$$

The direction of this force is *down*, so one could write this in vector form as $\vec{\boldsymbol{F}}_g = -mg\hat{\boldsymbol{y}}$ in a coordinate system such that up is the $+y$ direction. This rule follows from Newton's Law of Gravitation, the elementary law of nature in the list above, evaluated "near" the surface of the earth where it is approximately constant.

b) **The Spring** (Hooke's Law) in one dimension:

$$F_x = -k\Delta x \tag{38}$$

This force is directed back to the equilibrium point of unstretched spring, in the *opposite direction* to Δx, the displacement of the mass on the spring from equilibrium. This rule arises from the primarily electrostatic forces holding the atoms or molecules of the spring material together, which tend to linearly oppose *small* forces that pull them apart or push them together (for reasons we will understand in some detail later).

c) The **Normal Force**:

$$F_\perp = N \tag{39}$$

This points perpendicular and away from solid surface, magnitude sufficient to oppose the force of contact *whatever it might be!* This is an example of a *force of constraint* – a force whose magnitude is determined by the *constraint* that one solid object cannot generally interpenetrate another solid object, so that the solid surfaces exert whatever force is needed to prevent it (up to the point where the "solid" property itself fails). The physical basis is once again the electrostatic molecular forces holding the solid object together, and *microscopically* the surface deforms, however slightly, more or less like a spring.

d) **Tension** in an Acme (massless, unstretchable, unbreakable) string:

$$F_s = T \tag{40}$$

This force simply transmits an *attractive* force between two objects on opposite ends of the string, in the directions of the taut string at the points of contact. It is another constraint force with no fixed value. Physically, the string is like a spring once again – it microscopically is made of bound atoms or molecules that pull ever so slightly apart when the string is stretched until the restoring force balances the applied force.

e) **Static Friction**

$$f_s \le \mu_s N \tag{41}$$

(directed opposite towards net force parallel to surface to contact). This is another force of constraint, as large as it needs to be to keep the object in question travelling at the same speed as the surface it is in contact with, up to the *maximum* value static friction can exert before the object starts to slide. This force arises from mechanical interlocking at the microscopic level plus the electrostatic molecular forces that hold the surfaces themselves together.

f) **Kinetic Friction**

$$f_k = \mu_k N \tag{42}$$

(opposite to direction of relative sliding motion of surfaces and parallel to surface of contact). This force *does* have a fixed value when the right conditions (sliding) hold. This force arises from the forming and breaking of microscopic adhesive bonds between atoms on the surfaces plus some mechanical linkage between the small irregularities on the surfaces.

g) **Fluid Forces, Pressure**: A fluid in contact with a solid surface (or anything else) in general exerts a force on that surface that is related to the *pressure* of the fluid:

$$F_P = PA \tag{43}$$

which you should read as "the force exerted by the fluid on the surface is the pressure in the fluid times the area of the surface". If the pressure varies or the surface is curved one may have to use calculus to add up a total force. In general the direction of the force exerted is *perpendicular* to the surface. An object at rest in a fluid often has balanced forces due to pressure. The force arises from the molecules in the fluid literally bouncing off of the surface of the object, transferring momentum (and exerting an average force) as they do so. We will study this in some detail and will even derive a kinetic model for a gas that is in good agreement with real gases.

h) **Drag Forces**:

$$F_d = -bv^n \tag{44}$$

(directed opposite to relative velocity of motion through fluid, n usually between 1 (low velocity) and 2 (high velocity). This force also has a determined value, although one that depends in detail on the motion of the object. It arises first because the surface of an object moving through a fluid is literally bouncing fluid particles off in the leading direction while moving away from particles in the trailing direction, so that there is a differential pressure on the two surfaces.

Simple Motion in One or Two Dimensions

Homework for Week 1

Problem 1.

Physics Concepts

In order to solve the following physics problems for homework, you will need to have the following physics and math concepts first at hand, then in your long term memory, ready to bring to bear whenever they are needed. Every week (or day, in a summer course) there will be new ones.

To get them there efficiently, you will need to carefully organize what you learn as you go along. This organized summary will be a *standard, graded part of every homework assignment!*

Your homework will be graded in two *equal* parts. Ten points will be given for a complete crossreferenced summary of the physics concepts used in each of the assigned problems. One problem will be selected for grading in detail – usually one that well-exemplifies the material covered that week – for ten more points.

Points will be taken off for egregiously missing concepts or omitted problems in the concept summary. Don't just name the concepts; if there is an equation and/or diagram associated with the concept, put that down too. Indicate (by number) all of the homework problems where a concept was used.

This concept summary will eventually help you prioritize your study and review for exams! To help you understand what I have in mind, I'm building you a list of the concepts for *this* week, and indicating the problems that (will) need them.

- Writing a vector in cartesian coordinates. For example:

  $$\vec{\boldsymbol{A}} = A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}}$$

  Used in problems 2,3,4,5,6,7,8,9,10,12

- Decomposition of a vector at some angle into components in a (2D) coordinate system. Given a vector $\vec{\boldsymbol{A}}$ with length A at angle θ with respect to the x-axis:

  $$A_x = A\cos(\theta)$$
  $$A_y = A\sin(\theta)$$

  Used in problem 5,6,9,10,11,12

- Definition of trajectory, velocity and acceleration of a particle:

  The trajectory is the vector $\vec{\boldsymbol{x}}(t)$, the vector *position* of the particle as a function of the *time*.

The velocity of the particle is the (vector) rate at which its position changes as a function of time, or the time derivative of the trajectory:

$$\vec{\boldsymbol{v}} = \frac{\Delta \vec{\boldsymbol{x}}}{\Delta t} = \frac{d\vec{\boldsymbol{x}}}{dt}$$

The acceleration is the (vector) rate at which its velocity changes as a function of time, or the time derivative of the velocity:

$$\vec{\boldsymbol{a}} = \frac{\Delta \vec{\boldsymbol{v}}}{\Delta t} = \frac{d\vec{\boldsymbol{v}}}{dt}$$

Used in all problems.

- Inertial reference frame

  A set of coordinates in which (if you like) the laws of physics that describe the trajectory of particles take their simplest form. In particular a frame in which Newton's Laws (given below) hold in a consistent manner. A set of coordinates that is not itself accelerating with respect to all of the other non-accelerating coordinate frames in which Newton's Laws hold.

  Used in all problems (when I choose a coordinate system that is an inertial reference frame).

- Newton's First Law

  In an inertial reference frame, an object in motion will remain in motion, and an object at rest will remain at rest, unless acted on by a net force.

  If $\vec{\boldsymbol{F}} = 0$, then $\vec{\boldsymbol{v}}$ is a constant vector.

  A consequence, as one can see, of Newton's Second Law. Not used much yet.

- Newton's Second Law

  In an inertial reference frame, the net vector force on an object equals its mass times its acceleration.

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

  Used in every problem! **Very important!** Key! Five stars! *****

- Newton's Third Law

  If one object exerts a force on a second object (along the line connecting the two objects), the second object exerts an equal and opposite force on the first.

$$\vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji}$$

  Note used much yet.

- Differentiating x^n

$$\frac{dx^n}{dx} = nx^{n-1}$$

Not used much yet.

- Integrating $x^n dx$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Used in every problem where we implicitly use kinematic solutions to constant acceleration to find a trajectory.

Problems 2

- The force exerted by gravity near the Earth's surface

$$\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{y}}$$

(down).

Used in problems 2,3,4,5,6,8,9,10,11,12

Problems 2

- Centripetal accleration.

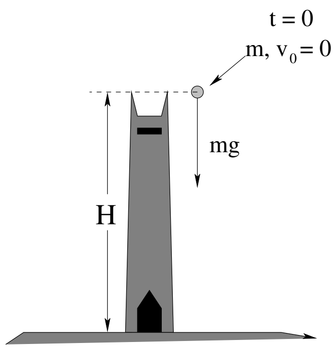
$$a_r = -\frac{v^2}{r}$$

Used in problems 11,13

This isn't a *perfect* example – if I were doing this by hand I would have drawn pictures to accompany, for example, Newton's second and third law, the circular motion acceleration, and so on.

I also included more concepts than are strictly needed by the problems – *don't hesitate* to add important concepts to your list even if none of the problems seem to need them! Some concepts (like that of inertial reference frames) are *ideas* and underlie problems even when they aren't actually/obviously used in an algebraic way in the solution!

Problem 2.



A ball of mass m is dropped at time $t = 0$ from the top of the Duke Chapel (which has height H) to fall freely under the influence of *gravity*.

a) How long does it take for the ball to reach the ground?

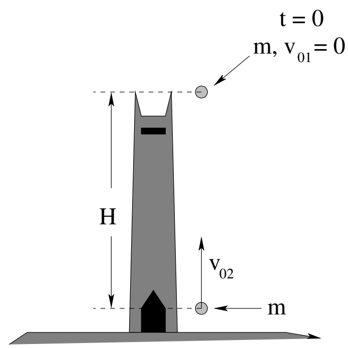
b) How fast is it going when it reaches the ground?

To solve this first problem, be sure that you use the following ritual:

- Draw a *good figure* – in this case a chapel tower, the ground, the ball falling. Label the distance H in the figure, indicate the force on the mass with a vector arrow labelled mg pointing *down*. This is called a *force diagram* (or sometimes a *free body diagram*). Note well! Solutions without a figure will lose points!
- *Choose coordinates!* In this case you could (for example) put an origin at the bottom of the tower with a y-axis going up so that the height of the object is $y(t)$.
- Write Newton's second law for the mass.
- Transform it into a (differential) equation of motion. This is the *math* problem that must be solved.
- In this case, you will want to integrate the *constant* $\frac{dv_y}{dt} = a_y = -g$ to get $v_y(t)$, then integrate $\frac{dy}{dt} = v_y(t))$ to get $y(t)$.
- Express the algebraic condition that is true when the mass reaches the ground, and solve for the *time* it does so, answering the first question.
- Use the answer to the first question (plus your solutions) to answer the second.

The first four steps in this solution will nearly always be the same for Newton's Law problems. Once one has the equation of motion, solving the *rest* of the problem depends on the force law(s) in question, and answering the questions requires a bit of insight that only comes from practice. So practice!

Problem 3.



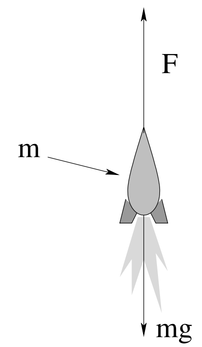
A ball of mass m is dropped at time $t = 0$ from rest ($v_{01} = 0$) from the top of the Duke Chapel (which has height H) to fall freely under the influence of *gravity*. At the same instant, a second ball of mass m is thrown *up* from the ground directly beneath at a speed v_{02} (so that if the two balls travel far enough, fast enough, they will collide). Neglect drag.

a) Draw a free body diagram for and compute the net force acting on each mass **separately**.

b) **From the equation of motion for each mass**, determine their one dimensional trajectory functions, $y_1(t)$ and $y_2(t)$.

c) Sketch *qualitatively correct* graphs of $y_1(t)$ and $y_2(t)$ on the same axes in the case where the two collide.

Problem 4.



A model rocket of mass m is launched vertically from rest at time $t = 0$ being pushed by an engine that produces a constant thrust force F (up). The engine runs for t seconds and then stops. Assume that the mass of the rocket remains more or less unchanged during this time.

a) Find the maximum height that the rocket reaches (neglecting air resistance).

b) Find the total time the rocket is in the air.

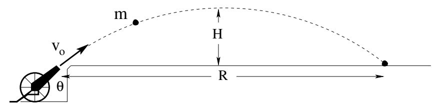
c) Find the speed of the rocket as it hits the ground.

d) Evaluate the numerical value of your *algebraic* answers to a-c if $m = 0.1$ kg, $F = 5$ N, and $t = 3$ seconds. You may use $g = 10$ m/sec^2 (now and for the rest of the course) for simplicity.

Discuss in your recitation groups: Are the assumptions (unchanging mass, constant thrust, no drag, $g = 10$ m/sec^2) *reasonable*? When should they be approximately true, when should they be very poor? Many physics problems you will solve this semester are *idealizations* – real world problems where we neglect this or that force or mass because they are (we hope) small enough to be "negligible". In some cases we do this because we could not solve the *actual* problem with a pen and piece of paper otherwise – we would have to use a computer or solve complex systems of differential equations to get an answer. In other cases the neglected elements really are negligible, and you would have to work very hard to detect a deviation from the "ideal" answer.

Try to think about how *reasonable* simplifying/ideal assumptions are throughout the course; as your skills improve, things we initially idealize away (like friction and drag) will be included. In other places we will be able to see *how* we could do better (but might shy away in an intro course from doing so).

Problem 5.



A cannon sits on a horizontal plain. It fires a cannonball of mass m at speed v_0 at an angle θ relative to the ground. Find:

a) The maximum height H of the cannonball's trajectory.

b) The time t_a the cannonball is in the air.

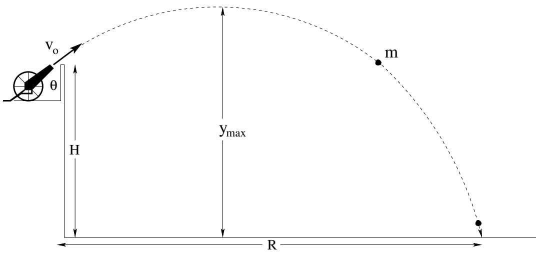
c) The range R of the cannonball.

Questions to discuss in recitation: How does the time the cannonball remains in the air depend on its maximum height? If the cannon is fired at different angles and initial speeds, does the cannonballs with the greatest range always remain in the air the longest? Use the trigonometric identity:

$$2\sin(\theta)\cos(\theta) = \sin(2\theta)$$

to express your result for the range. For a fixed v_0, how many angles (usually) can you set the cannon to that will have the same range?

Problem 6.



A cannon sits on at the top of a rampart of height (to the mouth of the cannon) H. It fires a cannonball of mass m at speed v_0 at an angle θ relative to the ground. Find:

a) The maximum height $y_{\max}$ of the cannonball's trajectory.

b) The time the cannonball is in the air.

c) The range of the cannonball.

Discussion: In your solution to 2) above you should have found *two* times, one of them negative. What does the negative time correspond to? (Does our mathematical solution "know" about the actual prior history of the cannonball?

You might find the quadratic formula useful in solving this problem. We will be using this *a lot* in this course, and on a quiz or exam you won't be given it, so be sure that you *really* learn it now in case you don't know or have forgotten it. The roots of a quadratic:

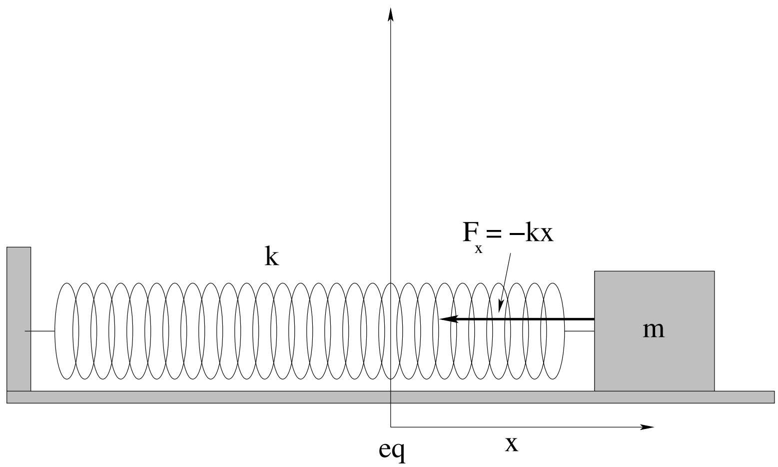
$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can actually derive this for yourself if you like (it helps you remember it). Just divide the whole equation by a and complete the square by adding and subtracting the right algebraic quantities, then factor.

Problem 7.



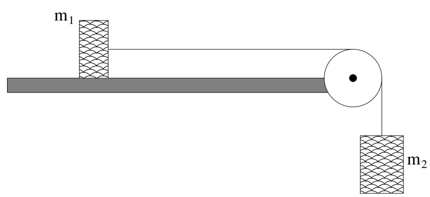
A mass m on a frictionless table is connected to a spring with spring constant k (so that the force on it is $F_x = -kx$ where x is the distance of the mass from its equilibrium postion. It is then pulled so that the spring is stretched by a distance x from its equilibrium position and at $t = 0$ is released.

Write Newton's Second Law and solve for the acceleration. Solve for the acceleration and write the result as a *second order, homogeneous differential equation* of motion for this system.

Discussion in your recitation group: Based on your experience and intuition with masses on springs, how do you expect the mass to move in time? Since $x(t)$ is not constant, and a is proportional to $x(t)$, a is a *function of time*! Do you expect the solution to resemble the kinds of solutions you derived in constant acceleration problems above *at all*?

The moral of this story is that not everything moves under the influence of a constant force! If the force/acceleration vary in time, we *cannot* use e.g. $x(t) = \frac{1}{2}at^2$! Yet this is a very common mistake made by intro physics students. Try to make sure that you are not one of them!

Problem 8.



A mass m_1 is attached to a second mass m_2 by an Acme (massless, unstretchable) string. m_1 sits on a frictionless table; m_2 is hanging over the ends of a table, suspended by the taut string from an Acme (frictionless, massless) pulley. At time $t = 0$ both masses are released.

a) Draw the force/free body diagram for this problem.

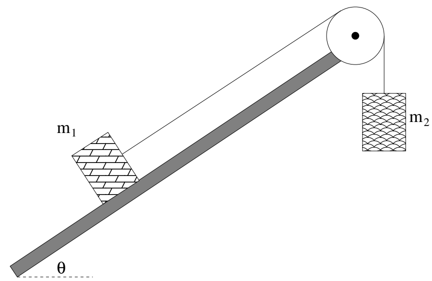
b) Find the acceleration of the two masses.

c) The tension T in the string.

d) How fast are the two blocks moving when mass m_2 has fallen a height H (assuming that m_1 hasn't yet hit the pulley)?

Discussion: Your answer should look something like: The *total unopposed* force acting on the system accelerates *both* masses. The string just *transfers* force from one mass to the other so that they accelerate *together!* This is a common feature to many problems involving multiple masses and internal forces, as we'll see and eventually formalize. Also, by this point you should be really internalizing the ritual for finding the speed of something when it hits – find the time, backsubstitute to find the speed/velocity. We could actually do this once and for all (algebraically) for constant accelerations and derive a formula that saves these steps1 However, very soon we will formally *eliminate* time from Newton's Second Law, and the result (work and energy) will work even for *non*-constant forces and accelerations, so don't do this yet.

Problem 9.



A mass m_1 is attached to a second mass $m_2 > m_1$ by an Acme (massless, unstretchable) string. m_1 sits on a frictionless inclined plane at an angle θ with the horizontal; m_2 is hanging over the high end of the plane, suspended by the taut string from an Acme (frictionless, massless) pulley. At time $t = 0$ both masses are released from rest.

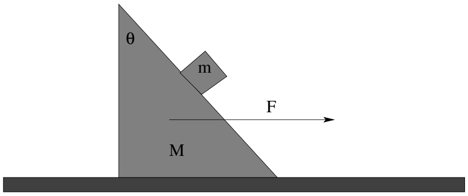
a) Draw the force/free body diagram for this problem.

b) Find the acceleration of the two masses.

c) Find the tension T in the string.

d) How fast are the two blocks moving when mass m_2 has fallen a height H (assuming that m_1 hasn't yet hit the pulley)?

Problem 10.

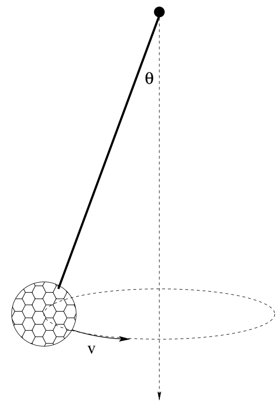


A block m is sitting on a frictionless inclined block with mass M at an angle θ_0 as shown. With what force F should you push on the large block in order that the small block will remain motionless with respect to the large block and neither slide up nor slide down?

Discussion: When your car or a jet airplane accelerates, you are "thrown back in your seat" – it feels like gravity has gotten stronger and points somewhat behind you instead of straight down. When the lower block accelerates the upper one, "gravity" for it starts to point behind it and down from a certain point of view (as we shall shortly see). What is the vector sum of the force of gravity and $-m\vec{a}$ for the upper block? In what direction does it point? Hmmmm....

BTW, I made the angle θ_0 sit in the upper corner just to annoy you and make you actually *think* about sines and cosines of angles. This is good for you – don't just memorize the trig for an inclined plane, understand it! Talk about it in your groups until you do!

Problem 11.



A tether ball of mass m is suspended by a rope of length L from the top of a pole. A youngster gives it a whack so that it moves with some speed v in a circle of radius $r = L\sin(\theta) < L$ around the pole.

a) Find an expression for the tension T in the rope as a function of m, g, and θ.

b) Find an expression for the speed v of the ball as a function of θ.

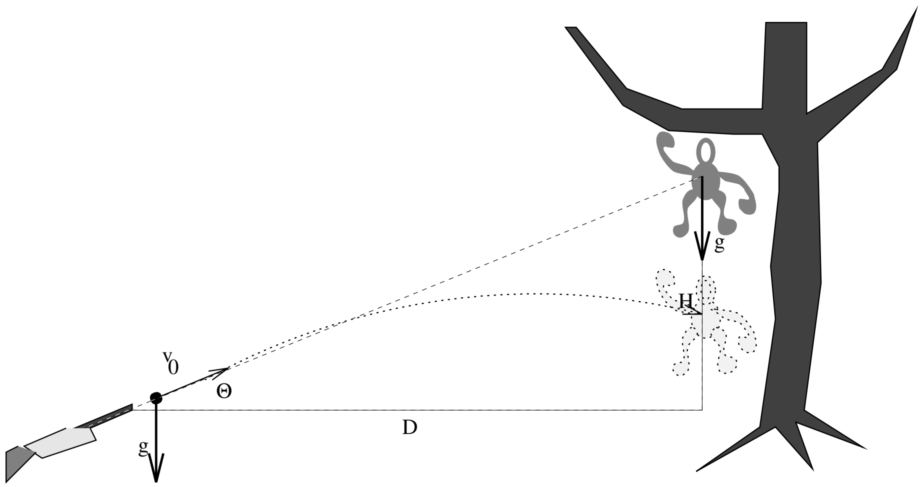
Discussion: Why don't you need to use L or v in order to find the tension T? Once the tension T is known, how does it constrain the rest of your solution?

By now you should have covered, and understood, the derivation of the True Fact that ***if*** a particle is moving in a circle of radius r, it *[***must***]* have a ***total*** acceleration towards the center of the circle of:

$$a_c = \frac{v^2}{r}$$

This acceleration (or rather, the acceleration times the mass, ma_c) is not a force!. The force that *produces* this acceleration has to come from the many *real forces of nature* pushing and pulling on the object (in this case tension in the string and gravity).

Problem 12.



A hunter aims his gun directly at a monkey in a distant tree. Just as she fires, the monkey lets go and drops in free fall towards the ground.

Show that the bullet still hits the monkey.

Discussion: There are some unspoken assumptions in this problem. For example, if the gun shoots the bullet too slowly, what will *really* happen? Also, real guns fire a bullet so fast that the trajectory is quite flat. Also, real hunters allow for the drop in their bullet and would fire above the monkey to hit it if it did *not* drop (the default assumption).

Be at peace. No real monkeys were harmed in this problem.

Problem 13.

A toy train engine of mass m is chugging its way around a circular track with radius R at a constant speed v. Draw a free body/force diagram for the train showing *all* of the forces acting on the train. Evaluate the *total vector force* acting on the engine as a function of its speed.

Discussion: Which forces are forces of constraint? Which force is responsible for making the train go in a circle? If you know how a train's wheel's are shaped and ride on the track draw a picture and draw forces that illustrate how a rail exerts both components of the force needed to hold a train up and curve its trajectory around in a circle on the interlocking wheels. If you *don't* know how a train's wheels are shaped, and how they fit on and inside of the rails, go find a train and take a look, or google up some good pictures and find out!

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

No optional problems this week.

Week 2: Newton's Laws: Continued

Examples

The following examples are typical physics problems not unlike what you might encounter on a quiz or hour exam. A number of them are selected to be your *homework* problems – examples you should simply know how to do, because if you do you will know how to solve many related or similar problems as well.

1) **Classsic inclined plane**

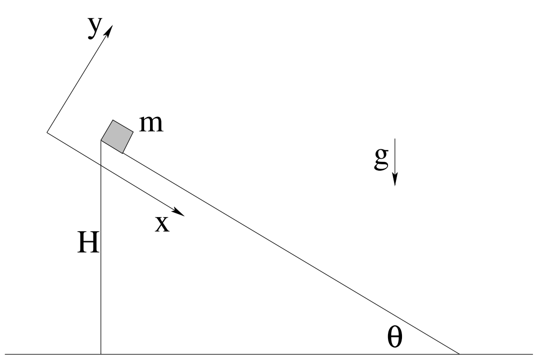


Figure 4:

$$\sum F_x = mg\sin(\theta) = ma_x \qquad (45)$$
$$\sum F_y = N - mg\cos(\theta) = ma_y = 0 \qquad (46)$$

or

$$a_x = g\sin(\theta) \qquad (47)$$
$$a_y = 0 \qquad (48)$$

With this in hand, we can now evaluate the trajectory of the particle (using the solutions developed for one dimensional motion) sliding on the incline from a knowledge of its initial conditions.

2) **Classic pulley problem** (or Atwood's machine).

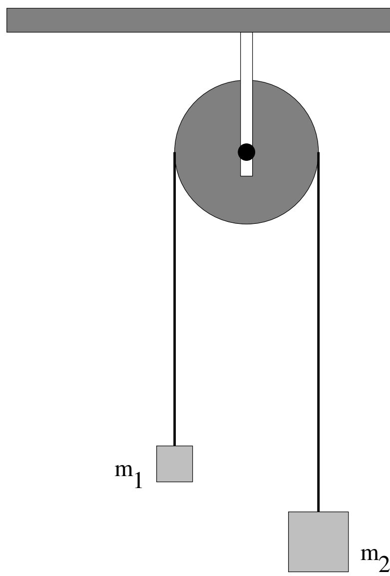


Figure 5:

We imagine the motion to be "one-dimensional" along a line wrapped around the *massless* pulley. (Don't worry, we do the same problem with a massive pulley later.) We then get (for each mass):

$$\sum F_1 = m_1 g - T = m_1 a \tag{49}$$

$$\sum F_2 = T - m_2 g = m_2 a. \tag{50}$$

If we solve these equations simultaneously, (by adding them, for example) we get:

$$m_1 g - m_2 g = m_1 a + m_2 a \tag{51}$$

$$a = \frac{m_1 - m_2}{m_1 + m_2} g \tag{52}$$

Again, we can now evaluate the trajectory of the pair of masses (using the solutions developed for one dimensional motion) rolling around the pulley from a knowledge of the initial conditions.

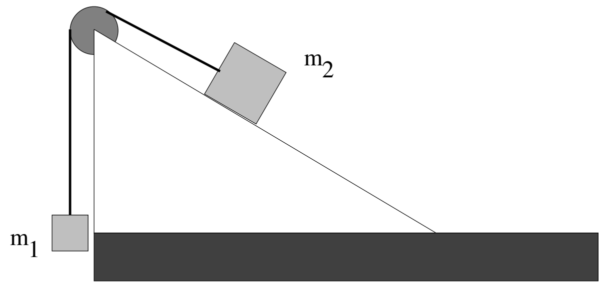


Figure 6:

3) **Inclined plane with pulley**

The pulley is massless, the plane frictionless. Combining the methods of the two problems, you should be able to show that:

$$m_2 g - m_1 \sin(\theta) g = m_1 a + m_2 a \tag{53}$$

so that

$$a = \frac{m_2 - m_1 \sin(\theta)}{m_1 + m_2} g \tag{54}$$

From a knowledge of a, you should be able to find T in the string, too.

4) **Spring and mass in equilibrium**

We will learn how to really *solve* this problem later. Right now we will just write down Newton's laws for this problem so we can find a. Let the x direction be *up*. Then:

$$\sum F_x = -k(x - x_0) - mg = ma_x \tag{55}$$

or (with $\Delta x = x - x_0$, so that Δx is negative as shown)

$$a_x = -\frac{k}{m}\Delta x - g \tag{56}$$

5) **Tether Ball** at angle θ, string of length L. Find T and v of ball.

We note that if the ball is moving in a circle of radius $r = L\sin\theta$, its centripetal acceleration *must* be $a_r = -\frac{v^2}{r}$. Since the ball is not moving up and down, the vertical forces must cancel. The only "real" forces acting on the ball are gravity and the tension T in the string. Thus in the y-direction we have:

$$\sum F_y = T\cos\theta - mg = 0 \tag{57}$$

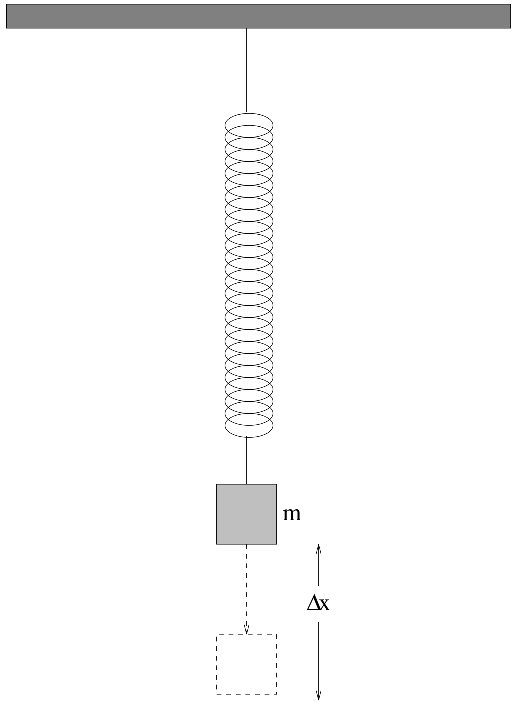


Figure 7:

and in the x-direction (the minus r-direction, as drawn) we have:

$$\sum F_r = T\sin\theta = ma_r = \frac{mv^2}{r}. \tag{58}$$

Thus

$$T = \frac{mg}{\cos\theta}, \tag{59}$$

$$v^2 = \frac{Tr\sin\theta}{m} \tag{60}$$

or

$$v = \sqrt{gL\sin\theta\tan\theta} \tag{61}$$

6) **Weight in an elevator**

Apparent weight in an elevator is pretty easy. The evevator accelerates up (or down) at some net rate a_y. Now, if you are riding in the elevator, you *must* be accelerating up with the *same* acceleration. Therefore the net force on you *must* be

$$\sum F_y = ma_y \tag{62}$$

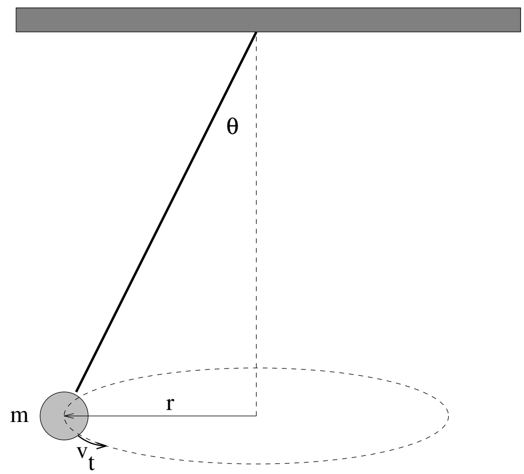


Figure 8:

That net force is made up of two "real" forces: The force of gravity which pulls you down, and the (normal) force exerted by all the molecules in the scale on the soles of your feet. This latter force is what the scale (which reads the net force applied to/by its top surface, remember Newton's third law) indicates as your "weight".

Thus:

$$\sum F_y = ma_y = N - mg \tag{63}$$

or

$$N = w = mg + ma_y \tag{64}$$

Friction and Drag Forces

So far, our picture of natural forces as being the *cause* of the acceleration of mass seems fairly successful. In time it will become second nature to you; you will intuitively seek a force behind all forms of change. However, our description thus far is fairly simplistic – we have massless strings, frictionless tables, drag-free air. That is, we are neglecting certain well-known and important forces that appear in real-world problems in order to concentrate on "ideal" problems that illustrate the methods simply.

It is time to restore some of the complexity to the problems we solve. The first thing we will add is friction.

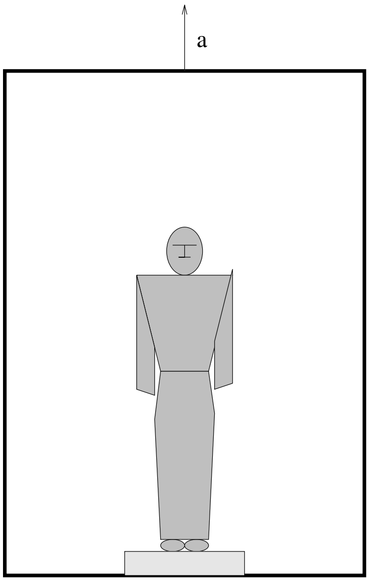


Figure 9:

Friction

Experimentally

1) $f_{s,\max} \leq \mu_s |N|$. The force exerted by *static friction* is less than or equal to the *coefficient of static friction* times the magnitude of the normal force exerted on the entire (homogeneous) surface of contact. It opposes any applied force to make the total force zero as long as it is able to do so.

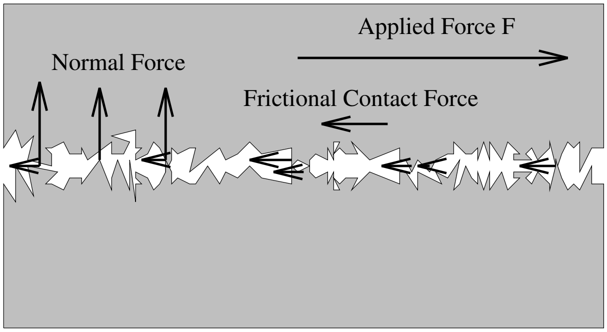
2) $f_k = \mu_k |N|$. The force exerted by *kinetic friction* (produced by two surfaces rubbing against each other in motion) is equal to the *coefficient of kinetic friction* times the magnitude of the normal force exerted on the entire (homogeneous) surface of contact. It opposes the direction of the motion of the two surfaces in such a way as to decrease the velocity along the surface of contact.

3) $\mu_k < \mu_s$

4) $\mu_k(v)$, but for "slow" speeds $\mu_k \sim$ constant.

5) μ_s and μ_k depend on materials, but *independent* of contact area.

We can understand this last observation by noting that the frictional force should depend on the *pressure* (the normal force/area $\equiv$ N/m^2) times the *area* in contact. But



then

$$f_k = \mu_k P \times A = \mu_k \frac{N}{A} \times A = \mu_k N \tag{65}$$

and we see that the frictional force will depend only on the total force, not the area.

Example 2.0.1: Inclined plane of length L with friction

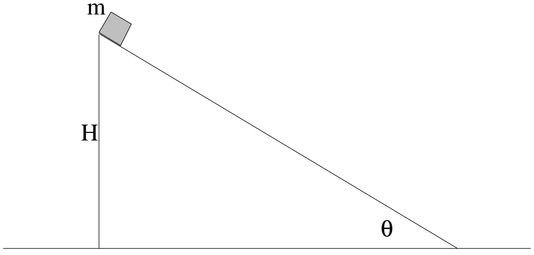


Figure 10: Block on inclined plane with both static and dynamic friction.

1) At what angle does the block barely overcome the force of static friction?
2) Started at rest from an angle greater than this, how fast is the block going when it reaches the bottom?

Static friction exerts *as much force as necessary* to keep the block at rest up to the maximum it can exert, $f_s(\max) = \mu_s N$. If we decompose the force into components parallel to and perpendicular to the plane, and note that the components perpendicular to the plane must cancel, we get:

$$\sum F_{||} = mg \sin\theta - \mu_s N = 0 \tag{66}$$

(for $\theta \leq \theta_{\rm max}$) and

$$\sum F_\perp = N - mg\cos\theta = 0 \qquad (67)$$

From the latter,

$$N = mg\cos\theta \qquad (68)$$

so that the first equation becomes:

$$\sum F_{||} = mg\sin\theta_{\rm max} - \mu_s mg\cos\theta_{\rm max} = 0 \qquad (69)$$

at the maximum possible angle. Solving for the angle, we get:

$$\theta_{\rm max} = \tan^{-1}\mu_s \qquad (70)$$

Once it is moving, $F_{||} > 0$ (down the plane) so we get:

$$\sum F_{||} = mg\sin\theta - \mu_s mg\cos\theta = ma_{||} \qquad (71)$$

(for $theta > theta_{\rm max}$) or

$$a_{||} = g\sin\theta - \mu_s g\cos\theta \qquad (72)$$

from which we can find whatever we like from the methods of chapters 1-3.

Example 2.0.2: Block hanging off of table with pulley pulls another block

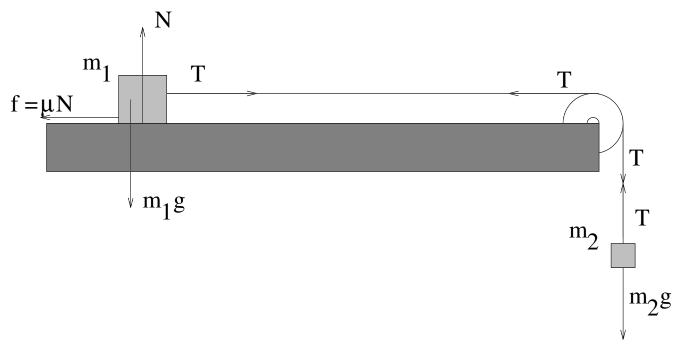


Figure 11: Atwood's machine, sort of, with one block on table with friction.

A block of mass $m_1 = 4$ kg sits on a table with $\mu_s = 0.5$, $\mu_k = 0.4$. It is attached by a massless unstretchable string running over a pulley to a block of mass m hanging off of a table. a) What is the largest mass that m can be before the system starts to move? b) Suppose that $m = 4$ kg. Describe the subsequent motion (find a).

The system is effectively one dimensional, with the string and pulley serving to "bend" the force around corners without loss. Just a smattering of what you need to solve the problem (I'm leaving some of it undone):

$$\sum F_{y1} = N - m_1 g = 0 \quad (73)$$

$$\sum F_{x1} = T - \mu_s N = 0 \quad (74)$$

$$\text{or} \sum F_{x1} = T - \mu_k N = m_1 a \quad (75)$$

$$\sum F_{y2} = T - m_2 g = m_2 a \quad (76)$$

$$(77)$$

One finds N, puts it in one of the next two equations (depending on what you are finding), and solves it and the last equation simultaneously (eliminating T) to find either m_2 or a and T. Once you have a, you own the solution. . .

Example 2.0.3: Find minimum braking distance for car stopping with and without skidding

Figure 12: Car stopping with and without skidding.

Find the minimum braking distance of a car travelling at 30 m/sec running on tires with $\mu_s = 0.5$ and $\mu_k = 0.3$:

1) equipped with ABS such that the tires do not skid, but rather roll (so that they exert the maximum static friction only);
2) the same car, but without ABS and with the wheels locked in a skid (kinetic friction only)

It is simplest to use $2a_x\Delta x = v_f^2 - v_0^2$, with $v_f = 0$ and $\Delta x = D$, the stopping distance.

The only forces exerted are gravity, the normal force, and the force of static (ABS) or kinetic (no-ABS) friction. To find a_x, we use Newton's Law:

$$\sum_x F_x = -\mu_{(s,k)} N = ma_x \tag{78}$$

$$\sum_y F_y = N - mg = ma_y = 0 \tag{79}$$

(where μ_s is for static friction and μ_k is for kinetic friction), or:

$$ma_x = -\mu_{(s,k)} mg \tag{80}$$

so

$$a_x = -\mu_{(s,k)} g \tag{81}$$

Thus

$$-2\mu_{(s,k)} g D = -v_0^2 \tag{82}$$

or

$$D = \frac{v_0^2}{2\mu_{(s,k)} g} \tag{83}$$

Numerically, $D_{\mathrm{ABS}} = 90$ m, and $D_{\mathrm{normal}} = 150$ m. Another nice feature of ABS is that when fully applied they will always stop your car in "exactly" the minimum distance. I'm spending \$800 for ABS for *my* next car…

Example 2.0.4: Car rounding a banked curve with friction

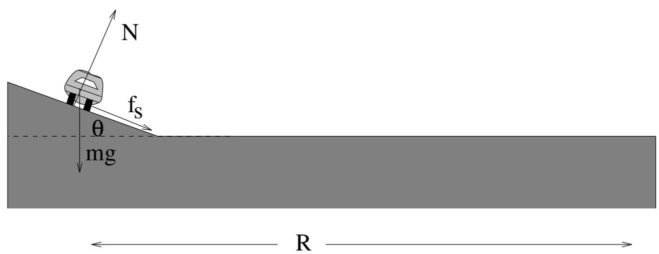


Figure 13: Friction and the normal force conspire to accelerate car towards the center of the circle in which it moves. Be sure to choose the right coordinates for this sort of problem!

A car is rounding a curve of $R = 50$m banked at an angle of 20○. The car is travelling at $v = 30$ m/sec. Find:

1) the normal force exerted by the road on the car;

2) the force of static friction exerted by the road on the tires.

Note that we don't know f_s; it is presumably less than $\mu_s N$, but we have to solve for N to be sure. We pick a direction for it down the plane, although if v is less than a certain value it will point up the plane (and our answer should reflect that). We use coordinates lined up with the eventual direction of $\vec{\boldsymbol{F}}_{tot}$: +x parallel to the ground (and R). We write Newton's second law:

$$\sum_x F_x = N\sin\theta + f_s\cos\theta = ma_x = \frac{mv^2}{R} \tag{84}$$

$$\sum_y F_y = N\cos\theta - mg - f_s\sin\theta = ma_y = 0 \tag{85}$$

and solve the y equation for N:

$$N = \frac{mg + f_s\sin\theta}{\cos\theta} \tag{86}$$

substitute into the x equation:

$$(mg + f_s\sin\theta)\tan\theta + f_s\cos\theta = \frac{mv^2}{R} \tag{87}$$

and finally solve for f_s:

$$f_s = \frac{\frac{mv^2}{R} - mg\tan\theta}{\sin\theta\tan\theta + \cos\theta} \tag{88}$$

From this we see that if

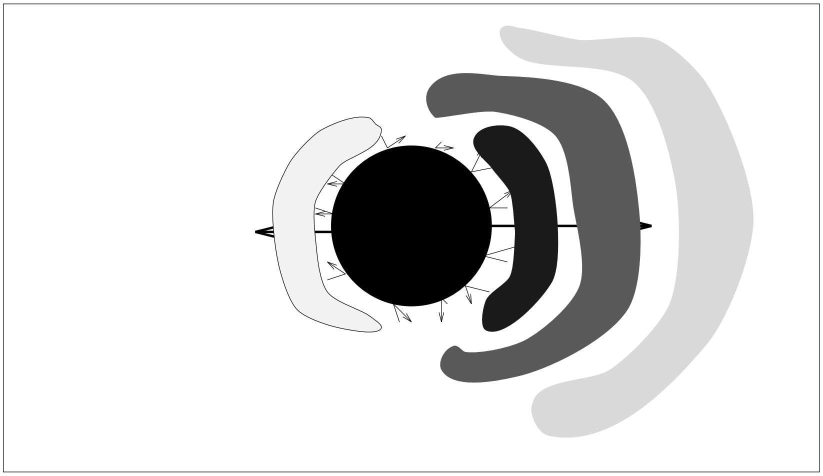
$$\frac{mv^2}{R} > mg\tan\theta \tag{89}$$

or

$$\frac{v^2}{Rg} > \tan\theta \tag{90}$$

then f_s is positive (down the incline), otherwise it is negative (up the incline). When $\frac{v^2}{Rg} = \tan\theta$, $f_s = 0$ and the car would round the curve even on ice.

Other questions I *might* have asked: Within what range of speeds will the car be able to round the curve, given $\mu_s = 0.3$? If $\mu_s = 0.5$, what is the smallest (largest) angle for which the car can round the curve?



Drag Forces

When an object is sitting at rest, the air around it presses on it fairly equally on all sides, as molecules on one side hit it, on the average, with as much force/area as molecules on the other side and the total cross-sectional area of the object seen from either side is the same (we'll analyze that statement in more detail later).

When the same object is moving with respect to the air (or the air is moving with respect to the object, i.e. – there is a wind) then the molecules hit on the side facing the direction of motion harder, on the average, then molecules on the other side and the recoil of these molecules then exerts, according to Newton's third law, an unbalanced force on the object. This "frictional force" of contact with the air is called a *drag force*, and we must account for it in our force diagrams of object moving in the air whenever the force it exerts is relevant (commensurate with the other forces in play).

Experimentally:

1) $\left|\vec{\boldsymbol{F}}_{\rm drag}\right| \propto v^n \approx bv^n$. The drag force is approximately proportional to some power of the speed of the object.

2) The drag force of a medium at rest always *opposes* the velocity; that is, it acts to slow the object down. If the medium is moving (that is, there is a wind or current) then the drag force tends to accelerate the object toward the speed of the medium.

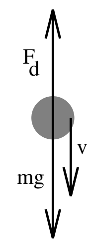
3) At low speeds, $n \approx 1$ and it tends toward $n = 2$ at high speeds.

We thus express the force law for as:

$$\vec{\boldsymbol{F}}_d = -b\frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}\,|\vec{\boldsymbol{v}}|^n \tag{91}$$

(to be really picky and formally correct). We'll frequently just write this as:

$$F_d = -bv^n \tag{92}$$



and keep track of the direction by hand.

Terminal velocity

One immediate consequence of this is that objects dropped in a graviational field in air do not just keep speeding up *ad infinitum*.

When first we drop an object it speeds up, but as it does so the drag force on it (opposing the increase caused by gravity) also increases. Eventually the two forces balance and the object drops at a constant speed. We call this speed *terminal velocity*. We see that (in the negative y-direction):

$$F_T = mg - bv_t^n = m\frac{dv}{dt} = 0 \tag{93}$$

when terminal velocity is reached or:

$$v_t = \left(\frac{mg}{b}\right)^{1/n} \tag{94}$$

For the simple case $n = 1$ we can actually solve the full equation of motion for at least the velocity as a function of time:

$$mg - bv = m\frac{dv}{dt} \tag{95}$$

or

$$\begin{aligned}
\frac{dv}{dt} &= g - \frac{b}{m}v && (96)\\
\frac{dv}{g - \frac{b}{m}v} &= dt && (97)\\
-\frac{m}{b}\int \frac{-\frac{b}{m}dv}{(g - \frac{b}{m}v)} &= \int dt && (98)\\
-\frac{m}{b}\ln\left(g - \frac{b}{m}v\right) &= t + C && (99)\\
g - \frac{b}{m}v &= Ce^{-\frac{b}{m}t} && (100)\\
v(t) &= \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right) && (101)
\end{aligned}$$

or

$$v(t) = v_t\left(1 - e^{-\frac{b}{m}t}\right) \tag{102}$$

(where the last step is for $v(0) = 0$ initial conditions). This equation describes exponential approach to a constant (terminal) velocity. There are various other ways to solve this differential equation (one of which I covered in class), but this one suffices. Note that if $n \neq 1$, the equation is also directly integrable, but the approach to a constant velocity will be algebraic, not exponential.

Example 2.0.5: Dropping the Ram

The UNC ram of mass 100 kg is carried by some naughty (but intellectually curious) Duke students up in an airplane to a height of 5000 m and thrown out. On the ground below a student armed with a radar gun measures and records the velocity of the ram as it plummets toward the vat of dark blue paint below[29]. Assuming that the fluffy, cute little ram experiences a drag force on the way down of $-bv$ with $b = 10kg/sec$:

1) Describe qualitatively the measurements recorded by the student.

2) Approximately how fast is the fat, furry creature going when it splashes into the paint, more or less permanently dying it dark blue?

The student sees an exponential approach to terminal velocity:

$$v(t) = \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right) \tag{103}$$

with $v_t = v(\infty) = \frac{mg}{b}$. This fall is far enough to "reach" $v_t = 100$ m/sec, since it takes many times $\frac{m}{b} = 10$ seconds to fall 5000 m.

[29]Note well: No sheep are harmed in this physics problem...

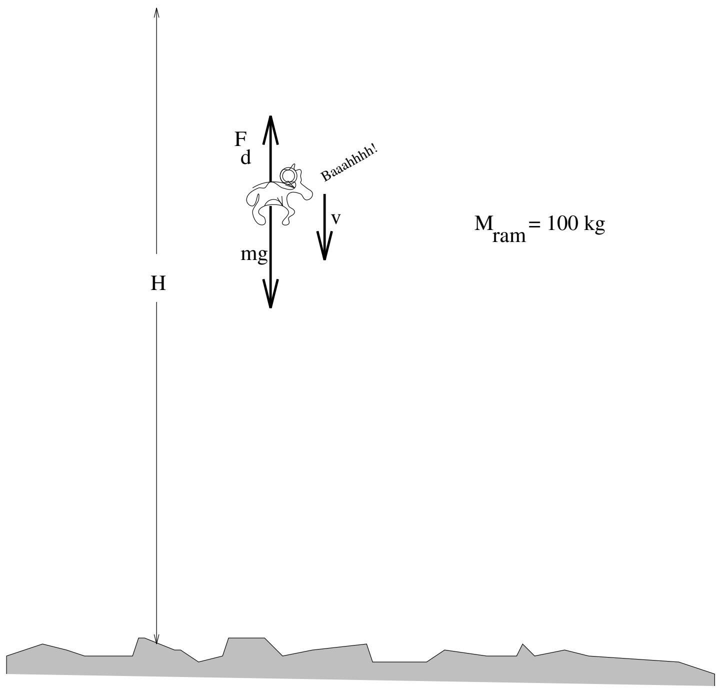


Figure 14: The UNC Ram is dropped from an airplane to measure its terminal velocity on the way down.

Pseudoforces

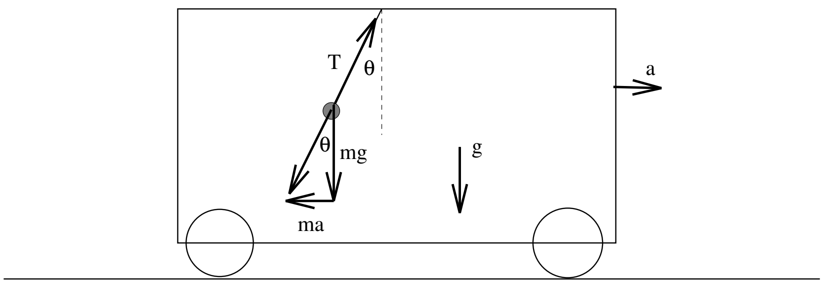
Pseudoforces are forces which aren't really there. Why, then, you might well ask, do we deal with them? The answer is because it is psychologically and occassionally computationally useful to do so.

Pseudoforces (think of them as Sears forces, not real forces – F. Zappa) are useful to describe the apparent total force acting on an object *in a non-inertial reference frame*. A non-inertial reference frame is one that is *accelerating*. Since the object may be accelerating inside the frame because of real forces acting on it, and the frame itself is accelerating, we have to add the two of these things to get the acceleration *relative* to the frame. The (mass times the) acceleration of the frame thus *looks* like an additional force that is acting on the object in the frame to produce its total acceleration in frame coordinates.

Various pseudoforces we will encounter in problems in the next few weeks are:

1) The force added or subtracted to a real force (i.e. – mg, or a normal force) in a frame accelerating uniformly (see picture above).

2) The "centrifugal" force that apparently acts on an object in a rotating frame. Note that this is just minus the real centripetal force that pushes the object toward the center. The centrifugal force is the normal force that a scale might read as it



provides the centripetal push.

3) Rotating frames actually account for lots of pseudoforces. There are also pseudoforces acting on objects falling towards or away from a rotating sphere. These forces describe the apparent deflection of a particle as its straight-line trajectory falls ahead of or behind the rotating frame (in which the "rest" velocity is a function of ω and r).

4) Finally, objects moving north or south along the surface of a rotating sphere also experience a similar deflection, for similar reasons. As a particle moves towards the equator, it is suddenly travelling too slowly for its new radius (and constant ω) and is apparently "deflected" west. As it travels away from the equator it is suddenly traveling too fast for its new radius and is deflected east. These effects combine to produce clockwise rotation of large air masses in the northern hemisphere and anticlockwise rotations in the southern hemisphere.

   Note Well: Hurricanes rotate *counterclockwise* in the northern hemisphere because the counterclockwise winds meet to circulate the other way around a *defect* at the center. This defect is called the "eye". Winds flowing into a center have to *go* somewhere. At the defect they must go up or down. In a hurricane the ocean warms air that rushes toward the center and rises. This warm wet air dumps (warm) moisture and cools. The cool air circulates far out and gets pulled back along the ocean surface, warming as it comes in. A hurricane is a *heat engine!*

5) The two forces just mentioned (pseudoforces in a rotating frame) are commonly called *coriolis forces* and are a major driving factor in the time evolution of weather patterns. They also complicate naval artillery trajectories, missile launches (as the coriolis forces combine with drag forces to produce very real and somewhat unpredictable deflections), and will one day make pouring a drink in a space station an interesting process (hold the cup just a bit antispinward, as things will not – apparently – fall in a straight line!).

Just For Fun: Hurricanes

Hurricanes are of great interest, at least in the Southeast United States where every fall one or more of them (on average) make landfall somewhere on the Atlantic or Gulf coast between Texas and North Carolina. Since they not infrequently do billions of dollars worth of damage and kill dozens of people (usually drowned due to flooding) it is worth taking a second to look over their Coriolis dynamics.

In the northern hemisphere, air circulates around high pressure centers in a generally clockwise direction as cool dry air "falls" out of them in all directions, deflecting west as it flows out south and east as it flow out north.

Air circulates around low pressure centers in a counterclockwise direction as air rushes to the center, warms, and lifts. Here the eastward deflection of north-travelling air *meets* the west deflection of south-travelling air and creates a whirlpool spinning opposite to the far curvature of the incoming air (often flowing in from a circulation pattern around a neighboring high pressure center).

If this circulation occurs over warm ocean water it picks up considerable water vapor and heat. The warm, wet air cools as it lifts in the central pattern of the low and precipitation occurs, releasing the energy of fusion into the rapidly expanding air as wind flowing *out* of the low pressure center at high altitude in the usual clockwise direction (the "outflow" of the storm). If the low remains over warm ocean water and no "shear" winds blow at high altitude across the developing eye and interfere with the outflow, a stable pattern in the storm emerges that gradually amplifies into a hurricane with a well defined "eye" where the air has very low pressure and no wind at all.

The details of the dynamics and energy release are only gradually being understood by virtue of intense study. An http://www.aoml.noaa.gov/hrd/tcfaq/A11.html on the Atlantic Oceanographic and Meteorological Laboratory's Hurricane http://www.aoml.noaa.gov/hrd/ website contains a lovely description of the structure of the eye and the inflowing rain bands.

Atlantic hurricanes *usually* move from southeast to northwest in the southern latitudes until they hook away to the north or northeast. Often they sweep away into the north atlantic to die as mere extratropical storms without ever touching land. When they do come ashore, though, they can pack winds well over a hundred miles an hour (faster than terminal velocity and thereby enough to lift a human right off their feet or a house right off its foundations) and can drop a foot of rain in a matter of hours across tens of thousands of square miles. The latter causes massive floods that are the most common cause of loss of life in hurricanes.

Hurricanes also can form in the Gulf of Mexico, the Carribean, or even the waters of the Pacific close to Mexico. Those hurricanes tend to be highly unpredictable in their behavior as they bounce around between surrounding air pressure ridges and troughs.

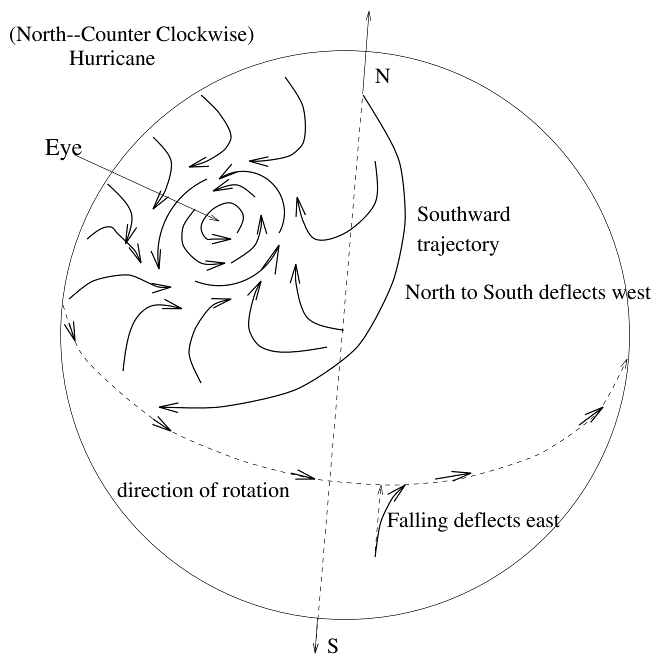


Figure 15: Coriolis dynamics associated with tropical storms. Air circulating clockwise (from surrounding higher pressure regions) meets at a center of low pressure and forms a counterclockwise "eye".

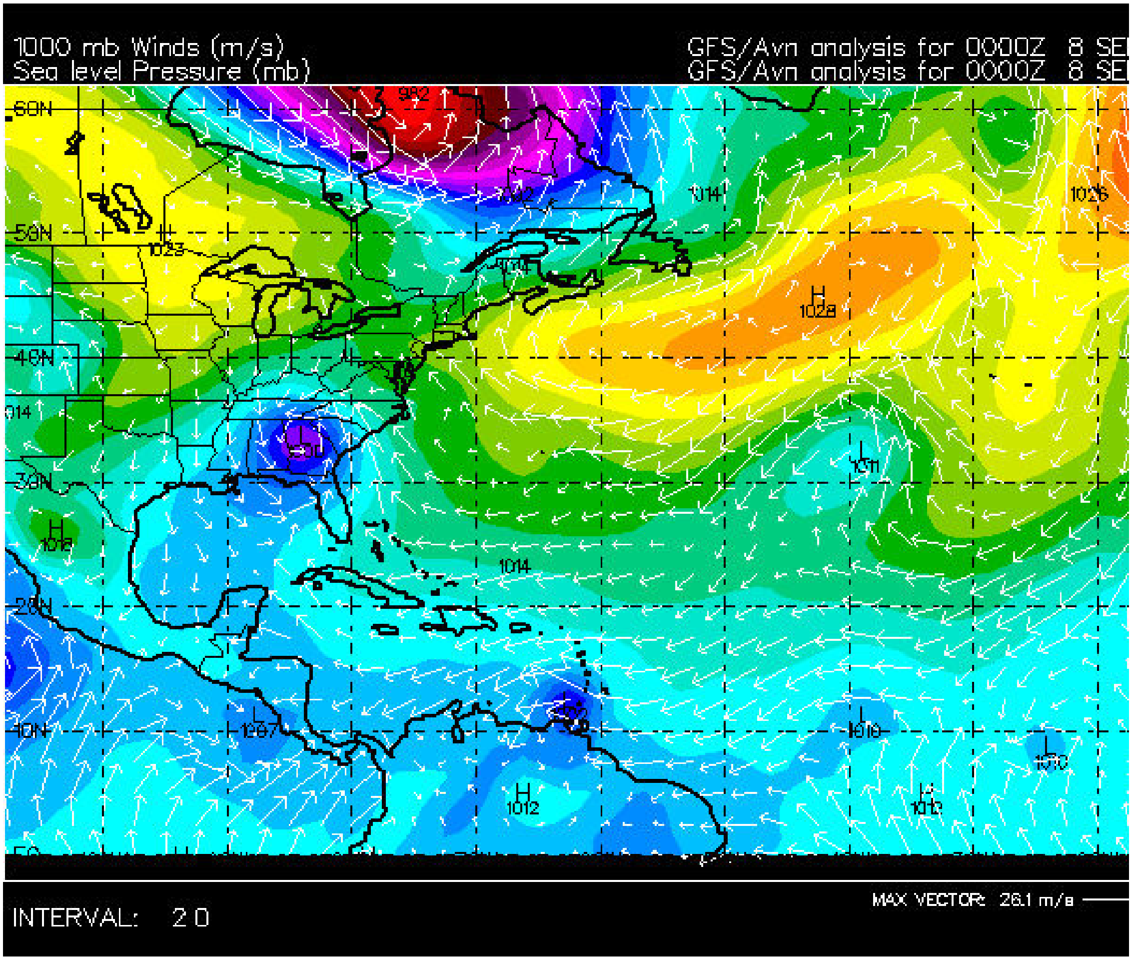


Figure 16: Pressure/windfield of the Atlantic on September 8, 2004. Two tropical storms are visible – the remnants of Hurricane Frances poised over the U.S. Southeast, and Hurricane Ivan just north of South America. Two more low pressure "tropical waves" are visible between South America and Africa – either or both could develop into tropical storms if shear and water temperature are favorable. The low pressure system in the middle of the Atlantic is extratropical and very unlikely to develop into a proper tropical storm.

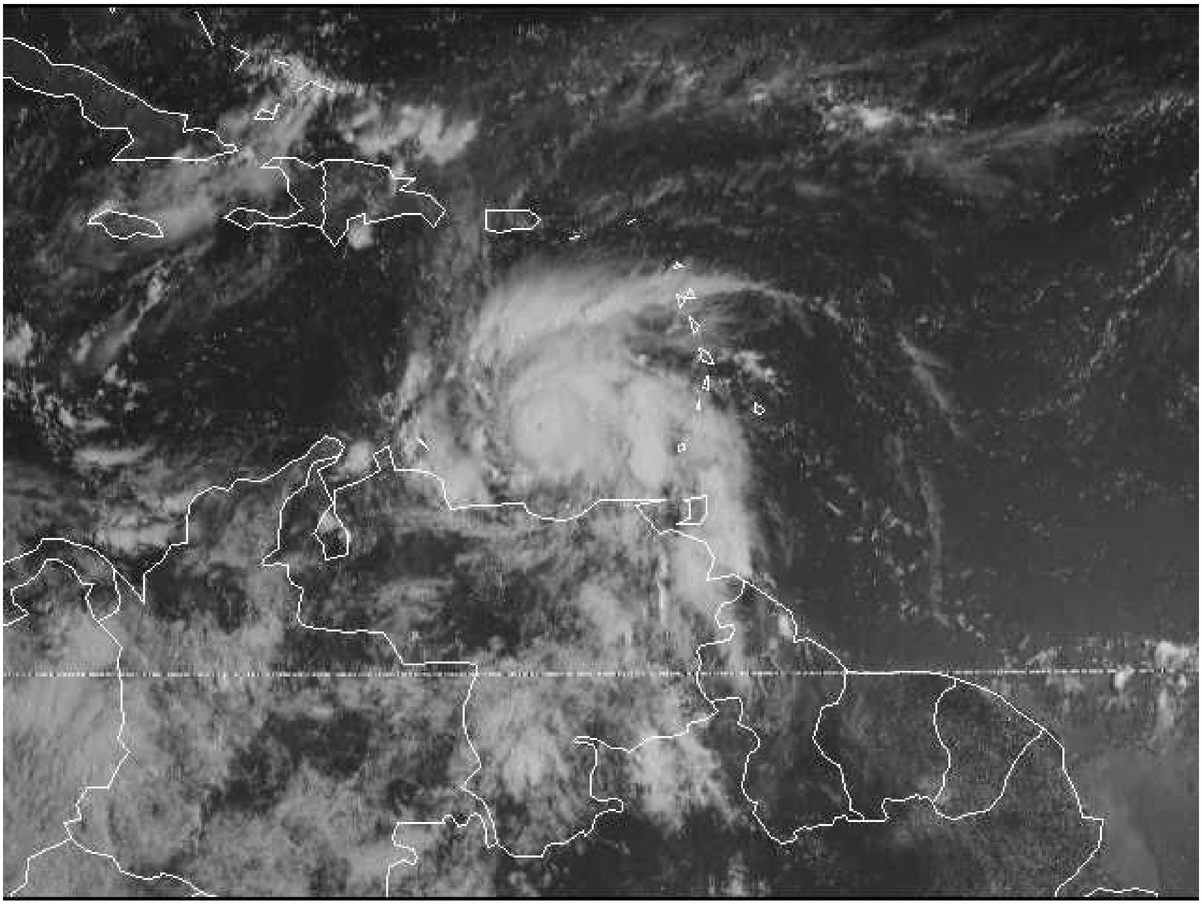


Figure 17: Satellite photo of Hurricane Ivan as of September 8, 2004. Note the roughly symmetric rain bands circulating in towards the center and the small but clearly defined "eye".

Homework for Week 2

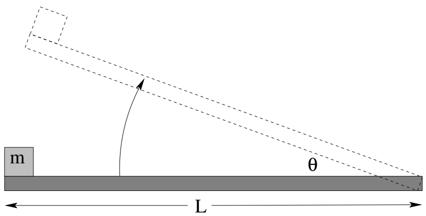
Problem 1.

Physics Concepts: As the first *required* step of solving the homework problems below, make a list of the following physics concepts or equations (that you *must know and understand* in order to solve them). Note that some of the necessary concepts will usually be from the previous weeks of the course. Also, one or two from previous weeks may be listed *just* for review and *not* be needed; you must not forget any of the earlier concepts as you learn new ones.

Note also that creating this list *is* part of your homework assignment and it must be handed in. As such, your homework (punched and added to e.g. a three ring binder) will gradually become a very effective study guide for hour exams and the final.

- Newton's Second Law

Problem 2.



A block of mass m sits **at rest** on a rough plank of length L that can be gradually tipped up until the block slides. The coefficient of static friction between the block and the plank is μ_s; the coefficient of dynamic friction is μ_k and as usual, $\mu_k < \mu_s$.

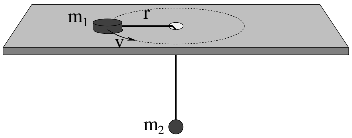
1) Find the angle θ_0 at which the block first starts to move.

2) Suppose that the plank is lifted to an angle $\theta > \theta_0$ (where the mass will definitely slide) and the mass is released from rest at time $t_0 = 0$. Find its acceleration a down the incline.

3) Finally, find the time t_f that the mass reaches the lower end of the plank.

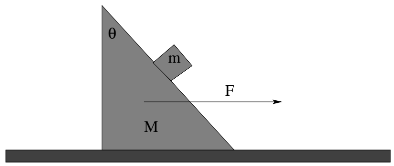
Problem 3.

A hockey puck of mass m_1 is tied to a string that passes through a hole in a frictionless



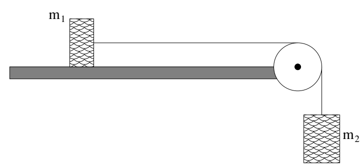
table, where it is also attached to a mass m_2 that hangs underneath. The mass is given a push so that it moves in a circle of radius r at constant speed v when mass m_2 hangs free beneath the table. Find r as a function of m_1, m_2, v, and g.

Problem 4.



A small square block m is sitting on a larger wedge-shaped block of mass M at an *upper* angle θ_0 such that the little block will slide on the big block if both are started from rest and no other forces are present. The large block is sitting on a frictionless table. The coefficient of static friction between the large and small blocks is μ_s. With what *range* of force F can you push on the large block to the right such that the small block will remain motionless with respect to the large block and neither slide up nor slide down?

Problem 5.

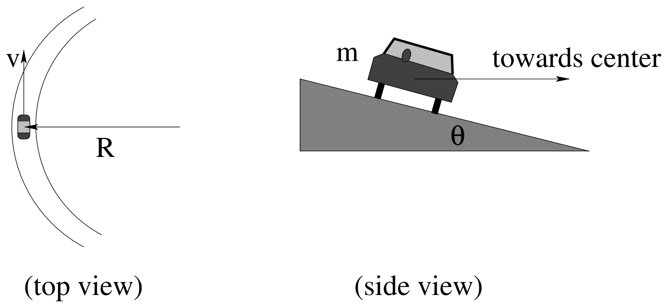


A mass m_1 is attached to a second mass m_2 by an Acme (massless, unstretchable) string. m_1 sits on a table with which it has coefficients of static and dynamic friction μ_s and μ_k respectively. m_2 is hanging over the ends of a table, suspended by the taut string from an Acme (frictionless, massless) pulley. At time $t = 0$ both masses are released.

1. What is the minimum mass $m_{2,\mathrm{min}}$ such that the two masses begin to move?

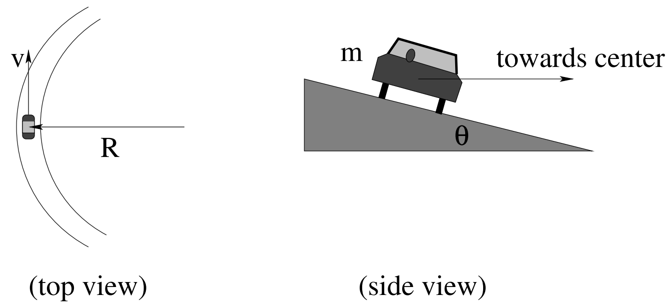
2. If $m_2 = 2m_{2,\mathrm{min}}$, determine how fast the two blocks are moving when mass m_2 has fallen a height H (assuming that m_1 hasn't yet hit the pulley)?

Problem 6.



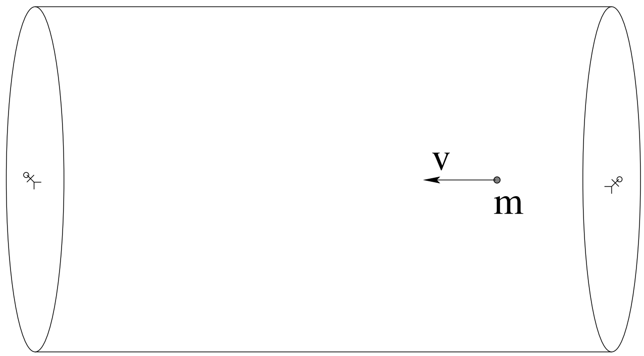
A car of mass m is rounding a banked curve that has radius of curvature R and banking angle θ. Find the speed v of the car such that it succeeds in making it around the curve without skidding on an extremely icy day when $\mu_s \approx 0$.

Problem 7.



A car of mass m is rounding a banked curve that has radius of curvature R and banking angle θ. The coefficient of static friction between the car's tires and the road is μ_s. Find the *range* of speeds v of the car such that it can succeed in making it around the curve without skidding.

Problem 8.

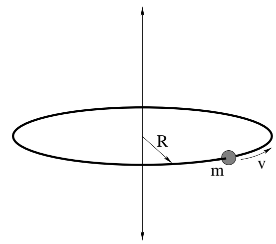


You and a friend are working inside a cylindrical new space station that is a hundred meters long and thirty meters in radius and filled with a thick air mixture. It is lunchtime and you have a bag of oranges. Your friend (working at the other end of the cylinder) wants one, so you throw one at him at speed v_0 at $t = 0$. Assume *Stokes* drag, that is $\vec{F}_d = -b\vec{v}$ (this is probably a poor assumption depending on the initial speed, but it makes the algebra relatively easy and qualitatively describes the motion well enough).

1) Derive an algebraic expression for the velocity of the orange as a function of time.

2) How long does it take the orange to lose half of its initial velocity?

Problem 9.

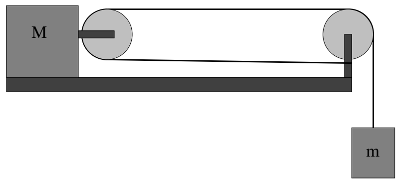


A bead of mass m is threaded on a metal hoop of radius R. There is a coefficient of kinetic friction μ_k between the bead and the hoop. It is given a push to start it sliding around the hoop with initial speed v_0. The hoop is located on the space station, so you can ignore gravity.

1) Find the normal force exerted by the hoop on the bead as a function of its speed.
2) Find the dynamical frictional force exerted by the hoop on the bead as a function of its speed.
3) Find its speed as a function of time. This involves using the frictional force on the bead in Newton's second law, finding its *tangential* acceleration on the hoop (which is the time rate of change of its speed) and solving the equation of motion.

All answers should be given in terms of m, μ_k, R, v (where requested) and v_0.

Problem 10.



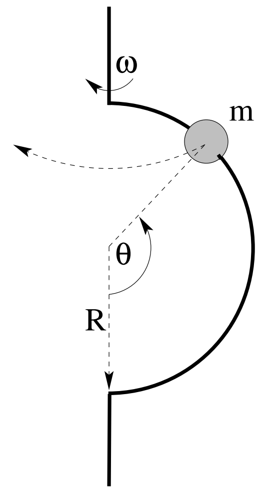
A block of mass M sits on a rough table. The coefficients of friction between the block and the table are μ_s and μ_k for static and kinetic friction respectively. A mass m is suspended from an Acme (massless, unstretchable, unbreakable) rope that is looped around the two pulleys as shown and attached to the support of the rightmost pulley. At time $t = 0$ the system is released at rest.

1) Find an expression for the *minimum* mass $m_{\rm min}$ such that the masses will begin to move.

2) Suppose $m = 2m_{\rm min}$ (twice as large as necessary to start it moving). Solve for the accelerations of *both masses.* Hint: Is there a constraint between how far mass M moves when mass m moves down a short distance?

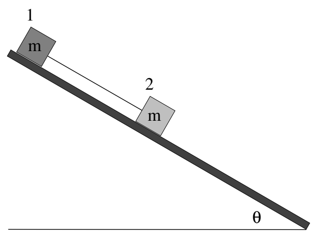
3) Find the speed of both masses after the small mass has fallen a distance H. Remember this answer and how hard you had to work to find it – next week we will find it much more easily.

Problem 11.



A small frictionless bead is threaded on a semicircular wire hoop with radius R, which is then spun on its vertical axis as shown above at angular velocity ω. Find the angle θ where the bead will remain stationary relative to the rotating wire.

Problem 12.



Two blocks, each with the same mass m but made of different materials, sit on a rough plane inclined at an angle θ such that they will slide (so that the component of their weight down the incline exceeds the maximum force exerted by static friction). The first (upper) block has a coefficient of *kinetic* friction of μ_{k1} between block and inclined plane; the second (lower) block has coefficient of kinetic friction μ_{k2}. The two blocks are connected by an Acme string.

Find the acceleration of the two blocks a_1 and a_2 down the incline:

1) when $\mu_{k1} > \mu_{k2}$;

2) when $\mu_{k2} > \mu_{k1}$.

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

No optional problems (yet) this week.

Week 3: Work and Energy

Work-Kinetic Energy Theorem

Recall that for **constant** acceleration in 1 dimension:

$$v_f^2 - v_i^2 = 2a\Delta x \tag{104}$$

where Δx is the displacement in the direction of the acceleration.

If we multiply by m (the mass of the object) and move the annoying 2 over to the other side, we can make the constant a into a constant force $F_x = ma$:

$$\begin{aligned} (ma)\Delta x &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &(105)\\ F_x\Delta x &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &(106) \end{aligned}$$

We now **define** the **work done by the force** F_x **on the mass** m to be:

$$\Delta W = F_x\Delta x. \tag{107}$$

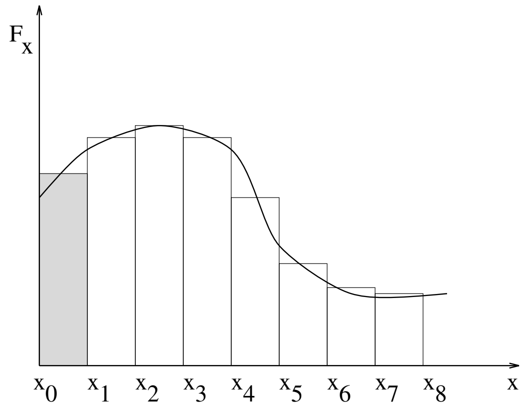
The work can be positive or negative. It has SI units of N-m = Joules = J.

Work is a form of **energy**. When we do work on an object with mass m, we change the quantity

$$E_k = \frac{1}{2}mv^2 \tag{108}$$

which is its **energy of motion** or **kinetic energy**. Note that this is a **relative** quantity – it depends upon the inertial frame in which it is measured. For example, the kinetic energy of a ball being tossed in a car is small relative to the car, but may be quite large relative to the ground flying by the car at 100 kph and extremely large relative to the sun as the earth whips around it.

Physics is the study of dynamics and forces that change the velocity (and hence the kinetic energy). The work done on a mass changes its kinetic energy by a certain amount independent of what it was to begin with. We thus start to see that dynamics (that is, our analysis of the effects of the forces involved) will not depend upon any constant we add to the energies being studied, as the depend only on the **difference** in energy between one state and another.



We now can formally state the **work-energy theorem** in the first form: **The work done on a mass by the total force acting on it is equal to the change in its kinetic energy,** or:

$$\Delta W = F_x \Delta x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta E_k \tag{109}$$

So far, we have proven this only for constant $F = ma$ and one dimensional motion. However, we can easily show it to be true more generally.

In this figure we see that we can break up a non-constant force acting through some distance into small intervals. In each interval the force is approximately constant. We thus get:

$$\begin{aligned} \Delta W_T &= F_1(x_1 - x_0) + F_2(x_2 - x_1) + \ldots && (110) \\ &= (\frac{1}{2}mv^2(x_1) - \frac{1}{2}mv^2(x_0)) + (\frac{1}{2}mv^2(x_2) - \frac{1}{2}mv^2(x_1)) + \ldots && (111) \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 && (112) \end{aligned}$$

where the internal changes in kinetic energy at the end of each interval but the first and last **cancel**. Finally, we let Δx go to zero in the usual way, and replace summation by integration. Thus:

$$\Delta W = \lim_{\Delta x \to 0} \sum_i F_i \Delta x = \int F dx = \Delta E_k \tag{113}$$

and we have generalized the theorem to include non-constant forces.

Finally, we can also show this to be true for each **component** of a force (and the kinetic energy associated with that component only). Because $v^2 = \vec{v} \cdot \vec{v} = v_x^2 + v_y^2 + v_z^2$, we can readily show that:

$$\Delta W = \vec{F} \cdot \Delta \vec{x} \rightarrow \int \vec{F} \cdot d\vec{x} = \Delta E_k \tag{114}$$

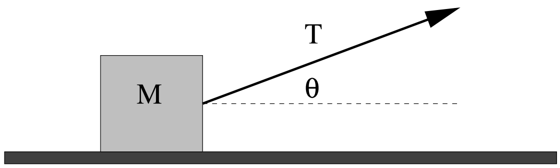
which is now our fully generalized work energy theorem in three dimensions.

Note well: Energy is a **scalar** and hence a bit easier to treat than vector quantities like forces.

Note well: Normal forces (perpendicular to the direction of motion) **do no work**:

$$\Delta W = \vec{\boldsymbol{F}}_{\perp} \cdot \Delta \vec{\boldsymbol{x}} = 0. \tag{115}$$

This **really** simplifies problem solving, as we shall see.



Examples

We should think about using **energy** (conservation or work-energy) whenever the answer requested for a given problem is **independent of time**. The reason for this is clear: we derived the work-energy theorem (and energy conservation) from the elimination of t from the dynamical equations.

1) Suppose we have a block of mass m being pulled by a string at a constant tension T at an angle θ with the horizontal along a rough table with coefficients of friction $\mu_s > \mu_k$. Typical questions might be: At what value of the tension does the block begin to move? If it is pulled with exactly that tension, how fast is it moving after it is pulled a horizontal distance L?

   We see that in the y-direction, $N + T\sin(\theta) - mg = 0$, or $N = mg - T\sin(\theta)$. In the x-direction, $F_x = T\cos(\theta) - \mu_s N = 0$ (at the point where block barely begins to move).

   Therefore:

$$T\cos(\theta) - \mu_s mg + T\mu_s \sin(\theta) = 0 \tag{116}$$

   or

$$T = \frac{\mu_s mg}{\cos(\theta) + \mu_s \sin(\theta)} \tag{117}$$

   With this value of the tension T, the work energy theorem becomes:

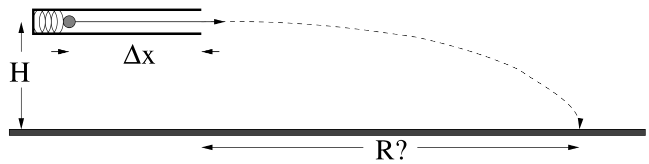
$$W = \Delta E_k \tag{118}$$

$$\left(T\cos(\theta) - \mu_k(mg - T\sin(\theta)\right) L = \frac{1}{2}mv_f^2 - 0 \quad (v_i = 0) \tag{119}$$

   or

$$v_f = \left(\frac{2\mu_s gL\cos(\theta)}{\cos(\theta) + \mu_s \sin(\theta)} - 2\mu_k gL + \frac{2\mu_k\mu_s gL\sin(\theta)}{\cos(\theta) + \mu_s \sin(\theta)}\right)^{\frac{1}{2}} \tag{120}$$

   Although it is difficult to check exactly, we can see that if $\mu_k = \mu_s$, $v_f = 0$ (or the mass doesn't accelerate). This is consistent with our value of T – the value at which the mass will exactly not move against μ_s alone.



2) We can also combine energy conservation with Newton's laws for different parts of the problem:

   Suppose we have a spring gun with a bullet of mass m compressing a spring with force constant k a distance Δx. When the trigger is pulled, the bullet is released from rest. It passes down a horizontal, frictionless barrel and comes out a distance H above the ground. What is the range of the gun?

   Part I: We find the speed of the bullet leaving the gun barrel from work-energy:

$$\begin{aligned} W &= \int_{x_1}^{x_0} -k(x-x_0)dx && (121)\\ &= -\frac{1}{2}k(x-x_0)^2|_{x_1}^{x_0} && (122)\\ &= \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}mv_f^2 - 0 && (123) \end{aligned}$$

   or

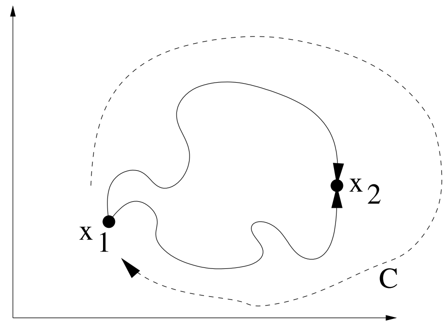
$$v_f = \sqrt{\frac{k}{m}}|\Delta x| \qquad (124)$$

   Part II: Given this speed (in the x-direction), find the range from Newton's Laws:

$$\vec{\boldsymbol{F}} = -mg\hat{y} \qquad (125)$$

   or $a_x = 0$, $a_y = -g$, $v_{0x} = v_f$, $v_{0y} = 0$, $x_0 = 0$, $y_0 = H$. Solving as usual, we find:

$$\begin{aligned} R &= v_{x0}t_0 && (126)\\ &= v_f\sqrt{\frac{2H}{g}} && (127)\\ &= \sqrt{\frac{2kH}{mg}}|\Delta x| && (128) \end{aligned}$$



Conservative Forces: Potential Energy

We have now seen two kinds of forces in action. One kind is like gravity. The work done by gravity doesn't depend on the path taken through the gravitational field – it only depends on the relative height of the two endpoints. The other kind is like friction. It not only depends on the path, it is (almost) always negative work, as typically friction dissipates kinetic energy as heat.

We define a **conservative** force to be one such that the work done **by the force** as you move a mass from point x_1 to point x_2 is independent of the path used to move between the points:

In this case (only), the work done going around an arbitrary closed path (starting and ending on the same point) will be identically zero!

$$W_{\text{loop}} = \oint_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = 0 \tag{129}$$

(Note that the two paths from $\vec{\boldsymbol{x}}_1$ to $\vec{\boldsymbol{x}}_2$ combine to form a closed loop C, where the work done going forward along one path is undone coming back along the other.)

Since the work done moving a mass m from an arbitrary starting point to any point in space is the **same** independent of the path, we can assign each point in space a numerical value: the work done by us on mass m, against the force, to reach it. This is the opposite of the work done by the force. We do it with this sign for reasons that will become clear in a moment. We call this **field** (a field is a function defined at each point in space) the **potential energy** of the mass m in the conservative force field $\vec{\boldsymbol{F}}$:

$$U(\vec{\boldsymbol{x}}) = -\int_{x_0}^{x} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = -W \tag{130}$$

Note Well that only one limit of integration depends on x; the other depends on where you choose to make the potential energy zero. This is a free choice. No physics can depend on where you choose the potential energy to be zero. Note that forces only depend on potential energy **differences** (or gradients, actually):

$$\Delta U = -\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} \tag{131}$$

$$dU = -\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} \tag{132}$$

or

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{\nabla}} U \tag{133}$$

Why do we bother? If a mass m is moving under the influence of a conservative force, then the work-energy theorem (114) looks like:

$$W_C = \Delta E_k \tag{134}$$

or

$$\Delta E_k - W_C = \Delta E_k + \Delta U = 0 \tag{135}$$

We can now state the principle of the **conservation of mechanical energy**: **The total mechanical energy (defined as the sum of its potential and kinetic energies) of a system moving in a conservative force field is constant.**

$$E_k + U = E_{\text{tot}} \tag{136}$$

(with E_{tot} a constant).

We can generalize this one further step by considering what happens if more than one (kind) of force is present (both conservative and nonconservative). In that case the argument above becomes:

$$W_{\text{tot}} = W_C + W_{NC} = \Delta E_k \tag{137}$$

or

$$W_{NC} = \Delta E_k - W_C = \Delta E_k + \Delta U \tag{138}$$

which we state as the **generalized energy conservation theorem**: **The work done by all the nonconservative forces acting on a system equals the change in its total mechanical energy.**

Note Well that even though energy is "lost" to (say) friction in the generalized energy conservation theorem, this energy appears as heat. It turns out that total energy is actually conserved at all times, when one takes into account conservative and nonconservative forces, as well as the work/energy that eventually appears as heat.

Gravity

The potential energy of an object experiencing a gravitational force (only) is:

$$U_g = -\int_0^y (-mg)dy = mgy \tag{139}$$

More generally, we write this as:

$$U(y) = mgy + U_0 \tag{140}$$

where U_0 is an arbitrary constant that sets the zero of gravitational potential energy. You will find it **very helpful** to choose a coordinate system (or set the zero of potential energy) in such as way as to make the problem as computationally simple as possible. That usually means arranging it so that either the initial of final potential energy is zero.

Springs

Springs also exert **reversible** (or conservative) forces. We can therefore define a potential energy function for them. We will essentially **always** choose the zero of potential energy to be the equilibrium position of the spring – other choices are possible, but will only rarely be useful:

$$U(x) = -\int_{x_0}^{x} -k(x - x_0)dx \tag{141}$$

$$= \frac{1}{2}kx^2 \tag{142}$$

or more generally:

$$U(x) = \frac{1}{2}kx^2 + U_0 \tag{143}$$

More Examples

1) A mass m is hanging by a massless thread of length L and is given an initial speed v_0 to the left. It swings up and stops at some maximum height H an angle θ. Find θ.

   We solve this by setting $E_i = E_f$ (total energy is conserved).

   Initial:
$$E_{\text{tot}} = \frac{1}{2}mv_0^2 + mg(0) = \frac{1}{2}mv_0^2 \tag{144}$$

   Final:
$$E_{\text{tot}} = \frac{1}{2}m(0)^2 + mgh = mgL(1 - \cos(\theta)) \tag{145}$$

   Set them equal and solve:
$$\cos(\theta) = 1 - \frac{v_0^2}{2gL} \tag{146}$$

   or
$$\theta = \cos^{-1}(1 - \frac{v_0^2}{2gL}). \tag{147}$$

2) Mass pushed by compressed spring that slides up smooth incline to height H? Show that $H = \frac{k\Delta x^2}{2mg}$.

3) Mass on rough incline of height H (given) slides down to bottom. Find v_f at the bottom? (Set $E_f - E_i = W_f$, where the latter is negative, and solve for v_f).

4) Loop the loop. What is the minimum height H such that the mass m loops-the-loop (stays on the track around the circle). Ans: $H = 5/2R$.

5) Block and tackle. Tension T doesn't do any net work on system, just redistributes energy. Potential energy of gravity is distributed throughout the system as kinetic energy.

Power

The energy in a given system is **not**, of course, usually constant in time. Energy is added to a given mass, or taken away, at some rate. We accelerate a car (adding to its mechanical energy). We brake a car (turning its kinetic energy into heat). There are many times when we are given the **rate** at which energy is added or removed in time, and need to find the total energy added or removed. This rate is called the **power**.

Power: The rate at which work is done, or energy released into a system.

This latter form lets us express it conveniently for time-varying forces:

$$dW = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = \vec{\boldsymbol{F}} \cdot \frac{dx}{dt}dt \tag{148}$$

or

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \tag{149}$$

so that

$$\Delta W = \Delta E_{\rm tot} = \int P dt \tag{150}$$

The units of power are clearly Joules/sec = Watts. Another common unit of power is “Horsepower”, 1 HP = 746 W. Note that the power of a car together with its drag coefficient determine how fast it can go. When energy is being added by the engine at the same **rate** at which it is being dissipated by drag and friction, the total mechanical energy of the car will remain constant in time.

Equilibrium

Recall that the **force** is given by the **negative gradient of the potential energy**:

$$\vec{F} = -\vec{\nabla} U \tag{151}$$

or (in each direction)

$$F_x = -\frac{dU}{dx} \tag{152}$$

or the force is the negative **slope** of the potential energy function in this direction. We can use this to rapidly evaluate the behavior of a system on a qualitative basis.

Consider the following potential energy curves:

In this first one, the slope is negative so F_x is positive. Near the origin the force is large, and farther out it gets weaker. This is a “repulsive” potential energy function.

This is a more useful one. This curve is typical of a “molecular potential”. At short ranges it is strongly repulsive. There is a minimum that corresponds to a stable equilibrium point of separation. There is an unstable equilibrium point somewhat farther out (where $\vec{F} = 0$ but where any motion around this point will cause the mass to “fall away” to infinity or into the attractive potential). Beyond this the force points out.

Clearly, the force points the way we intuitively expect things to “slide” down these potential energy curves. In fact, as something moves into a region of lower potential energy, its kinetic energy must increase, so that its velocity increases. This means that there is a force pointing in the direction of the potential energy's decrease, which is what we **mean** when we write the gradient relation above.

Homework for Week 3

Problem 1.

Physics Concepts: As the first *required* step of solving the homework problems below, make a list of the following physics concepts or equations (that you *must know and understand* in order to solve them). Note that some of the necessary concepts will usually be from the previous weeks of the course. Also, one or two from previous weeks may be listed *just* for review and *not* be needed; you must not forget any of the earlier concepts as you learn new ones.

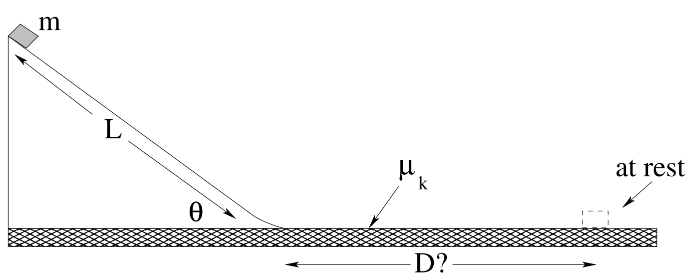
Note also that creating this list *is* part of your homework assignment and it must be handed in. As such, your homework (punched and added to e.g. a three ring binder) will gradually become a very effective study guide for hour exams and the final.

- Newton's Second Law

Problem 2.

Derive the Work-Kinetic Energy (WKE) theorem in one dimension from Newton's second law. Show that if you sum the WKE theorem for the x, y, and z directions separately you get the WKE theorem in three dimensions.

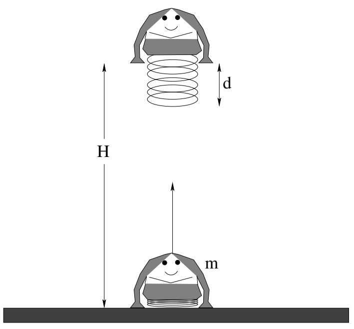
Problem 3.



A block of mass m slides down a *smooth* (frictionless) incline of length L that makes an angle θ with the horizontal as shown. It then reaches a *rough* surface with a coefficient of kinetic friction μ_k.

Use the concepts of work and/or mechanical energy to find the distance D the block slides across the rough surface before it comes to rest.

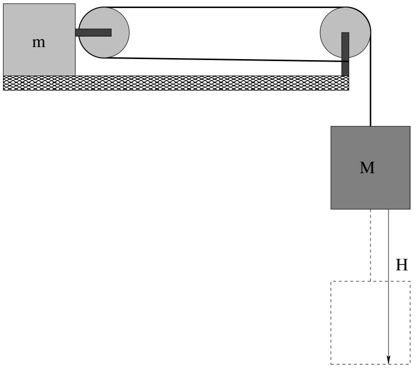
Problem 4.



A simple child's toy is a jumping frog made up of an approximately massless spring of uncompressed length d and spring constant k that propels a molded "frog" of mass m. The frog is pressed down onto a table (compressing the spring by d) and at $t = 0$ the spring is released so that the frog leaps high into the air.

Use work and/or mechanical energy to determine how high the frog leaps.

Problem 5.

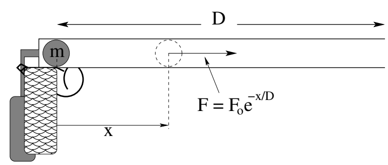


A block of mass m sits on a rough table. The coefficients of friction between the block and the table are μ_s and μ_k for static and kinetic friction respectively. A much larger mass M (easily heavy enough to overcome static friction) is suspended from an

Acme (massless, unstretchable, unbreakable) rope that is looped around the two pulleys as shown and attached to the support of the rightmost pulley. At time $t = 0$ the system is released at rest.

Use work and/or mechanical energy (where the latter is *very easy* since the internal work done by the tension in the string *cancels*) to find the speed of both masses after the large mass M has fallen a distance H. Note that you will still need to use the constraint between the coordinates that describe the two masses. Remember how hard you had to "work" to get this answer last week? When time isn't important, energy is better!

Problem 6.



A simple schematic for a paintball gun with a barrel of length D is shown above; when the trigger is pulled carbon dioxide gas under pressure is released into the approximately frictionless barrel behind the paintball (which has mass m). The expanding, cooling gas exerts a force on the ball of magnitude:

$$F = F_0 e^{-\frac{x}{D}}$$

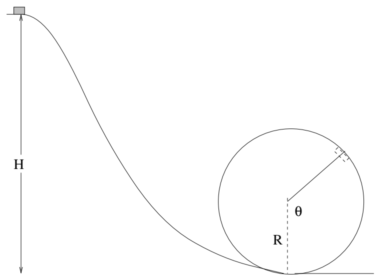
on the ball to the *right*, where x is measured from the paintball's initial position as shown.

1) Find the work done on the paintball by the force as the paintball is accelerated down the barrel.

2) Use the work-kinetic-energy theorem to compute the kinetic energy of the paintball after it has been accelerated.

3) Find the speed with which the paintball emerges from the barrel after the trigger is pulled.

Problem 7.



A block of mass M sits at the top of a frictionless hill of height H leading to a circular loop-the-loop of radius R.

1) Find the minimum height H_{min} for which the block *barely* goes around the loop staying on the track at the top. (Hint: What is the condition on the normal force when it "barely" stays in contact with the track? This condition can be thought of as "free fall" and will help us understand circular orbits later, so don't forget it.).

   Discuss within your recitation group why your answer is a scalar number times R and how this *kind* of result is usually a good sign that your answer is probably right.

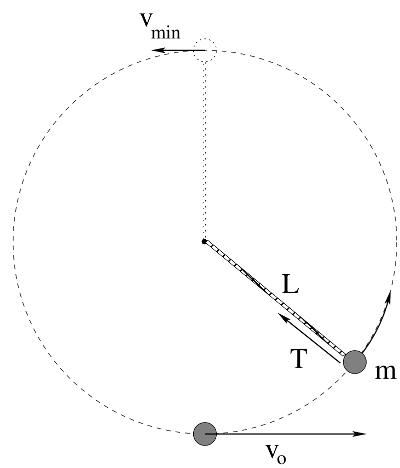
2) If the block is started at this position, what is the normal force exerted by the track at the *bottom* of the loop, where it is greatest?

   If you have ever ridden roller coasters with loops, use the fact that your apparent weight *is* the normal force exerted on you by your seat if *you* are looping the loop in a roller coaster and discuss with your recitation group whether or not the results you derive here are in accord with your experiences. If you haven't, consider riding one *aware* of the forces that are acting on you and how they affect your perception of weight and change your direction on your next visit to e.g. Busch Gardens to be, in a bizarre kind of way, a *physics assignment.* (Now c'mon, how many classes have you ever taken that assign riding roller coasters, even as an optional activity?:-)

Problem 8.

A ball of mass m is attached to a (massless, unstretchable) string and is suspended from a pivot. It is moving in a vertical circle of radius R such that it has speed v_0 at the bottom as shown. The ball is in a vacuum; neglect drag forces and friction in this problem.

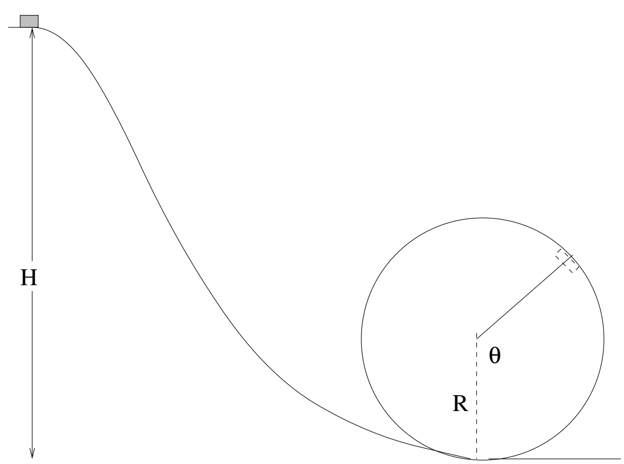
1) Find an expression for the tension T at the top of the loop as a function of m, v_0, g, R, assuming that it is still moving in a circle when it gets there.



2) From this expression, find the minimum speed v_{min} that the ball must have at the top to barely loop the loop *on the circular trajectory* (with a precisely limp string with tension $T = 0$).

3) Determine the speed v_0 the ball must have at the bottom to arrive at the top with this minimum speed. You may use either work or potential energy for this part of the problem.

Problem 9.



A block of mass M sits at the top of a frictionless hill of height H. It slides down and around a loop-the-loop of radius R to an angle θ (an arbitrary one is shown in the figure above, which may or may not be the easiest one to use to answer this question).

Find the magnitude of the normal force as a function of the angle θ. From this, deduce an expression for the angle θ_0 at which the block will *leave* the track if the block is started at a height $H = 2R$.

Problem 10.

In the figure above we see two cars, one moving at a speed v_0 and an identical car moving at a speed $2v_0$. The cars are moving at a *constant* speed, so their motors are pushing them forward with a force that precisely cancels the drag force exerted by the air. This drag force is quadratic in their speed:

$$F_d = -bv^2$$

(in the opposite direction to their velocity) and we assume (incorrectly) that this is the *only* force acting on the car in the direction of motion besides that provided by the motor itself, neglecting various other sources of friction or inefficiency.

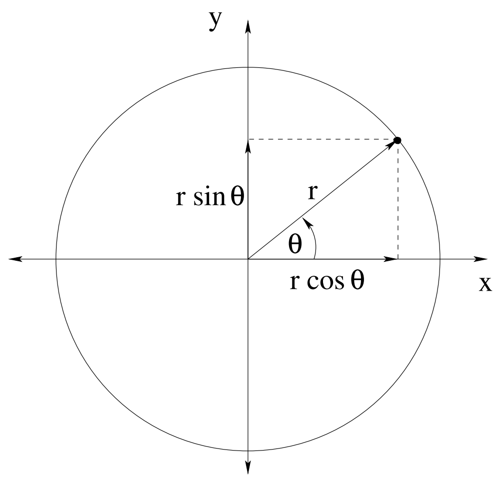
Prove that the engine of the car that is travelling twice as fast has to be providing *eight times as much power* to maintain the higher constant speed. Does this mean that it uses eight times as much gasoline to go the same horizontal distance? Assuming that gasoline is being turned into work at the same rate (a quantity we will eventually call the "efficiency" of the gas motors) in both cars, how much more gasoline does the faster car consume than the slower one to go some fixed distance D? Discuss this thoroughly in your groups, as this is *practical* information in our energy-conscious age. Given your experiences driving various distances at various speeds and paying for gasoline, do you think that the assumption of equal efficiency independent of speed is likely to be correct?

Problem 11.

This is a guided exercise exploring the kinematics of circular motion and the relation between Cartesian and Plane Polar coordinates. In the figure above, note that:

$$\vec{\boldsymbol{r}} = r\cos(\theta(t))\,\hat{\boldsymbol{x}} + r\sin(\theta(t))\,\hat{\boldsymbol{y}}$$

where r is the radius of the circle and $\theta(t)$ is an *arbitrary* continuous function of time



describing where a particle is on the circle at any given time. This is equivalent to:

$$
\begin{aligned}
x(t) &= r\cos(\theta(t)) \\
y(t) &= r\sin(\theta(t))
\end{aligned}
$$

(going from (r, θ) plane polar coordinates to (x, y) cartesian coordinates and the corresponding:

$$
\begin{aligned}
r &= \sqrt{x(t)^2 + y(t)^2} \\
\theta(t) &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
$$

You will find the following two definitions useful:

$$
\begin{aligned}
\omega &= \frac{d\theta}{dt} \\
\alpha &= \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}
\end{aligned}
$$

The first you should already be familiar with as the *angular velocity*, the second is the *angular acceleration*. Recall that the tangential speed $v_t = r\omega$; similarly the tangential acceleration is $a_t = r\alpha$ as we shall see below.

Work through the following exercises:

1) Find the velocity of the particle $\vec{\boldsymbol{v}}$ in cartesian vector coordinates.

2) Form the dot product $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{r}}$ and show that it is *zero*. This proves that the velocity vector is perpendicular to the radius vector for any particle moving on a circle!

3) Show that the total acceleration of the particle $\vec{\boldsymbol{a}}$ in cartesian vector coordinates can be written as:

$$\vec{\boldsymbol{a}} = -\omega^2\vec{\boldsymbol{r}} + \frac{\alpha}{\omega}\vec{\boldsymbol{v}}$$

Since the direction of $\vec{v}$ is tangent to the circle of motion, we can identify these two terms as the results:

$$a_r = -\omega^2 r = -\frac{v_t^2}{r}$$

(now derived in terms of its cartesian components) and

$$a_t = \alpha r.$$

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

No optional problems (yet) this week.

Week 4: Systems of Particles, Momentum and Collisions

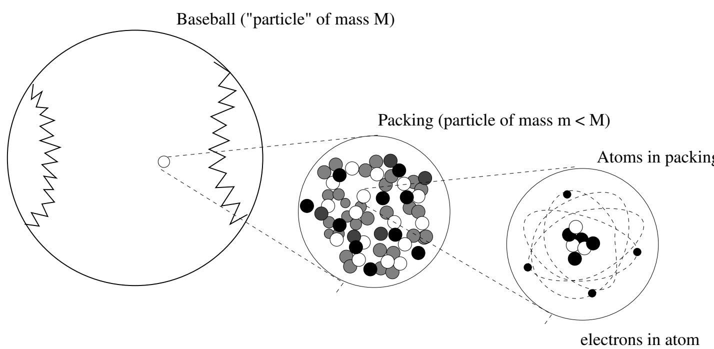


Figure 18: An object such as a baseball is not really a particle. It is made of *many, many* particles – even the atoms it is made of are made of many particles *each*. Yet it *behaves* like a particle as far as Newton's Laws are concerned. Now we find out why.

Systems of Particles

The world of one particle is fairly simple. Something pushes on the particle, and it accelerates. Stop pushing, it coasts or remains still. Do work on it and it speeds up. Do negative work on it and it slows down. Increase or decrease its potential energy; decrease or increase its kinetic energy.

However, the real world is not so simple. For one thing, every push works two ways – all forces act symmetrically between objects – no object experiences a force all by itself. For another, real objects are not particles – they are made up of lots of "particles" themselves. Finally, even if we ignore the internal constituents of an object, we seem to inhabit a universe with lots of objects.

Somehow we know intuitively that the details of the motion of every electron in a baseball are irrelevant to the behavior of the baseball as a whole. Clearly, we need to deduce ways of taking a collection of particles and determining its collective behavior. Ideally, this process should be one we can iterate, so that we can treat collections of collections – a box of baseballs, under the right circumstances (falling out of an airplane, for example) might also be expected to behave within reason like a single object independent of the motion of the baseballs inside, or the motion of the atoms in the baseballs, or the motion of the electrons in the atoms.

We will obtain this collective behavior by **averaging**, or *summing over* at successively larger scales, the physics that we know applies at the smallest scale to things that *really are* particles and discover to our surprise that it applies equally well to collections of those particles, subject to a few new definitions and rules.

Newton's Laws for a System of Particles – Center of Mass

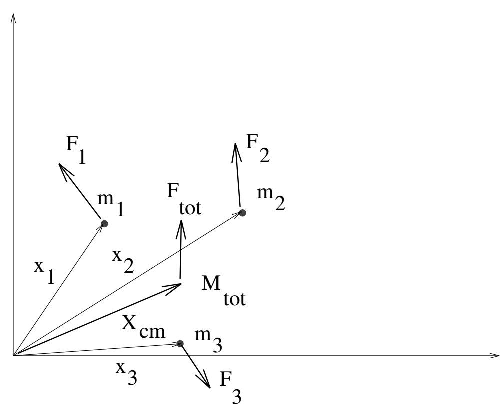


Figure 19: A system of $N = 3$ particles is shown above, with various forces $\vec{\boldsymbol{F}}_i$ acting on the masses (which therefore each their own accelerations $\vec{\boldsymbol{a}}_i$). From this, we construct a *weighted average* acceleration of the system, in such a way that Newton's Second Law is satisfied for the *total* mass.

Suppose we have a system of N particles, each of which is experiencing a force. Some (part) of those forces are "external" – they come from outside of the system. Some (part) of them may be "internal" – equal and opposite force pairs between particles that help hold the system together (solid) or allow its component parts to interact (liquid or gas).

We would like the total force to act on the total mass of this system as if it were a "particle". That is, we would like for:

$$\vec{\boldsymbol{F}}_{\text{tot}} = M_{\text{tot}}\vec{\boldsymbol{A}} \tag{153}$$

where vA is the "acceleration of the system". This is easily accomplished.

Newton's Second Law for a system of particles is written as:

$$\vec{\boldsymbol{F}}_{\rm tot} = \sum_i \vec{\boldsymbol{F}}_i = \sum_i m_i \frac{d^2\vec{\boldsymbol{x}}_i}{dt^2} \tag{154}$$

We now perform the following Algebra Magic:

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\rm tot} &= \sum_i \vec{\boldsymbol{F}}_i &(155)\\
&= \sum_i m_i \frac{d^2\vec{\boldsymbol{x}}_i}{dt^2} &(156)\\
&= \left(\sum_i m_i\right)\frac{d^2\vec{\boldsymbol{X}}}{dt^2} &(157)\\
&= M_{\rm tot}\frac{d^2\vec{\boldsymbol{X}}}{dt^2} = M_{\rm tot}\vec{\boldsymbol{A}} &(158)
\end{aligned}$$

Note well the introduction of a new coordinate, vX. This introduction isn't "algebra", it is a *definition*. Let's isolate it so that we can see it better:

$$\sum_i m_i \frac{d^2\vec{\boldsymbol{x}}_i}{dt^2} = M_{\rm tot}\frac{d^2\vec{\boldsymbol{X}}}{dt^2} \tag{159}$$

Basically, *if* we define an $\vec{\boldsymbol{X}}$ such that this relation is true *then* Newton's second law is recovered for the entire system of particles "located at $\vec{\boldsymbol{X}}$" as if that location were indeed a particle of mass $M_{\rm tot}$ itself.

We can rearrange this a bit as:

$$\frac{d\vec{\boldsymbol{V}}}{dt} = \frac{d^2\vec{\boldsymbol{X}}}{dt^2} = \frac{1}{M_{\rm tot}}\sum_i m_i \frac{d^2\vec{\boldsymbol{x}}_i}{dt^2} = \frac{1}{M_{\rm tot}}\sum_i m_i \frac{d\vec{\boldsymbol{v}}_i}{dt} \tag{160}$$

and can *integrate twice* on both sides (as usual, but we only do the integrals formally). The first integral is:

$$\frac{d\vec{\boldsymbol{X}}}{dt} = \vec{\boldsymbol{V}} = \frac{1}{M_{\rm tot}}\sum_i m_i\vec{\boldsymbol{v}}_i + \vec{\boldsymbol{V}}_0 = \frac{1}{M_{\rm tot}}\sum_i m_i \frac{d\vec{\boldsymbol{x}}_i}{dt} + \vec{\boldsymbol{V}}_0 \tag{161}$$

and the second is:

$$\vec{\boldsymbol{X}} = \frac{1}{M_{\rm tot}}\sum_i m_i\vec{\boldsymbol{x}}_i + \vec{\boldsymbol{V}}_0 t + \vec{\boldsymbol{X}}_0 \tag{162}$$

Note that this equation is exact, but we have had to introduce *two constants of integration* that are completely arbitrary: $\vec{\boldsymbol{V}}_0$ and $\vec{\boldsymbol{X}}_0$.

These constants represent the exact same freedom that we have with our *inertial frame of reference* – we can put the origin of coordinates anywhere we like, and we will get the same equations of motion even if we put it somewhere and describe everything in a uniformly moving frame. We should have *expected* this sort of freedom in our definition of a coordinate that describes "the system" because we have precisely the same freedom in our choice of *coordinate system* in terms of which to describe it.

In many problems, however, we don't want to use this freedom. Rather, we want the *simplest* description of the system itself, and push all of the freedom concerning constants of motion over to the coordinate choice itself (where it arguably "belongs"). We therefore select just *one* (the simplest one) of the infinity of possibly consistent rules represented in our definition above that would preserve Newton's Second Law and call it by a special name: ***The Center of Mass!***

We define the position of the center of mass to be:

$$M\vec{\boldsymbol{X}}_{\rm cm} = \sum_i m_i \vec{\boldsymbol{x}}_i \tag{163}$$

(with $M = \sum_i m_i$). If we consider the "location" of the system of particles to be the center of mass, then Newton's Second Law will be satisfied for the system as if it were a particle, and the location in question will be exactly what we intuitively expect: the "middle" of the (collective) object or system, weighted by its distribution of mass.

Not all systems we treat will appear to be made up of point particles. Most solid objects or fluids appear to be made up of a *continuum* of mass, a *mass distribution*. In this case we need to do the sum by means of *integration*, and our definition becomes:

$$M\vec{\boldsymbol{X}}_{\rm cm} = \int \vec{\boldsymbol{x}} dm \tag{164}$$

(with $M = \int dm$). The latter form comes from treating every little differential chunk of a solid object like a "particle", and adding them all up. Integration, recall, is just a way of adding them up.

Of course this leaves us with the recursive problem of the fact that "solid" objects are *really* made out of lots of point-like elementary particles and their fields. It is worth very briefly presenting the standard "coarse-graining" argument that permits us to treat solids and fluids like a continuum of smoothly distributed mass – and the limitations of that argument.

Coarse Graining: Continuous mass distributions

Suppose we wish to find the center of mass of a small cube of some uniform material – such as gold, why not? We know that *really* gold is made up of gold atoms, and that gold atoms are make up of (elementary) electrons, quarks, and various massless field particles that bind the massive particles together. In a cube of gold with a mass of 197 grams, there are roughly 6×10^{23} atoms, each with 79 electrons and 591 quarks for a total of 670 elementary particles per atom. This is then about 4×10^{26} elementary particles in a cube just over 2 cm per side.

If we tried to actually *use the sum form* of the definition of center of mass to evaluate it's location, and ran the computation on a computer capable of performing one trillion floating point operations per second, it would take several hundred trillion seconds (say ten million years) and – unless we knew the exacly location of every quark – would *still* be approximate, no better than a guess.

We do far better by *averaging*. Suppose we take a small chunk of the cube of gold – one with cube edges 1 millimeter long, for example. This still has an enormous number of elementary particles in it – so many that if we shift the boundaries of the chunk a tiny bit *many* particles – many *whole atoms* are moved in or out of the chunk. Clearly we are justified in talking about the "average number of atoms" or "average amount of mass of gold" in a tiny cube like this.

A millimeter is still absurdly large on an atomic scale. We could make the cube 1 *micron* (1×10^{-6} meter, a thousandth of a millimeter) and because atoms have a "generic" size around one Angstrom – 1×10^{-10} meters – we would expect it to contain around $(10^{-6}/10^{-10})^3 = 10^12$ atoms. Roughly a trillion atoms in a cube too small to see with the naked eye (and each atom still has almost 700 elementary particles, recall). We could go down at least 1-2 *more* orders of magnitude in size and still have millions of particles in our chunk!

A chunk 10 nanometers to the side is fairly accurately located in space on a scale of meters. It has enough elementary particles in it that we can meaningfully speak of its "average mass" and use this to define the *mass density* at the point of location of the chunk – the mass per unit volume at that point in space – with at least 5 or 6 significant figures (one part in a million accuracy). In most real-number computations we might undertake in the kind of physics learned in this class, we wouldn't pay attention to more than 3 or 4 significant figures, so this is plenty.

The point is that this chunk is now small enough to be considered *differentially small* for the purposes of doing *calculus*. This is called *coarse graining* – treating chunks big on an atomic or molecular scale but small on a macroscopic scale. To complete the argument, in physics we would generally consider a small chunk of matter in a solid or fluid that we wish to treat as a smooth distribution of mass, and write at first:

$$\Delta m = \rho \Delta V \tag{165}$$

while reciting the following magical formula to ourselves:

> **The mass of the chunk is the mass per unit volume ρ times the volume of the chunk.**

We would then think to ourselves: "Gee, ρ is *almost* a uniform function of location for chunks that are small enough to be considered a differential as far as doing sums using integrals are concerned. I'll just coarse grain this and use integration to evaluation all sums." Thus:

$$dm = \rho \, dV \tag{166}$$

We do this all of the time, in this course. This semester we do it repeatedly for mass distributions, and sometimes (e.g. when treating planets) will coarse grain on a much larger scale to form the "average" density on a planetary scale. On a planetary scale, barring chunks of neutronium or the occasional black hole, a cubic kilometer "chunk" is still "small" enough to be considered differentially small – we usually won't need to integrate over every single distinct pebble or clod of dirt on a much smaller scale. Next semester we will do it repeatedly for electrical charge, as after all all of those gold atoms are made up of *charged* particles so there are just as many charges to consider as there are elementary particles. Our models for electrostatic fields of continuous charge and electrical currents in wires will all rely on this sort of coarse graining.

Before we move on, we should say a word or two about two other common distributions of mass. If we want to find e.g. the center of mass of a flat piece of paper cut out into (say) the shape of a triangle, we *could* treat it as a "volume" of paper and integrate over its thickness. However, it is probably a pretty good bet from *symmetry* that unless the paper is very inhomogeneous across its thickness, the center of mass in the flat plane is in the middle of the "slab" of paper, and the paper is already so thin that we don't pay much attention to its thickness as a general rule. In this case we basically integrate out the thickness in our minds (by multiplying ρ by the paper thickness t) and get:

$$\Delta m = \rho \Delta V = \rho t \Delta A = \sigma \Delta A \tag{167}$$

where $\sigma = \rho t$ is the (average) *mass per unit area* of a chunk of paper with *area* ΔA. We say our (slightly modified) magic ritual and poof! We have:

$$dm = \sigma \; dA \tag{168}$$

for two dimensional *areal distributions* of mass.

Similarly, we often will want to find the center of mass of things like wires bent into curves, things that are *long and thin*. By now I shouldn't have to explain the following reasoning:

$$\Delta m = \rho \Delta V = \rho A \Delta x = \lambda \Delta x \tag{169}$$

where A is the (small!) cross section of the solid wire and $\lambda = \rho A$ is the *mass per unit length* of the chunk of wire, magic spell, cloud of smoke, and when the smoke clears we are left with:

$$dm = \lambda \; dA \tag{170}$$

In all of these cases, note well, ρ, σ, λ can be *functions of the coordinates!* They are not necessarily *constant*, they simply describe the (average) mass per unit volume at the point in our object or system in question, subject to the coarse-graining limits. Those limits are pretty sensible ones – if we are trying to solve problems on a length scale of angstroms, we *cannot use these averages* because the laws of large numbers won't apply. Or rather, we can and do still use these *kinds* of averages in quantum theory (because even on the scale of a *single atom* doing all of the discrete computations proves to be a problem) but then we do so knowing up front that they are approximations and that our answer will be "wrong".

Examples of evaluating the center of mass

In order to use the idea of center of mass (CM) in a problem, we need to be able to evaluate it. For a system of discrete particles, the sum definition is all that there is – you brute-force your way through the sum (decomposing vectors into suitable coordinates and adding them up).

For a solid object that is symmetric, the CM is "in the middle". But where's that? To precisely find out, we have to be able to use the integral definition of the CM:

$$M\vec{\boldsymbol{X}}_{\rm cm} = \int \vec{\boldsymbol{x}} dm \tag{171}$$

(with $M = \int dm$, and $dm = \rho dV$ or $dm = \sigma dA$ or $dm = \lambda dl$ as discussed above).

Let's try a few examples:

Example 4.0.1: Center of mass of a continuous rod

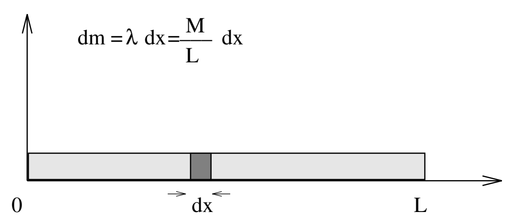


Figure 20:

Let us evaluate the center of mass of a continuous rod of length L and total mass M, to make sure it is in the middle:

$$M\vec{\boldsymbol{X}}_{\rm cm} = \int \vec{\boldsymbol{x}} dm = \int_0^L \lambda x dx \tag{172}$$

where

$$M = \int dm = \int_0^L \lambda dx = \lambda L \tag{173}$$

(which defines λ, if you like) so that

$$M\vec{\boldsymbol{X}}_{\rm cm} = \lambda \frac{L^2}{2} = M\frac{L}{2} \tag{174}$$

and

$$\vec{\boldsymbol{X}}_{\rm cm} = \frac{L}{2}. \tag{175}$$

Gee, that was easy. Let's try a hard one.

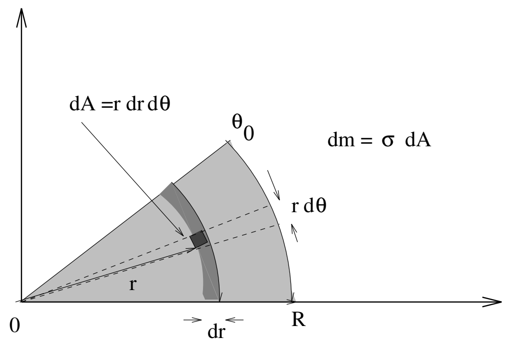


Figure 21:

Example 4.0.2: Center of mass of a circular wedge

Let's find the center of mass of a circular wedge (a shape like a piece of pie, but very flat). It is two dimensional, so we have to do it one coordinate at a time. We start from the same place:

$$MX_{\rm cm} = \int x dm = \int_0^R \int_0^{\theta_0} \sigma x dA = \int_0^R \int_0^{\theta_0} \sigma r^2 \cos\theta dr d\theta \tag{176}$$

where

$$M = \int dm = \int_0^R \int_0^{\theta_0} \sigma dA = \int_0^R \int_0^{\theta_0} \sigma r dr d\theta = \sigma \frac{R^2 \theta_0}{2} \tag{177}$$

(which defines σ, if you like) so that

$$MX_{\rm cm} = \sigma \frac{R^3 \sin\theta_0}{3} \tag{178}$$

from which we find (with a bit more work than last time but not much) that:

$$X_{\rm cm} = \frac{2R^3 \sin\theta_0}{3R^2\theta}. \tag{179}$$

Amazingly enough, this has units of R (length), so it might just be right. To check it, do $Y_{\rm cm}$ on your own!

Example 4.0.3: Breakup of Projectile in Midflight

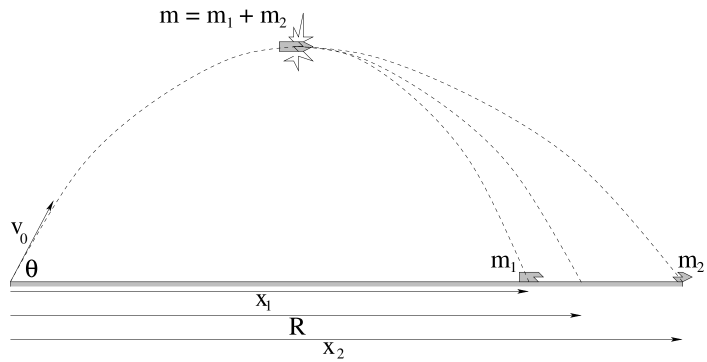


Figure 22: A projectile breaks up in midflight. The center of mass follows the original trajectory of the particle, allowing us to predict where one part lands if we know where the other one lands, as long as the explosion exerts no vertical component of force on the two particles.

Suppose that a projectile breaks up horizontally into two pieces of mass m_1 and m_2 in midflight. Given θ, v_0, and x_1, predict x_2.

The idea is: The trajectory of the center of mass obeys Newton's Laws for the entire projectile and lands in the same place that it would have, because no external forces *other* than gravity act. The projectile breaks up horizontally, which means that both pieces will land at the same time, with the center of mass in between them. We thus need to find the point where the center of mass would have landed, and solve the equation for the center of mass in terms of the two places the projectile fragments land for one, given the other. Thus:

Find R. As usual:

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \tag{180}$$

$$t_R(v_0 \sin\theta - \frac{1}{2}gt_R) = 0 \tag{181}$$

$$t_R = \frac{2v_0 \sin\theta)}{g} \tag{182}$$

$$R = (v_0 \cos\theta)t_R = \frac{2v_0^2 \sin\theta\cos\theta}{g}. \tag{183}$$

R is the position of the center of mass. We write the equation making it so:

$$m_1x_1 + m_2x_2 = (m_1 + m_2)R \tag{184}$$

and solve for the unknown x_2.

$$x_2 = \frac{(m_1 + m_2)R - m_1x_1}{m_2} \tag{185}$$

Momentum

Momentum is a useful idea that follows naturally from our decision to treat collections as objects. It is a way of combining the mass (which is a characteristic of the object) with the velocity of the object. We **define** the **momentum** to be:

$$\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}} \tag{186}$$

Thus (since the mass of an object is generally constant):

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = m\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}\left(m\vec{\boldsymbol{v}}\right) = \frac{d\vec{\boldsymbol{p}}}{dt} \tag{187}$$

is another way of writing Newton's second law. In fact, this is the way Newton actually **wrote** Newton's second law – he did not say "$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$" the way we have been reciting. We emphasize this connection because it makes the path to solving for the trajectories of constant mass particles a bit easier, not because things really make more sense that way.

Note that there exist systems (like rocket ships, cars, etc.) where the mass is **not** constant. As the rocket rises, its thrust (the force exerted by its exhaust) can be constant, but it continually gets lighter as it burns fuel. Newton's second law (expressed as $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) **does** tell us what to do in this case – but only if we treat each little bit of burned and exhausted gas as a "particle", which is a pain. On the other hand, Newton's second law expressed as $\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt}$ still works fine and makes perfect sense – it simultaneously describes the loss of mass and the increase of velocity as a function of the mass correctly.

Clearly we can repeat our previous argument for the sum of the momenta of a collection of particles:

$$\vec{\boldsymbol{P}}_{\rm tot} = \sum_i \vec{\boldsymbol{p}}_i = \sum_i m\vec{\boldsymbol{v}}_i \tag{188}$$

so that

$$\frac{d\vec{\boldsymbol{P}}_{\rm tot}}{dt} = \sum_i \frac{d\vec{\boldsymbol{p}}_i}{dt} = \sum_i \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{F}}_{\rm tot} \tag{189}$$

Differentiating our expression for the position of the center of mass above, we also get:

$$\frac{d\sum_i m_i\vec{\boldsymbol{x}}_i}{dt} = \sum_i m_i\frac{d\vec{\boldsymbol{x}}_i}{dt} = \sum_i \vec{\boldsymbol{p}}_i = \vec{\boldsymbol{P}} = M_{\rm tot}\vec{\boldsymbol{v}}_{\rm cm} \tag{190}$$

Conservation of Momentum

We are now in a position to state and trivially prove the **Law of Conservation of Momentum**. It reads:

> In the absence of **external** forces the total momentum of a system (of particles) is constant.

Please learn this law *exactly* as it is written here. The condition $\vec{\boldsymbol{F}}_{\rm tot} = 0$ is *essential* – otherwise, as you can see, $\vec{\boldsymbol{F}}_{\rm tot} = \frac{d\vec{\boldsymbol{P}}_{\rm tot}}{dt}$!

The proof is almost a one-liner at this point:

$$\vec{\boldsymbol{F}}_{\rm tot} = \sum_i \vec{\boldsymbol{F}}_i = 0 \tag{191}$$

implies

$$\frac{d\vec{\boldsymbol{P}}_{\rm tot}}{dt} = 0 \tag{192}$$

so that $\vec{\boldsymbol{P}}_{\rm tot}$ is a constant if the forces all sum to zero. This is not quite enough. We need to note that for the **internal** forces (between the ith and jth particles in the system, for example) from Newton's third law we get:

$$\vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji} \tag{193}$$

so that

$$\vec{\boldsymbol{F}}_{ij} + \vec{\boldsymbol{F}}_{ji} = 0 \tag{194}$$

pairwise, between **every** pair of particles in the system. That is, although internal forces may not be zero (and generally are not, in fact) the changes the cause in the momentum of the system cancel. We can thus subtract:

$$\vec{\boldsymbol{F}}_{\rm internal} = \sum_{i,j} \vec{\boldsymbol{F}}_{ij} = 0 \tag{195}$$

from $\vec{\boldsymbol{F}}_{\rm tot} = \vec{\boldsymbol{F}}_{\rm external} + \vec{\boldsymbol{F}}_{\rm internal}$ to get:

$$\vec{\boldsymbol{F}}_{\rm external} = \frac{d\vec{\boldsymbol{P}}_{\rm tot}}{dt} = 0 \tag{196}$$

As we shall see, the idea of momentum and its conservation greatly simplify doing a wide range of problems, just like energy and its conservation did in the last chapter.

Center of Mass Reference Frame

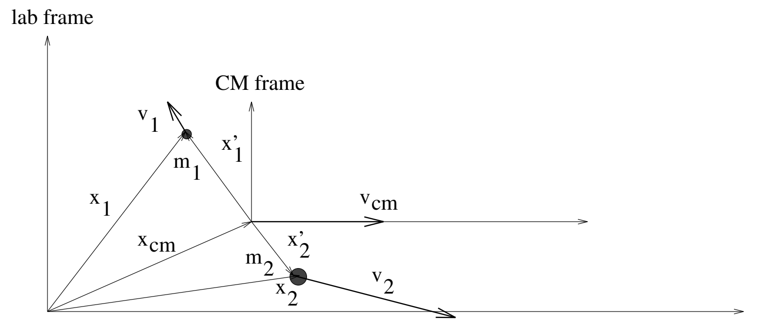


Figure 23: The coordinates of the "center of mass reference frame", a very useful inertial reference frame for solving collisions and understanding rigid rotation.

The **Center of Mass Reference Frame** is **very** convenient for analyzing certain physical phenomena (collisions in free space, rigid rotation, static equilibrium). It is defined below, and its use exemplified in the first of these instances.

In the discussion below, $\vec{x}_i$, $\vec{v}_i$ and $\vec{p}_i$ refer to the "lab" frame (where in general $\vec{P}_{\rm tot} = M_{\rm tot}\vec{v}_{\rm cm} \neq 0$) while $\vec{x}'_i$, $\vec{v}'_i$ and $\vec{p}'_i$ (with primes $'$) refer to the velocity and momentum are in the "CM" frame whose origin is at $\vec{X}_{\rm cm} = \vec{v}_{rmcm}t$ (starting the two origins at the same point at $t = 0$).

Recall that in this case the coordinate transformation between two frames (the new one with origin at $\vec{x}_{\rm cm}$) is:

$$\vec{x}'_i = \vec{x}_i - \vec{x}_{\rm cm} = \vec{x}_i - \vec{v}_{\rm cm}t \tag{197}$$

The reason that this is useful is that (differentiating with respect to time once as previously noted in the first few lectures)

$$\vec{v}'_i = \vec{v}_i - \vec{v}_{\rm cm} \tag{198}$$

provided $\vec{v}_{\rm cm}$ is constant in time. This is **true** provided only that the net external force is zero:

$$\vec{P}_{\rm tot} = M\vec{v}_{\rm cm} = \text{constant vector.} \tag{199}$$

We thus see that a "free" system (one with no net external forces acting) indeed defines an inertial reference frame wherein Newton's laws are expected to hold.

To start, let's apply this idea for just two particles. The ideas are readily extended to N particles, although the algebra and arithmetic can get pretty hairy for complicated problems. First we find the velocities of the particles **in** the CM frame

$$\vec{v}'_1 = \vec{v}_1 - \vec{v}_{\rm cm}, \quad \vec{v}'_2 = \vec{v}_2 - \vec{v}_{\rm cm} \tag{200}$$

where

$$\vec{P}_{\rm tot} = (m_1 + m_2)\vec{v}_{\rm cm} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \tag{201}$$

so that

$$\vec{v}_{\rm cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}. \tag{202}$$

Now **in** the CM frame,

$$\begin{aligned}
\vec{P}'_{\rm tot} &= \vec{p}'_1 + \vec{p}'_2 = m_1\vec{v}'_1 + m_2\vec{v}'_2 \\
&= m_1\vec{v}_1 + m_2\vec{v}_2 - \left(\frac{m_1+m_2}{m_1+m_2}\right)(m_1\vec{v}_1 + m_2\vec{v}_2) \\
&= 0!
\end{aligned} \tag{203}$$

so the CM frame (by definition) is the frame where the total momentum is **zero**. (Note: See if you can generalize this proof to work for an arbitrary number of particles by summing $\sum_i$ instead of just 1 and 2.) From this we can readily conclude that:

$$\vec{P}_{\rm tot} = M_{\rm tot}\vec{v}_{\rm cm} + \vec{P}'_{\rm tot} \tag{204}$$

or the total momentum in the lab is the momentum of the "system" plus its internal momentum (relative to the CM), which sums to zero.

We can thus speak of the internal "relative" momentum $\vec{p}'$ *in* the CM frame as well as **total** momentum $\vec{P}_{\rm tot}$ **of** the CM frame. The former adds to zero. The latter describes the uniform motion of the whole frame with no pseudoforces.

A **typical problem** using the CM frame requires you to transform the problem from the lab frame (where it is difficult) into the CM frame where it is much easier), solve it in the CM frame, and transform the result back into the lab frame. We will see applications of this idea later.

The CM frame (and this problem-solving methodology) is also useful when a **uniform** force acts on **all** the masses in the frame. For example, the frame itself can freely "fall" under the influence of a uniform external force (like gravity) without destroying the internal relations within the frame! This is because

$$\sum \vec{F}_{\rm ext} = \frac{d\vec{P}_{\rm tot}}{dt} = M_{\rm tot}\frac{d\vec{v}_{\rm cm}}{dt} \tag{205}$$

can be integrated for the trajectory $\vec{X}_{\rm cm}(t)$ just as if it were a particle, in the cases where the total external force and the total mass $M_{\rm tot}$ are known. In this case, as well, one can solve for the motion **of** the CM as if it were a particle, evaluate the motion **in** the CM from a knowlege of its internal forces or conservation properties, and transform the combined result back into the lab to find the individual trajectories $\vec{x}_i(t)$ or anything else that might be desired.

This trick "works" but is **considerably** more complicated for rotating systems because of the non-uniform pseudoforces introduced into the rotating frame. We will examine only simple cases of it (for rigid objects, where the rigidity cancels most of the complexity) in a few chapters.

Impulse

The momentum transferred during a collision that takes a time T is

$$\vec{I} = \Delta\vec{p} = \int_0^T \vec{F}(t)dt \tag{206}$$

This transferred momentum is called the "impulse". The function $\vec{F}(t)$ may be very complicated; we frequently don't care much about its details but only wish to know the change in momentum it produces, which is why the notion of **impulse** was invented. Impulse can be easily measured independent of the details of the collision.

Sometimes (such as when we develop the kinetic theory of gases later!) we will want to know what the **average** force exerted during a collision (that takes a known time T) is. By definition, it is

$$\vec{F}_{\rm avg} = \frac{1}{T}\int_0^T \vec{F}(t)dt = \frac{\vec{I}}{T}. \tag{207}$$

The average force is thus the transferred momentum divided by the time it takes to transfer it, which comes very close to its definition in calculus.

Collisions

Evaluating the dynamics and kinetics of collisions is a big part of contemporary physics – so big that we call it by a special name: **Scattering Theory**. The idea is to take some initial (presumed known) state of an about-to-collide "system", to let it collide, and to either determine something about the nature of the force that acted during the collision from the measured final state of the system or to predict, from the measured final state of some of the particles, the final state of the rest.

Sound confusing? It's not, really, but it can be **complicated** because there are lots of things that might make up an initial and final state.

To simplify matters so that we can do the algebra in this course, we will restrict our attention to collisions in one dimension. Because we are also not prepared to integrate equations of motion for a realistic interparticle potential at this time, we will further restrict our attention to collisions in one of two categories: **Elastic collisions** conserve the total mechanical energy from the time before the particles interact to after they are through interacting. It is assumed that the collision will occur so rapidly that no external forces will have time to affect this conservation of energy by doing work on the composite system during the time that the particles DO interact. **Inelastic collisions** occur when the particles collide and stick together, forming a single composite "particle" for the final state. In this case all the kinetic energy in the CM system disappears (and usually appears as heat) – the collision does not conserve total mechanical energy.

We will remark upon collisions in more than one dimension in passing, but you will only be expected to be able to solve particularly simple scattering situations in more than one dimension.

In general, **all** collisions will be assumed to take place so fast that no external forces can do meaningful work on the system during the collision. This is called the **Impulse Approximation**. This means that **momentum is conserved** during **all** collisions in this approximation, which we will consistently make (for good reason) in this course. If the collision is every "slow" enough for external forces to significantly act during the time of the collision, we would have to work much harder *and* would require a detailed knowledge of all of the forces acting during that time – we wouldn't even speak of this as a "collision"; rather we'd just be solving Newton's Second Law for all of the masses involved using a known interaction.

In the impulse approximation, as we'll see, we can reduce a *complicated* force of interaction to its most basic parameters – the change in momentum it causes and the (short) time over which it occurs. Life will be simple, life will be good. Momentum conservation (as an equation or set of equations) will yield one or more relations between the various momentum components of the initial and final state, and hence will enable us to solve them simultaneously for one or more unknowns.

For *elastic* collisions, we will add a relation based on the conservation of total **kinetic energy** immediately before and after the collision. It is assumed that all other contributions to the total mechanical energy are identical before and after if not just zero, again a variant of the impulse approximation. Two of your homework problems will treat exceptions by explicitly *giving* you a conservative, "slow" interaction force (gravity and an inclined plane slope and a spring) that *mediates* the "collision". You can use these as mental models for what really happens in elastic collisions on a much faster and more violent time frame.

For *inelastic* collisions, we will assume that the two particles form a single "particle" as a final state with the same total momentum as the system had before the collision. In these collisions, kinetic energy is *always lost*. Since energy itself is technically conserved, we can ask ourselves: Where did it go? The answer is usually: Into *heat!*

Inelastic collisions are much easier to solve than elastic ones, because there are fewer degrees of freedom in the final state (only one velocity, not two). However, we will treat them last, because there is more to be learned from elastic (or partially inelastic) collisions where there is more going on. First, however, let's think about the impulse approximation.

Elastic Collisions in One Dimension

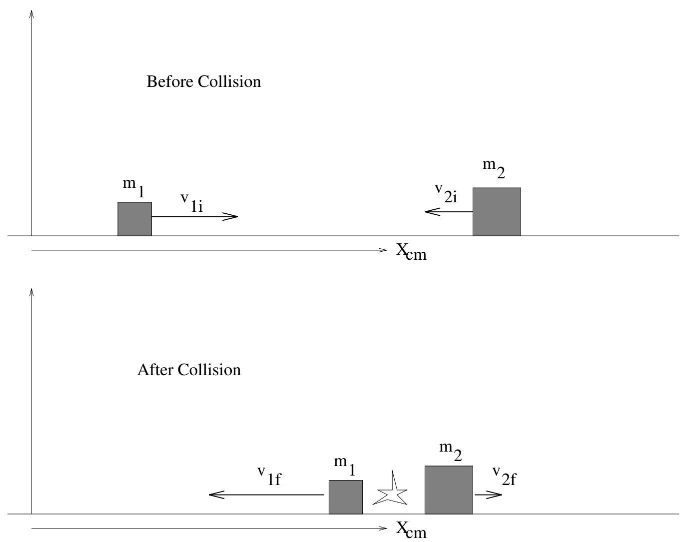


Figure 24: Before and after snapshots of an elastic collision in one dimension, illustrating the important quantities.

$$\begin{aligned} \vec{\boldsymbol{p}}_{1i} + \vec{\boldsymbol{p}}_{2i} &= \vec{\boldsymbol{p}}_{1f} + \vec{\boldsymbol{p}}_{2f} \\ m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \quad (208) \\ E_{k1i} + E_{k2i} &= E_{k1f} + E_{k2f} \\ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 &= \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (209) \end{aligned}$$

are the two relations that follow from momentum and energy conservation (in the one, e.g. – x – direction). Assuming we know m_1, m_2, v_{1i} and v_{2i}, can we find v_{1f} and v_{2f}?

Yes. There are three ways to proceed. One is to directly solve these two equations simultaneously. This involves solving an annoying quadratic and we will avoid it. The second is to note that we can do the following rearrangement of the energy and momentum equations:

$$\begin{aligned} m_1 v_{1i}^2 - m_1 v_{1f}^2 &= m_2 v_{2f}^2 - m_2 v_{2i}^2 \\ m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) &= m_2 (v_{2f} - v_{2i})(v_{2i} + v_{2f}) \quad (210) \end{aligned}$$

(from energy conservation) and

$$m_1 (v_{1i} - v_{2f}) = m_2 (v_{2f} - v_{2i}) \quad (211)$$

(from momentum conservation). When we divide the first of these by the second, we get:

$$\begin{aligned}(v_{1i} + v_{1f}) &= (v_{2i} + v_{2f}) \\ (v_{2f} - v_{1f}) &= -(v_{2i} - v_{1i}) \end{aligned} \tag{212}$$

or **the relative speed of approach before a collision equals the relative speed of recession after a collision**.

Although it isn't obvious, this equation is independent from the momentum conservation equation and can be used with it to solve for v_{1f} and v_{2f}, e.g. –

$$\begin{aligned} v_{2f} &= v_{1f} - (v_{2i} - v_{1i}) \\ m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2(v_{1f} - (v_{2i} - v_{1i})) \\ (m_1 + m_2) v_{1f} &= 2 m_2 v_{2i} + (m_1 - m_2) v_{1i} \\ v_{1f} &= \frac{m_1 - m_2}{(m_1 + m_2)} v_{1i} + \frac{2 m_2}{(m_1 + m_2)} v_{1i} \end{aligned} \tag{213}$$

(back substitute for v_{2f}).

This is satisfying and easy to evaluate, but still hard to understand. To **understand** the collision, it is easiest to **put everything into the CM frame, evaluate the collision, and then put the results back into the lab frame!** Let's try this:

In the CM frame,

$$u_{1i} = v_{1i} - v_{\rm cm} \tag{214}$$

$$u_{2i} = v_{2i} - v_{\rm cm} \tag{215}$$

$$v_{\rm cm} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \tag{216}$$

so that the momentum conservation equation becomes:

$$m_1 u_{1i} + m_2 u_{2i} = m_1 u_{1f} + m_2 u_{2f} \tag{217}$$

$$p'_{1i} + p'_{2i} = p'_{1f} + p'_{2f} = 0 \tag{218}$$

Thus $p'_i = p'_{1i} = -p'_{2i}$ and $p_f = p'_{1f} = p'_{2f}$. The energy conservation equation (in terms of the p's) becomes:

$$\frac{p_i'^2}{2m_1} + \frac{p_i'^2}{2m_2} = \frac{p_f'^2}{2m_1} + \frac{p_f'^2}{2m_2} \tag{219}$$

$$\frac{p_i'^2}{2} \frac{1}{m_1 + m_2} = \frac{p_f'^2}{2} \frac{1}{m_1 + m_2} \tag{220}$$

$$p_i'^2 = p_f'^2 \tag{221}$$

so that we can conclude that

$$p'_{1f} = \pm p'_{1i} \tag{222}$$

The + sign satisfies all the conditions, but means the the particles "missed". If they hit, their internal momenta relative to the CM reverses! Thus:

$$\begin{aligned} m_1 u_{1f} &= -m_1 u_{1i} && (223) \\ u_{1f} &= -u_{1i} && (224) \\ v_{1f} - v_{\rm cm} &= -(v_{1i} - v_{\rm cm}) && (225) \\ v_{1f} &= -v_{1i} + 2v_{\rm cm} && (226) \\ &= -\frac{-(m_1+m_2)v_{1i} + 2(m_1 v_{1i} + m_2 v_{2i})}{m_1+m_2} && (227) \\ &= -\frac{(m_1-m_2)v_{1i} + 2m_2 v_{2i}}{m_1+m_2} && (228) \end{aligned}$$

(and ditto for v_{2f}) which can be seen to equal the result above by inspection. It is, however, a bit easier to understand!

Let's collect these two final equations (from any of the three derivations) into one place and look at them:

$$\begin{aligned} v_{1f} &= -v_{1i} + 2v_{\rm cm} && (229) \\ v_{2f} &= -v_{2i} + 2v_{\rm cm} && (230) \\ & && (231) \end{aligned}$$

Note that they are *completely symmetric* – we obviously don't care which of the two particles is *labelled* "1" or "2", so the answer should be the same for both. If you are a physics major, you should be prepared to derive this result one of the various ways it can be derived (I'd strongly suggest the last way, using the CM frame). If you are e.g. a life science major or engineer, you should derive this result for yourself *at least once, at least one of the ways* (again, I'd suggest that last one) but then you are welcome to memorize/learn the resulting equations well enough to use them.

That way if you *forget* them – I have little confidence in memorization as a learning tool for mountains of complicated material – you have a *prayer* of being able to rederive them on a test.

The BB Limits

It is also worthwhile thinking about the "ball bearing and bowling ball (BB) limits" of elastic scattering. These are basically the asymptotic results one obtains when one hits a stationary bowling ball (large mass, BB) with rapidly travelling ball bearing (small mass, bb). We all know that if you shoot a bb gun at a bowling ball, it will bounce off as fast as it comes in (almost) and the bowloing ball will hardly recoil[30]. Given that $v_{\rm cm}$ in this case is more or less equal to v_{BB}, that is, $v_{\rm cm} \approx 0$ (just a bit greater), note that this is *exactly* what the solution predicts.

What happens if you throw a bowling ball at a stationary bb? Well, we know perfectly well that the BB in this case will just continue barrelling along at more or less $v_c m$ (still

[30]...and you'll put your eye out – kids, do *not* try this at home!

roughly equal to the velocity of the more massive bowling ball) – when your car hits a bug with the windshield, it doesn't significantly slow down. The bb (or the bug)? According to our results above, it will bounce off the BB and recoil forward at *twice the speed* of the BB. Note well that both of these results preserve the idea derived above that the relative velocity of approach equals the relative velocity of recession!

Finally, there is the "pool ball limit" of the elastic collision of equal masses. When the cue ball strikes another ball head on (with no English) as pool players well know the cue ball stops dead and the other ball continues on at the original speed of the cue ball. Note that this, too, is exactly what the equations predict, since $v_{\rm cm} = v_{1i}/2$.

Any answer you derive (such as this one) ultimately has to pass the test of common-sense agreement with your everyday experience. This one seems to, so difficult as the derivation is, it appears to be correct!

Pool is a good example of a game of "approximately elastic collisions" because the hard balls used in the game have a very elastic coefficient of restitution, another way of saying that the surfaces of the balls behave like very small, very hard springs and store and re-release the kinetic energy of the collision from a conservative impulse type force. What happens if the collision between two balls is *not* along a line? Well, then we have to take into account momentum conservation in *two* dimensions. So alas, sorry fellow human students, you have to bite the bullet and at least think a bit about collisions in more than one dimension.

Elastic Collisions in 2-3 Dimensions

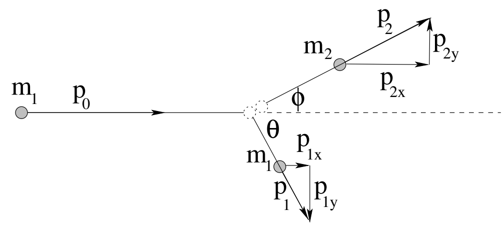


Figure 25: The geometry for an elastic collision in a two-dimensional plane.

As we can see, elastic collisions in one dimension are "good" because we can completely solve them using only kinematics – we don't care about the *details* of the interaction between the colliding entities; we can find the final state from the initial state for all possible elastic forces and the only differences that will depend on the forces will be things like how *long* it takes for the collision to occur.

In 2+ dimensions we at the very least have to work much harder to solve the problem. We can – barely, subject to some assumptions – use nothing but vector momentum conservation and energy conservation to solve the problem independent of most of the details of the interaction, but now we have more things to solve for. We also cannot make the result *completely* insensitive to the details of the interaction unless it is really a kind of "hard sphere" collision where the impact approximation really holds at the point and at the scale where the primary bending of the incoming momentum takes place.

This is good and bad. The good is that we can *learn* things about the interaction from the results of a collision experiment. The bad is that for the most part the algebra infolved in solving multidimensional collisions is beyond the scope of this course. Physics majors will have to sweat blood later to work all this out, but non-majors or majors taking their intro course get a bit of a break, here.

There are a few things that *are* within the scope of the course. After all, there are some things we *can* say, in general, about elastic collisions in 2-3 dimensions even in this course.

Coordinates of the Problem

We will suppose that our collision is between mass m_1, coming in with momentum $\vec{p}_0$ from the left as shown above, and a stationary "target" mass m_2. More generally, m_2 could itself be moving. There are six "degrees of freedom" associated with the incoming particles before the collision – three momentum coordinates per mass – in addition to the masses themselves.

After the collision, we will assume that we still have the same two particles with the same two masses. m_1 will recoil with momentum $\vec{p}_1$, m_2 will recoil with momentum $\vec{p}_2$. We want to find these two final momenta (or equivalently, the velocities of the two outgoing particles). Either way, we have six[31] unknown numbers to solve for: The three vector cooordinates for each of the two momenta/velocities.

If we make some fairly general assumptions about the interaction – such as that it is elastic, is either attractive or repulsive and acts only along a line connecting the two masses at some distance, then the outgoing particles both lie in a *plane* – the so-called *scattering plane*. This gets rid of two degrees of freedom in our final answer by setting the final momenta in the direction perpendicular to the scattering plane equal to zero after the collision. This leaves us with four – the two components of the momentum of each particle in the 2D scattering plane.

We this have a bunch of numbers to use as input: $m_1, m_2, v_{1x}, v_{1y}, v_{1z}, v_{2x}, v_{2y}, v_{2z}$ – and four numbers to produce as output *in the scattering plane* (assuming that we can somehow identify it): $p_{1x}, p_{1y}, p_{2x}, p_{2y}$. Again, the geometry of this is shown above.

Depending on something called the *impact parameter* – the distance off-axis of the target that the incident particle strikes – the outgoing particles can scatter into many possible angles. As noted, we cannot quite predict which angles without knowning *any* of the details of the interaction and the impact parameter b. However, we do have some equations to help us solve the problem of finding *some* of these numbers given a few of the others.

Conservation Laws

We expect both energy and momentum to be conserved in any elastic collision. This gives us the following set of equations:

$$p_{0x} = p_{1x} + p_{2x} \tag{232}$$

$$p_{0y} = p_{1y} + p_{2y} \tag{233}$$

$$p_{0z} = p_{1z} + p_{2z} \tag{234}$$

$$\frac{p_0^2}{2m_1} = E_0 = E_1 + E_2 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{m_2} \tag{235}$$

We have four equations, and four unknowns, so we *might* hope to be able to solve it quite generally. However, we don't *really* have that many equations – if we assume that the scattering plane is the $x - y$ plane, then necessarily $p_{0z} = 0$ and this equation tells us nothing. We need more information in order to be able to solve the problem.

Let's see what we *can* tell in this case. Examine figure 25. Note that we have introduced two angles: θ and ϕ for the incident and target particle's outgoing angle with respect to the incident direction. Using them and setting $p_{0y} = p_{0z} = 0$ (and assuming that the target is at rest initially and has no momentum at all initially) we get:

[31] This is already a bit of an approximation, because in real nuclear or atomic scattering, the outgoing particles might be more than two in number, they might have different masses from the incident particles, and some of the outgoing particles might even be "invisible". We're doing only the simple "billiard ball" sort of BB collision problem in more than 1 dimension.

$$p_{0x} = p_{1x} + p_{2x} = p_1 \cos(\theta) + p_2 \cos(\phi) \tag{236}$$
$$p_{0y} = p_{1y} + p_{2y} = 0 = -p_1 \sin(\theta) + p_2 \sin(\phi) \tag{237}$$

In other words, the momentum in the x-direction is conserved, and the momentum in the y-direction (after the collision) cancels. The latter is a powerful relation – if we know the y-momentum of one of the outgoing particles, we know the other. If we know the magnitudes/energies of both, we know an important relation between their angles.

This, however, puts us no closer to being able to solve the general problem (although it does help with a special case that is on your homework). To make real progress, it is necessarily to once again change to the center of mass reference frame by subtracting $\vec{v}_{\rm cm}$ from the velocity of both particles. We can easily do this:

$$\vec{p}'_{i1} = m_1(\vec{v}_0 - \vec{v}_{\rm cm}) = m_1 u_1 \tag{238}$$
$$\vec{p}'_{i2} = -m_2\vec{v}_{\rm cm} = m_2 u_2 \tag{239}$$

so that $\vec{p}'_{i1} + \vec{p}'_{i2} = \vec{p}'_{\rm tot} = 0$ in the center of mass frame as usual. The initial energy in the center of mass frame is just:

$$E_i = \frac{p^{2'}_{i1}}{2m_1} + \frac{p^{2'}_{i2}}{2m_2} \tag{240}$$

Since $p'_{i1} = p'_{i2} = p'_i$ (the magnitudes are equal) we can simplify this a bit further:

$$E_i = \frac{p^{2'}_{i}}{2m_1} + \frac{p^{2'}_{i}}{2m_2} = \frac{p^{2'}_{i}}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p^{2'}_{i}}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right) \tag{241}$$

After the collision, we can see by inspection of

$$E_f = \frac{p^{2'}_{f}}{2m_1} + \frac{p^{2'}_{f}}{2m_2} = \frac{p^{2'}_{f}}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p^{2'}_{f}}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right) = E_i \tag{242}$$

that $p'_{f1} = p'_{f2} = p'_f = p'_i$ will cause energy to be conserved, just as it was for a 1 dimensional collision. All that can change, then, is the *direction* of the incident momentum in the center of mass frame. In addition, since the total momentum in the center of mass frame is by definition zero before and after the collision, if we know the *direction* of either particle after the collision in the center of mass frame, the other is the opposite:

$$\vec{p}'_{f1} = -\vec{p}'_{f2} \tag{243}$$

We have then "solved" the collision as much as it can be solved. We cannot *uniquely* predict the direction of the final momentum of either particle in the center of mass (or any other) frame without knowing more about the interaction and e.g. the incident impact parameter. We can predict the magnitude of the outgoing momenta, and if we know the outgoing direction alone of either particle we can find everything – the magnitude and direction of the other particle's momentum and the magnitude of the momentum of the particle whose angle we measured.

As you can see, this is all pretty difficult, so we'll leave it at this point as a partially solved problem, ready to be tackled again for specific interactions or collision models in a future course.

Inelastic Collisions

A fully inelastic collision is where two particles collide and *stick together*. As always, momentum is conserved in the impact approximation, but now *kinetic energy is not!* In fact, we will see that macroscopic kinetic energy is always *lost* in an inelastic collision, either to heat or to some sort of mechanism that traps and reversibly stores the energy.

These collisions are much easier to understand and analyze than elastic collisions. That is because there are *fewer degrees of freedom* in an inelastic collision – we can easily solve them even in 2 or 3 dimensions. The whole solution is developed from

$$\vec{\boldsymbol{p}}_{i,\mathrm{ntot}} = m_1\vec{\boldsymbol{v}}_{1i} + m_2\vec{\boldsymbol{v}}_{2i} = (m_1 + m_2)\vec{\boldsymbol{v}}_f = (m_1 + m_2)\vec{\boldsymbol{v}}_{\mathrm{cm}} = \vec{\boldsymbol{p}}_{f,\mathrm{tot}} \qquad (244)$$

In other words, in a fully inelastic collision, the velocity of the outgoing combined particle is the velocity of the *center of mass* of the system, which we can easily compute from a knowledge of the initial momenta or velocities and masses. Of course! How obvious! How easy!

From this relation you can easily find $\vec{\boldsymbol{v}}_f$ in any number of dimensions, and answer many related questions. The collision is "solved". However, there are a number of different kinds of *problems* one can solve given this basic solution – things that more or less tag additional physics problems on to the end of this initial one and use its result as their *starting* point, so you have to solve two or more subproblems in one long problem, one of which is the "inelastic collision". This is best illustrated in some archetypical examples.

Example 4.0.4: One-dimensional Fully Inelastic Collision (only)

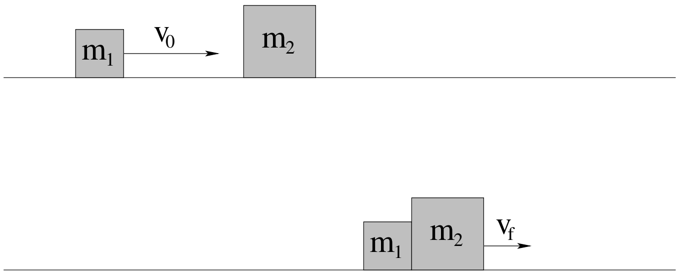


Figure 26: Two blocks of mass m_1 and m_2 collide and stick together on a frictionless table.

In figure 26 above, a block m_1 is sliding across a frictionless table at speed v_0 to strike a second block m_2 initially at rest, whereupon they stick together and move together as one thereafter at some final speed v_f.

Before, after, and during the collision, gravity acts but is opposed by a normal force. There is no friction or drag force doing any work. The only forces in play are the *internal* forces mediating the collision and making the blocks stick together. We therefore know that **momentum is conserved** in this problem independent of the features of that internal interaction. Even if friction or drag forces *did* act, as long as the collision took place "instantly" in the impact approximation, momentum would still be conserved from immediately before to immediately after the collision, when the impulse Δp of the collision force would be much, much larger than any change in momentum due to the drag over the same small time Δt.

Thus:

$$p_i = m_1 v_0 = (m_1 + m_2) v_f = p_f \tag{245}$$

or

$$v_f = \frac{m v_0}{(m_1 + m_2)} (= v_{\rm cm}) \tag{246}$$

A traditional question that accompanies this is: How much kinetic energy was lost in the collision? We can answer this by simply figuring it out.

$$\begin{aligned}
\Delta K &= K_f - K_i = \frac{p_f^2}{2(m_1+m_2)} - \frac{p_i^2}{2m_1} \\
&= \frac{p_i^2}{2}\left(\frac{1}{(m_1+m_2)} - \frac{1}{m_1}\right) \\
&= \frac{p_i^2}{2}\left(\frac{m1-(m1+m2)}{m_1(m_1+m_2)}\right) \\
&= -\frac{p_i^2}{2m_1}\left(\frac{m2}{(m_1+m_2)}\right) \\
&= -\left(\frac{m2}{(m_1+m_2)}\right) K_i
\end{aligned} \tag{247}$$

where we have expressed the result as a *fraction of the initial kinetic energy!*

Note well the BB limits: For a light bb (m_1) striking a massive BB (m_2), nearly *all* the energy is lost. This sort of collision between an asteroid (bb) and the earth (BB) caused at least one of the mass extinction events, the one that ended the Cretaceous and gave mammals the leg up that they needed in a world dominated (to that point) by dinosaurs. For a massive BB (m_1) stricking a light bb (m_2) very little of the energy of the massive object is lost. Your truck hardly slows when it smushes a bug "inelastically" against the windshield. In the equal billiard ball bb collision ($m_1 = m_2$), exactly one half of the initial kinetic energy is lost.

A similar collision in 2D is given for your homework, where a truck and a car inelastically collide and then slide down the road together. In this problem friction works, but *not during the collision!* Only after the "instant" (impact approximation) collision do we start to worry about the effect of friction.

Example 4.0.5: Ballistic Pendulum

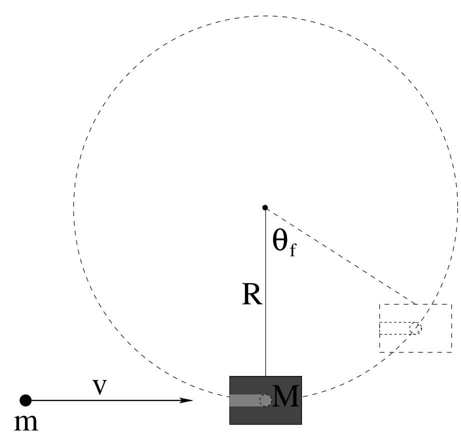


Figure 27: The "ballistic pendulum", where a bullet strikes and sticks to/in a block, which then swings up to a maximum angle θ_f before stopping and swinging back down.

The classic ballistic pendulum question gives you the mass of the block M, the mass of the bullet m, the length of a string or rod suspending the "target" block from a free pivot, and the initial velocity of the bullet v_0. It then asks for the maximum angle θ_f through which the pendulum swings after the bullet hits and sticks to the block (or alternatively, the maximum height H through which it swings). Variants abound – on your homework you might be asked to find the *minimum speed* v_0 the bullet must have in order the the block whirl around in a circle on a never-slack string, or on the end of a rod. Still other variants permit the bullet to pass through the block and emerge with a different (smaller) velocity. You should be able to do them all, if you completely understand this example (and the other physics we have learned up to now, of course).

There is an actual lab that is commonly done to illustrate the physics; in this lab one typically measures the maximum horizontal displacement of the block, but it amounts to the same thing once one does the trigonometry.

The solution is simple:

- **During the collision momentum is conserved** in the impact approximation, which in this case basically implies that the block has no time to swing up appreciably "during" the actual collision.
- **After the collision mechanical energy is conserved**. Mechanical energy is *not* conserved during the collision (see solution above of straight up inelastic collision).

One can replace the second sub-problem with *any other problem* that requires a knowledge of either v_f or K_f immediately after the collision as its initial condition. Ballistic loop-the-loop problems are entirely possible, in other words!

At this point the algebra is almost anticlimactic: The collision is one-dimensional (in the x-direction). Thus (for block M and bullet m) we have momentum conservation:

$$p_{m,0} = mv_0 = p_{M+m,f} \tag{248}$$

Now if we were foolish we'd evaluate $v_{M+m,f}$ to use in the next step: mechanical energy conservation. Being smart, we instead do the kinetic part of mechanical energy conservation in terms of momentum:

$$\begin{aligned} E_0 = \frac{p_{B+b,f}^2}{2(M+m)} &= \frac{p_{b,0}^2}{2(M+m)} \\ &= E_f = (M+m)gH \\ &= (M+m)gR(1-\cos\theta_f) \end{aligned} \tag{249}$$

Thus:

$$\theta_f = \cos^{-1}\left(1 - \frac{(mv_0)^2}{2(M+m)^2 gR}\right) \tag{250}$$

which only has a solution if mv_0 is less than some maximum value. What does it mean if it is greater than this value (there is no inverse cosine of an argument with magnitude bigger than 1)? Will this answer "work" if $\theta > \pi/2$, for a string? For a rod? For a track?

Don't leave your common sense at the door when solving problems using algebra!

Example 4.0.6: Partially Inelastic Collision

Let's briefly consider the previous example in the case where the bullet passes *through* the block and emerges on the far side with speed $v_1 < v_0$ (both given). How is the problem going to be different?

Not at all, not really. Momentum is still conserved during the collision, mechanical energy after. The *only two differences* are that we have to evaluate the speed of the block after the collision from *this* equation:

$$p_0 = m_1 v_0 = Mv_f + m_1 v_1 = p_m + p_1 = p_f \tag{251}$$

so that:

$$v_f = \frac{\Delta p}{M} = \frac{m_1(v_0 - v_1)}{M} \tag{252}$$

We can read this as "the momentum transferred to the block is the momentum lost by the bullet" because momentum is conserved. Given v_f of the block *only*, you should be able to find e.g. the kinetic energy lost in this collision or θ_f or whatever in any of the many variants involving slightly different "after"-collision subproblems.

You should be able to do them all.

Kinetic Energy in the CM Frame

If we extend this treatment to the **kinetic energy** E_k, we also obtain an interesting statement:

$$E_{k,\mathrm{tot}} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{p_i^2}{2m_i}. \tag{253}$$

Now we recall (from above) that $\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_i' + \vec{\boldsymbol{v}}_{\mathrm{cm}}$ so that

$$\vec{\boldsymbol{p}}_i = m_i \vec{\boldsymbol{v}}_i = m_i \vec{\boldsymbol{v}}_i' + m_i \vec{\boldsymbol{v}}_{\mathrm{cm}} \tag{254}$$

and

$$E_{ki} = \frac{p_i^2}{2m_i} = \frac{m_i v_i'^2}{2m_i} + \frac{2m_i^2 \vec{\boldsymbol{v}}_i' \cdot \vec{\boldsymbol{v}}_{\mathrm{cm}}^2}{2m_i} + \frac{m_i^2 V_{\mathrm{cm}}^2}{2m_i}. \tag{255}$$

If we sum this:

$$\begin{aligned} E_{k,\mathrm{tot}} &= \sum_i E_{ki} = \sum_i \frac{p_i'^2}{2m_i} + \frac{P_{\mathrm{tot}}^2}{2M_{\mathrm{tot}}} + \vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \underbrace{\sum_i m_i \vec{\boldsymbol{v}}_i'}_{=\sum_i \vec{\boldsymbol{p}}_i' \;=0} \\ &= E_k(\text{in cm}) + E_k(\text{of cm}) \end{aligned} \tag{256}$$

We thus see that the total kinetic energy in the lab frame is the sum of the kinetic energy of all the particles in the CM frame **plus** the kinetic energy of the CM frame itself (viewed as single "object"). To put it another way, the kinetic energy of a baseball flying through the air is the kinetic energy of the "baseball itself" (the entire system viewed as a particle) plus the kinetic energy of all the particles that make up the baseball measured in the CM frame of the baseball itself. This is comprised of **rotational kinetic energy** (which we will shortly treat) plus all the general vibrational (atomic) kinetic energy that is what we would call **heat**.

We see that we can indeed break up big systems into smaller systems and vice versa!

Homework for Week 4

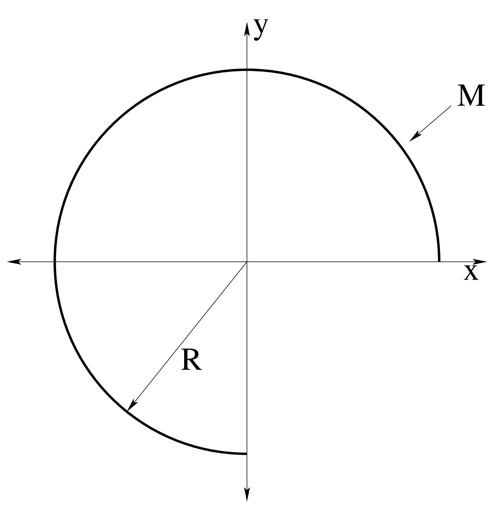
Problem 1.

Physics Concepts: As the first *required* step of solving the homework problems below, make a list of the following physics concepts or equations (that you *must know and understand* in order to solve them). Note that some of the necessary concepts will usually be from the previous weeks of the course. Also, one or two from previous weeks may be listed *just* for review and *not* be needed; you must not forget any of the earlier concepts as you learn new ones.

Note also that creating this list *is* part of your homework assignment and it must be handed in. As such, your homework (punched and added to e.g. a three ring binder) will gradually become a very effective study guide for hour exams and the final.

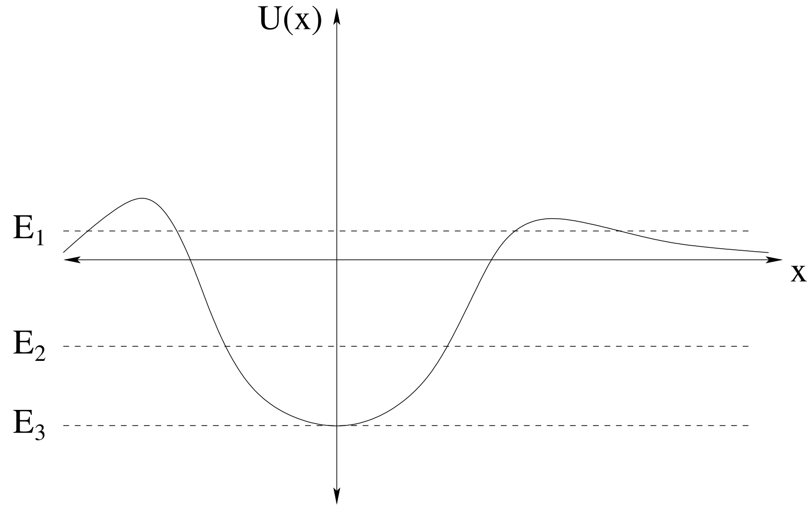
- Newton's Second Law

Problem 2.



In the figure above, a uniformly thick piece of wire is bent into 3/4 of a circular arc as shown. Find the center of mass of the wire in the coordinate system given, using integration to find the x_{cm} and y_{cm} components separately.

Problem 3.



This problem will help you learn required concepts such as:

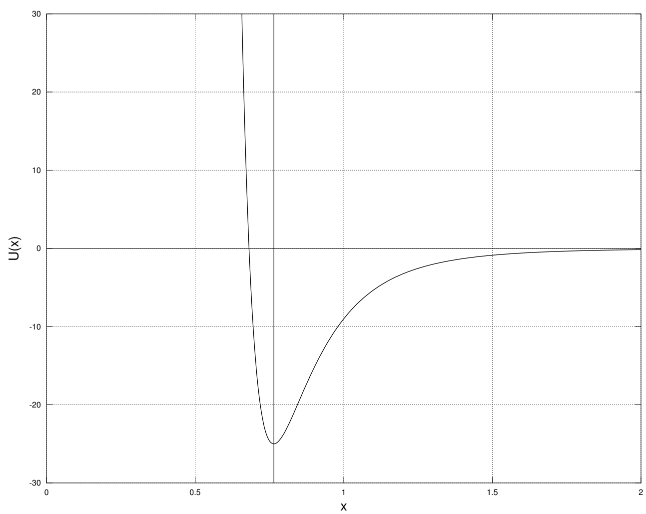
- Potential energy/total energy diagrams
- Finding the force from the potential energy
- Identifying turning points and stable/unstable equilibribum points on a graph of the potential energy
- Identifying classically forbidden versus allowed domains of motion (in one dimension) on an energy diagram

so please review them before you begin.

1) On (a copy of) the diagram above, place a small letter 'u' to mark points of unstable equilibrium.
2) Place the letter 's' to mark points of stable equilibrium.
3) On the curve itself, place a few arrows in each distinct region indicating the *direction* of the force. Try to make the lengths of the arrows proportional in a *relative* way to the arrow you draw for the largest magnitude force.
4) For the three energies shown, mark the *turning points* of motion with the letter 't'.
5) For energy E_2, place $\overbrace{\qquad\qquad}^{\text{allowed}}$ to mark out the *classically allowed region* where the particle might be found. Place $\underbrace{\qquad\qquad}_{\text{forbidden}}$ to mark out the *classically forbidden region* where the particle can never be found.

Be sure that you are comfortable finding turning points and allowed/forbidden domains for the other two total energies shown and indeed *any* total energy that might be given, but you don't need to mark the latter onto your diagram to keep it from getting *too* messy.

Problem 4.



This problem will help you learn required concepts such as:

- Energy conservation and the use of $E = K + U$ in graphs.
- Finding the force from the potential energy.
- Finding turning points and stable/unstable equilibrium points from algebraic expressions for the potential as well as visualizing the result on a graph.

so please review them before you begin.

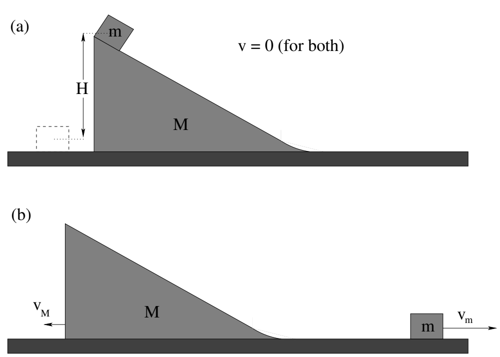
An object moves in the force produced by a potential energy function:

$$U(x) = \frac{1}{x^{12}} - \frac{10}{x^6}$$

This is a one-dimensional representation of an actual important physical potential, the *Lennard-Jones* potential. This "12-6"Lennard-Jones potential models the dipole-induced dipole Van der Waals interaction between two atoms or molecules in a gas, but the "12-10" form can also model hydrogen bonds in physical chemistry. Note well that the force is *strongly repulsive* inside the $E = 0$ turning point x_t (which one can think of as where the atoms "collide") but weakly attractive for all $x > x_0$, the position of stable equilibrium.

1) Write an algebraic expression for $F_x(x)$.
2) Find x_0, the location of the *stable equilibrium distance* predicted by this potential.
3) Find $U(x_0)$ the *binding energy* for an object located at this distance.
4) Find x_t, the turning point distance for $E = 0$. This is essentially the sum of the radii of the two atoms (in suitable coordinates – the parameters used in this problem are not intended to be physical).

Problem 5.



This problem will help you learn required concepts such as:

- Newton's Third Law
- Momentum Conservation
- Fully Elastic Collisions

so please review them before you begin.

A small block with mass m is sitting on a large block of mass M that is sloped so that the small block can slide down the larger block. There is no friction between the two blocks, no friction between the large block and the table, and no drag force. The *center of mass* of the small block is located a height H above where it would be if it were sitting on the table, and both blocks are started at rest (so that the *total momentum* of this system is *zero*, note well!)

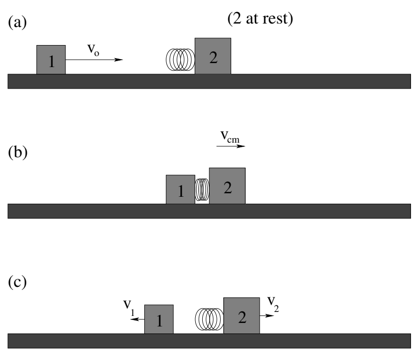
1) Are there any *net* external forces acting in this problem? What quantities do you expect to be conserved?

2) Using suitable conservation laws, find the velocities of the two blocks after the small block has slid down the larger one and they have separated.

3) To check your answer, consider the limiting case of $M \to \infty$ (where one rather expects the larger block to pretty much not move). Does your answer to part b) give you the usual result for a block of mass m sliding down from a height H on a *fixed* incline?

4) This problem doesn't *look* like a collision problem, but it easily could be half of one. Look carefully at your answer, and see if you can determine what initial velocity one should give the two blocks so that they would move *together* and precisely come to rest with the smaller block a height H above the ground. If you put the two halves together, you have solved a fully elastic collision in one dimension in the case where the center of mass velocity is zero!

Problem 6.



This problem will help you learn required concepts such as:

- Newton's Third Law
- Momentum Conservation
- Energy conservation
- Impulse and average force in a collision

so please review them before you begin.

This problem is intended to walk you through the concepts associated with collisions in one dimension.

In (a) above, mass m_1 approaches mass m_2 at velocity v_0 to the right. Mass m_2 is initially at rest. An ideal massless spring with spring constant k is attached to mass m_2

and we will assume that it will not be fully compressed from its uncompressed length in this problem.

Begin by considering the forces that act, neglecting friction and drag forces. What will be conserved throughout this problem?

1) In (b) above, mass m_1 has collided with mass m_2, compressing the spring. At the particular instant shown, *both masses are moving with the same velocity to the right.* Find this velocity. What physical principle do you use?

2) Also find the compression Δx of the spring at this instant. What physical principle do you use?

3) The spring has sprung back, pushing the two masses apart. Find the final velocities of the two masses. Note that the diagram assumes that $m_2 > m_1$ to guess the final directions, but in general your answer should make sense regardless of their relative mass.

4) So *check* this. What are the two velocities in the "BB limits" – the $m_1 \gg m_2$ (bowling ball strikes ball bearing) and $m_1 \ll m_2$ (ball bearing strikes bowling ball) limits? In other words, does your answer make dimensional and intuitive *sense*?

5) In this particular problem one could in principle solve Newton's second law because the elastic collision force is *known*. In general, of course, it is not known, although for a very *stiff* spring this model is an excellent one to model collisions between hard objects. Assuming that the spring is sufficiently stiff that the two masses are in contact for a very short time Δt, write a simple expression for the *impulse* imparted to m_2 and *qualitatively* sketch $F_{\rm av}$ over this time interval compared to $F_x(t)$.

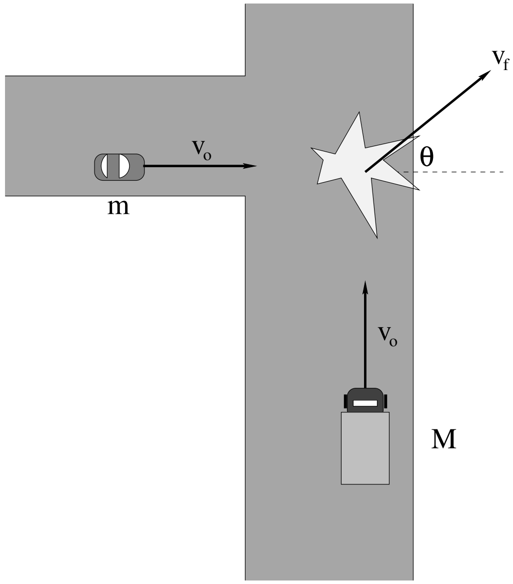
Problem 7.

This problem will help you learn required concepts such as:

- Newton's Third Law
- Momentum Conservation
- Fully Inelastic Collisions

so please review them before you begin.

In the figure above, a large, heavy truck with mass M speeds through a red light to collide with a small, light compact car of mass m. Both cars fail to brake and are travelling at the speed limit (v_0) at the time of the collision, and their metal frames tangle together in the collision so that after the collision they move as one big mass.



1) Which exerts a larger force on the other, the car or the truck?

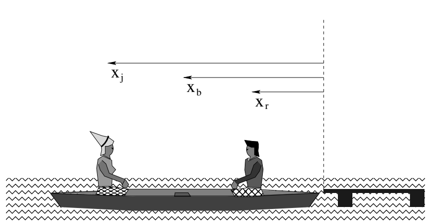
2) Which transfers a larger momentum to the other, the car or the truck?

3) What is the final velocity of the wreck immediately after the collision (please give (v_f, θ))?

4) How much kinetic energy was lost in the collision?

5) If the tires blow and the wreckage has a coefficient of kinetic friction μ_k with the ground after the collision, *set up an expression* in terms of v_f that will let you solve for how far the wreck slides before coming to a halt. You do not need to substitute your expression from part c) into this and get a final answer, but you should definitely be *able* to do this on a quiz or exam.

Problem 8.



This problem will help you learn required concepts such as:

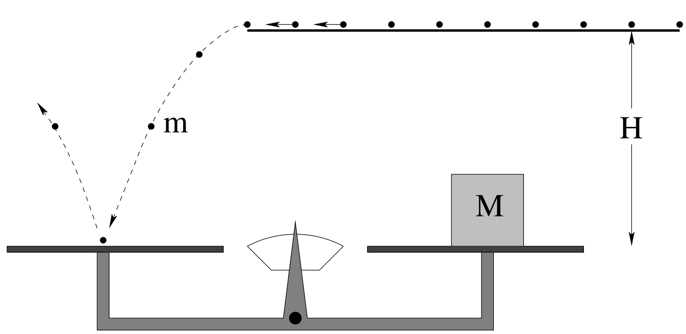
- Center of Mass
- Momentum Conservation
- Newton's Third Law

so please review them before you begin.

Romeo and Juliet are sitting in a boat at rest next to a dock, looking deeply into each other's eyes and wishing that they were fishing. Juliet, overcome with emotion, gets up to give Romeo a kiss. She walks at a constant speed v relative to the water from her end of the boat to his, and it takes her Δt to reach him. Assume that the masses and positions of Romeo, Juliet and the boat are (m_r, x_r), (m_j, x_j), (m_b, x_b) initially, where x is measured from the dock as shown.

1) While she is moving, the boat and Romeo are moving at speed v' in the opposite direction. What is the ratio v'/v?
2) Find v and v' in terms of your answer to a) and the distance $D = x_j - x_r$.
3) How far has the boat moved away from the dock when she reaches him?
4) Make sure that your answer makes sense by thinking about the following limits: $m_b \gg m_r = m_j$; $m_b \ll m_r = m_j$.

Problem 9.



This problem will help you learn required concepts such as:

- Newton's Second Law
- Gravitation
- Newton's Third Law

- Impulse and Average Force
- Fully Elastic Collisions

so please review them before you begin.

In the figure above, a feeder device provides a steady stream of beads of mass m that fall a distance H and bounce elastically off of one of the hard metal pans of a beam balance scale (and then fall somewhere else into a hopper and disappear from our problem). N beads per second come out of the feeder to bounce off of the pan. Our goal is to derive an expression for M, the mass we should put on the other pan to balance the *average force* exerted by this stream of beads[32]

1) First, the easy part. The beads come off of the feeder with an initial velocity of $\vec{v} = v_{0x}\hat{x}$ in the x-direction only. Find the y-component of the velocity v_y when a single bead hits the pan after falling a height H.
2) Since the beads bounce elastically, the x-component of their velocity is unchanged and the y-component *reverses*. Find the change of the *momentum* of this bead $\Delta\vec{p}$ during its collision.
3) Compute the *average force* being exerted on the stream of beads by the pan over a second (assuming that $N \gg 1$, so that many beads strike the pan per second).
4) Use Newton's Third Law to deduce the average force exerted by the beads on the pan, and from this determine the mass M that would produce the same force on the other pan to keep the scale in balance.

Optional Problems

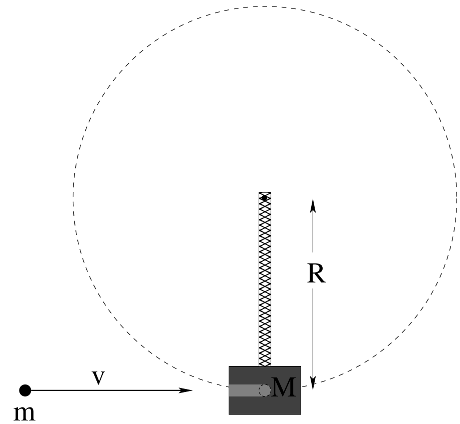
The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

Ballistic Pendulum Loops the Loop

This problem will help you learn required concepts such as:

- Conservation of Momentum
- Conservation of Energy
- Disposition of energy in fully inelastic collisions
- Circular motion
- The different kinds of constraint forces exerted by rigid rods versus strings.

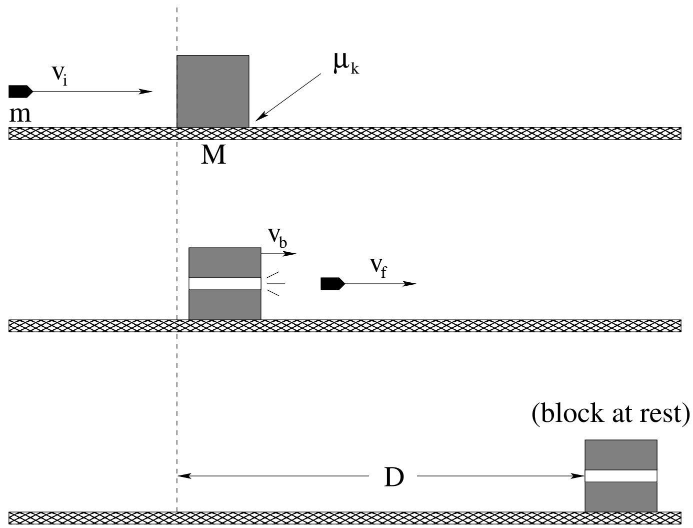
[32] This is very similar (conceptually) to the way a gas microscopically exerts a force on a surface that confines it; we will later use this idea to understand the pressure exerted by a fluid and to derive the kinetic theory of gases and the ideal gas law $PV = NkT$, which is why I assign it in particular now.



so please review them before you begin.

A block of mass M is attached to a rigid massless rod of length R (pivot to center-of-mass of the block/bullet distance at collision) and is suspended from a pivot. A bullet of mass m travelling at velocity v strikes it as shown and is quickly stopped by friction in the hole so that the two masses move together as one thereafter. Find the minimum speed v_r that the bullet must have in order to swing through a complete circle after the collision.

Bullet Goes Through Block on Rough Surface



This problem will help you learn required concepts such as:

- Momentum Conservation
- The Impact Approximation
- Elastic versus Inelastic Collisions
- The Non-conservative Work-Mechanical Energy Theorem

so please review them before you begin.

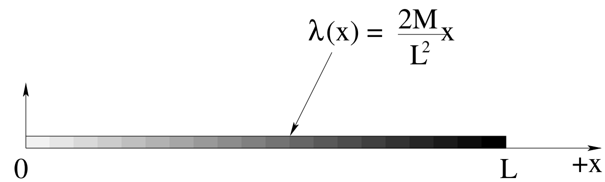
In the figure above a bullet of mass m is travelling at initial speed v_i to the right when it strikes a larger block of mass M that is resting on a rough horizontal table (with coefficient of friction between block and table of μ_k). Instead of "sticking" in the block, the bullet blasts its way *through* the block (without changing the mass of the block significantly in the process). It emerges with the smaller speed v_f, still to the right.

1) Find the speed of the block v_b *immediately after* the collision (but before the block has had time to slide any significant distance on the rough surface).

2) Find the (kinetic) energy lost during this collision. Where did this energy go?

3) How far down the rough surface D does the block slide before coming to rest?

Find the Center of Mass of a Non-Uniform Rod



This problem will help you learn required concepts such as:

- Center of Mass of Continuous Mass Distributions
- Integrating Over Distributions

so please review them before you begin.

In the figure above a rod of total mass M and length L is portrayed that has been machined so that it has a mass per unit length that increases *linearly* along the length of the rod:

$$\lambda(x) = \frac{2M}{L^2}x$$

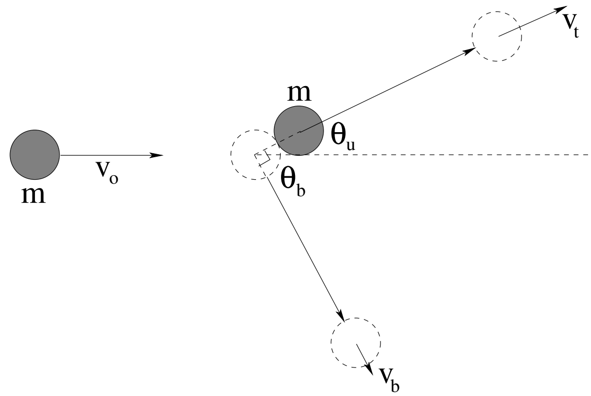
This might be viewed as a very crude model for the way mass is distributed in something like a baseball bat or tennis racket, with most of the mass near one end of a long object and very little near the other (and a continuum in between).

Treat the rod as if it is really one dimensional (we know that the center of mass will be in the center of the rod as far as y or z are concerned, but the rod is so thin that we can imagine that $y \approx z \approx 0$) and:

1) verify that the total mass of the rod is indeed M for this mass distribution;

2) find $x_{\rm cm}$, the x-coordinate of the center of mass of the rod.

Fully Elastic Collision Between Billiard Balls



This problem will help you learn required concepts such as:

- Vector Momentum Conservation
- Fully Elastic Collisions in Two Dimensions

so please review them before you begin.

In the figure above, two identical billiard balls of mass m are sitting in a zero gravity vacuum (so that we can neglect drag forces and gravity). The one on the left is given a push in the x-direction so that it elastically strikes the one on the right (which is at rest) off center at speed v_0. The top ball recoils along the direction shown at a speed v_t and angle θ_t relative to the direction of incidence of the bottom ball, which is deflected so that it comes out of the collision at speed v_b at angle θ_b relative to this direction.

1) Use conservation of momentum to show that in this special case that the two masses are equal:
$$\vec{\boldsymbol{v}}_0 = \vec{\boldsymbol{v}}_u + \vec{\boldsymbol{v}}_b$$
and draw this out as a triangle.

2) Use the fact that the collision was *elastic* to show that
$$v_0^2 = v_u^2 + v_b^2$$
(where these speeds are the lengths of the vectors in the triangle you just drew).

3) Identify this equation and triangle with the *pythagorean theorem* proving that in this case $\vec{\boldsymbol{v}}_t \perp \vec{\boldsymbol{v}}_b$ (so that
$$\theta_u = \theta_b + \pi/2$$

Using these results, one can actually solve for $\vec{\boldsymbol{v}}_u$ and $\vec{\boldsymbol{v}}_b$ given only v_0 and either of θ_u or θ_b. Reasoning very similar to this is used to analyze the results of e.g. nuclear scattering experiments at various laboratories around the world (including Duke)!

Week 5: Torque and Rotation

Summary

I need to fill in this whole week/chapter, really.

Homework for Week 5

Problem 1.

Physics Concepts: As the first *required* step of solving the homework problems below, make a list of the following physics concepts or equations (that you *must know and understand* in order to solve them). Note that some of the necessary concepts will usually be from the previous weeks of the course. Also, one or two from previous weeks may be listed *just* for review and *not* be needed; you must not forget any of the earlier concepts as you learn new ones.

Note also that creating this list *is* part of your homework assignment and it must be handed in. As such, your homework (punched and added to e.g. a three ring binder) will gradually become a very effective study guide for hour exams and the final.

- Torque and Newton's Second Law for rotating objects
- Newton's second law for translating objects
- The Moment of Inertia of:
  - A Particle
  - A Collection of Particles
  - An integral over a continuous distribution of mass

  about any given axis of rotation.

- The Parallel Axis Theorem
- The Perpendicular Axis Theorem
- The Rolling Constraint
- Rotational Kinetic Energy
- Work in Rotating Systems
- Conservation of Mechanical Energy
- Static Friction (in the context of the Rolling Constraint).
- Torque on objects due to gravity.

Problem 2.

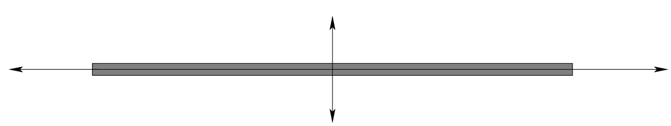
This problem will help you learn required concepts such as:

- Definition/Evaluation of Moment of Inertia
- Parallel Axis Theorem

so please review them before you begin.

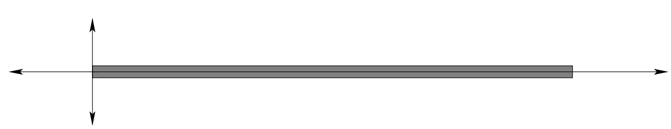
This problem asks you to solve several short mini-problems on computing moments of inertia.

a)

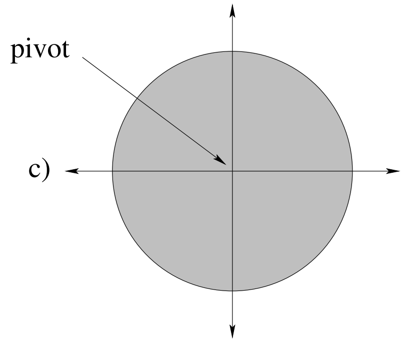


1) Evaluate the moment of inertia of a uniform rod of mass M and length L about its center of mass by direct integration.

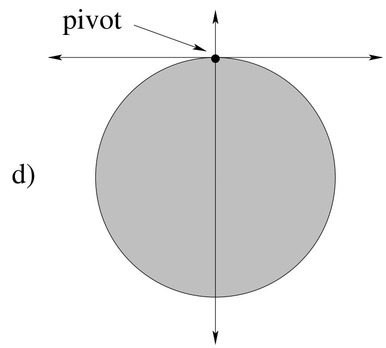
b)



2) Evaluate the moment of inertia of a uniform rod of mass M and length L about one end by direct integration. *Also* evaluate it using the parallel axis theorem. Which is easier?

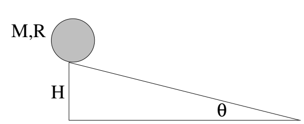


3) Evaluate the moment of inertia of a uniform disk of mass M and radius R about its axis of symmetry by direct integration (this can be set up as a "one dimensional integral" and hence is not too difficult).



4) Evaluate the moment of inertia of this disk around a pivot at the edge of the disk (for example, a thin nail stuck through a hole at the outer edge) using the parallel axis theorem. Would you care to do the actual integral to find the moment of inertia of the disk in this case?

Problem 3.



This problem will help you learn required concepts such as:

- Newton's Second Law for Translation and Rotation

- Moment of Inertia
- Conservation of Mechanical Energy
- Static Friction

so please review them before you begin.

A round object with mass m and radius R is released from rest to roll without slipping down an inclined plane of height H at angle θ relative to horizontal. The object has a moment of inertia $I = \beta m R^2$ (where β is a dimensionless number such as $\frac{1}{2}$ or $\frac{2}{5}$, that might describe a disk or a solid ball, respectively).

1) Begin by relating v (the speed of the center of mass) to the angular velocity (for the rolling object). You will use this (and the related two equations for s and θ and a and α) repeatedly in rolling problems.

2) Using Newton's second law in both its linear and rotational form plus the rolling constraint, show that the acceleration of the object is:

$$a = \frac{g\sin(\theta)}{1+\beta}$$

3) Using conservation of mechanical energy, show that it arrives at the bottom of the incline with a velocity:

$$v = \sqrt{\frac{2gH}{1+\beta}}$$

4) Show that the condition for the greatest angle for which the object will roll *without slipping* is that:

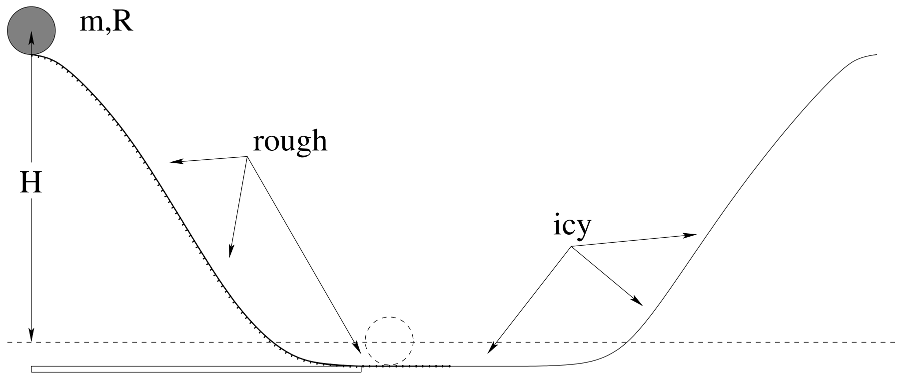
$$\tan(\theta) \le (1 + \frac{1}{\beta})\mu_s$$

where μ_s is the coefficient of static friction between the object and the incline.

Problem 4.

This problem will help you learn required concepts such as:

- Conservation of Mechanical Energy
- Rotational Kinetic Energy
- Rolling Constraint.



so please review them before you begin.

A disk of mass m and radius R rolls without slipping down a rough slope of height H onto an icy (frictionless) track at the bottom that leads up a second icy/frictionless hill as shown.

1) How fast is the disk moving at the bottom of the first incline? How fast is it rotating (what is its angular velocity)?

2) Does the disk's angular velocity change as it leaves the rough track and moves onto the ice (in the middle of the flat stretch in between the hills)?

3) How far up the second hill (vertically, find H') does the disk go before it stops rising?

Problem 5.

This problem will help you learn required concepts such as:

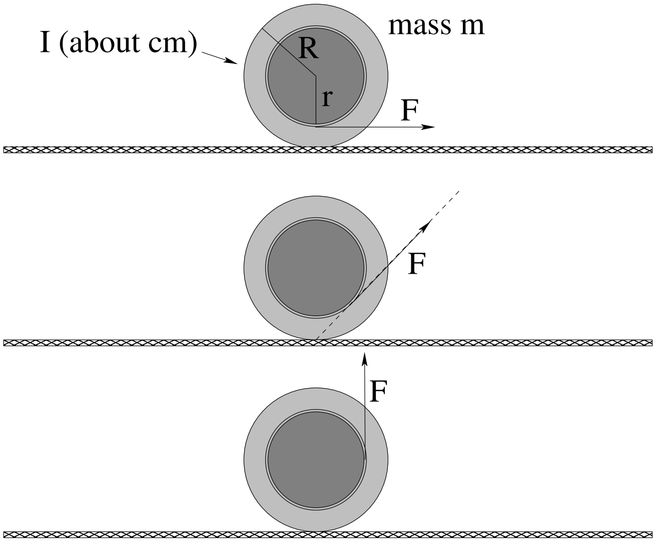
- Direction of torque
- Rolling Constraint

so please review them before you begin.

In the figure above, a yo-yo of mass m is wrapped with string around the inner spool. The yo-yo is placed on a rough surface (with coefficient of friction $\mu_s = 0.5$) and the string is pulled with force $F \ll 0.5\ mg$ in the three directions shown.

1) For each picture, indicate the direction that the yo-yo will accelerate.

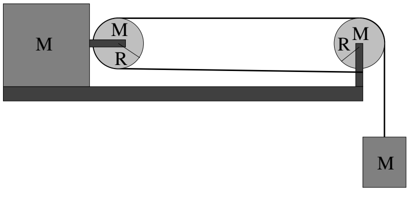
2) For each picture, indicate the direction that static friction will point.



3) For each picture, find the magnitude of the force exerted by static friction and the magnitude of the acceleration of the yo-yo. To do this, you will need to use symbols to indicate the radius of the inner spool r, the radius of the outer spool R, and the moment of inertia of the yo-yo about its center of mass, $I = \beta m R^2$. Use $\beta \approx 1/3$ (reasonable for an arrangement like this) to see if your answer for a "makes sense".

Note that you do not need to know r, R or I to answer the first two questions beyond what you can directly see in the geometry of of the problem. Hint: Consider the direction of the torque about the point of contact with the ground...

Problem 6.



This problem will help you learn required concepts such as:

- Moment of Inertia
- Newton's Second Law (translational and rotational)
- Conservation of Mechanical Energy

so please review them before you begin.

A block of mass M sits on a smooth (frictionless) table. A mass M is suspended from an Acme (massless, unstretchable, unbreakable) rope that is looped around the two pulleys, each with the same mass M and radius R as shown and attached to the support of the rightmost pulley. The two pulleys each have the moment of inertia of a disk, $I = \frac{1}{2}MR^2$ and the rope rolls on the pulleys without slipping. At time $t = 0$ the system is released at rest.

1) Draw free body diagrams for each of the four masses. Don't forget the forces exerted by the bar that attaches the pulley to the mass on the left or the pulley on the right to the table! Note that some of these forces will cancel due to the constraint that the center of mass of certain masses moves only in one direction or does not move at all. Note also that the torque exerted by the *weight* of the pulley around the near corder of the mass on the table is not not enough to tip it over.

2) Write the relevant form(s) of Newton's Second Law for each mass, translational and rotational as needed, separately, wherever the forces in some direction do *not* cancel. Also write the constraint equations that relate the accelerations (linear and angular) of the masses and pulleys.

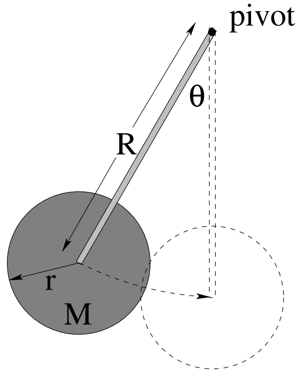
3) Find the acceleration of the hanging mass M (only) in terms of the givens. Note that you can't *quite* just add all of the equations you get after turning the torque equations into force equations, but if you solve the equations simultaneously, systematically eliminating all internal forces and tensions, you'll get a simple enough answer.

4) Find the speed of the hanging mass M after it has fallen a height H, using conservation of total mechanical energy. Show that it is consistent with the "usual" constant-acceleration-from-rest answer $v = \sqrt{2aH}$ for the acceleration found in c).

Problem 7.

This problem will help you learn required concepts such as:

- Newton's Second Law for rotating objects
- Moment of Inertia
- Parallel Axis Theorem
- Work in rotating systems
- Rotational Kinetic Energy



- Kinetic Energy in Lab versus CM

so please review them before you begin.

In the picture above, a physical pendulum is constructed by hanging a disk of mass M and radius r on the end of a massless rigid rod in such a way that the center of mass of the disk is a distance R away from the pivot and so that the whole disk pivots with the rod. The pendulum is pulled to an initial angle θ_0 (relative to vertically down) and then released.

1) Find the *torque* about the pivot exerted on the pendulum by gravity at an arbitrary angle θ.

2) Integrate the torque from $\theta = \theta_0$ to $\theta = 0$ to find the total work done by the gravitational torque as the pendulum disk falls to its lowest point. Note that your answer should be $MgL(1 - \cos(\theta_0)) = MgH$ where H is the initial height above this lowest point.

3) Find the moment of inertia of the pendulum about the pivot (using the parallel axis theorem).

4) Set the work you evaluated in b) equal to the rotational kinetic energy of the disk $\frac{1}{2}I\omega^2$ using the moment of inertia you found in c). Solve for $\omega = \frac{d\theta}{dt}$ when the disk is at its lowest point.

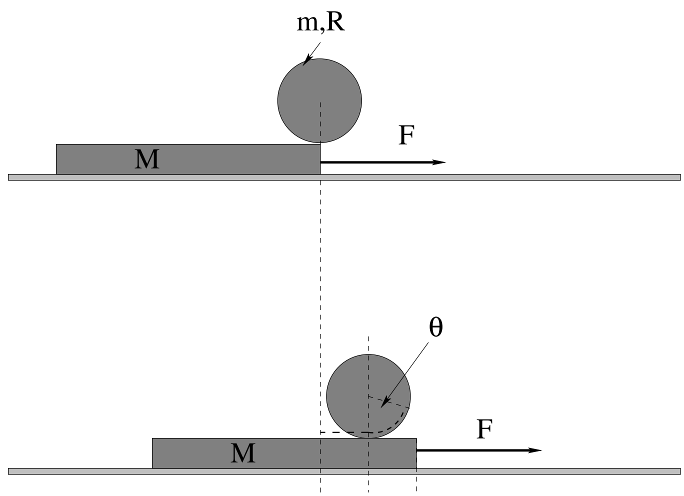
5) Show that this kinetic energy is equal to the kinetic energy of the moving center of mass of the disk $\frac{1}{2}Mv^2$ plus the kinetic energy of the disk's rotation about its own center of mass, $\frac{1}{2}I_{\rm cm}\omega^2$, at the lowest point.

As an optional additional step, write Newton's Second Law in rotational coordinates $\tau = I\alpha$ (using the values for the magnitude of the torque and moment of inertia you determined above) and solve for the angular acceleration as a function of the angle θ. In a few weeks we will learn to solve this equation of motion for small angle oscillations, so it is good to practice obtaining it from the basic physics now.

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

Disk Rolls Back on Slab



This problem will help you learn required concepts such as:

- Newton's Second Law (linear and rotational)
- Rolling Constraint
- Static and Kinetic Friction

so please review them before you begin.

A disk of mass m is resting on a slab of mass M, which in turn is resting on a frictionless table. The coefficients of static and kinetic friction between the disk and the slab are μ_s and μ_k, respectively. A small force $\vec{F}$ to the right is applied to the slab as shown, then gradually increased.

1) When $\vec{F}$ is small, the slab will accelerate to the right and the disk will roll on the slab without slipping. Find the acceleration of the slab, the acceleration of the disk, and the *angular* acceleration of the disk as this happens, in terms of m, M, R, and the magnitude of the force F.

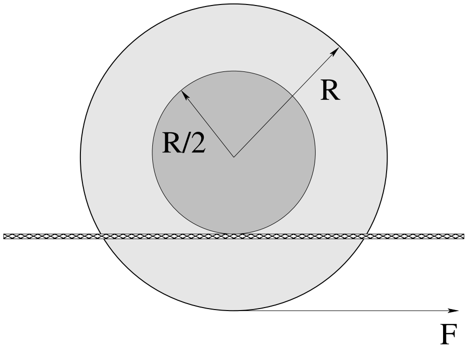
2) Find the maximum force $F_{\max}$ such that it rolls without slipping.

3) If F is greater than this, solve once again for the acceleration and angular acceleration of the disk and the acceleration of the slab.

Hint: The hardest single thing about this problem isn't the physics (which is really pretty straightfoward). It is visualizing the coordinates as the center of mass of the disk moves with a different acceleration as the slab. I have drawn *two* figures above to help you with this – the lower figure represents a possible position of the disk after the slab has moved some distance to the right and the disk has rolled back (relative to the slab! It has moved *forward* relative to the ground! Why?) without slipping. Note the dashed radius to help you see the angle through which it has rolled and the various dashed lines to help you relate the distance the slab has moved x_s, the distance the center of the disk has moved x_d, and the angle through which it has rolled θ. Use this relation to connect the acceleration of the slab to the acceleration and angular acceleration of the disk.

If you can do this one, good job!

Rolling Cable Spool



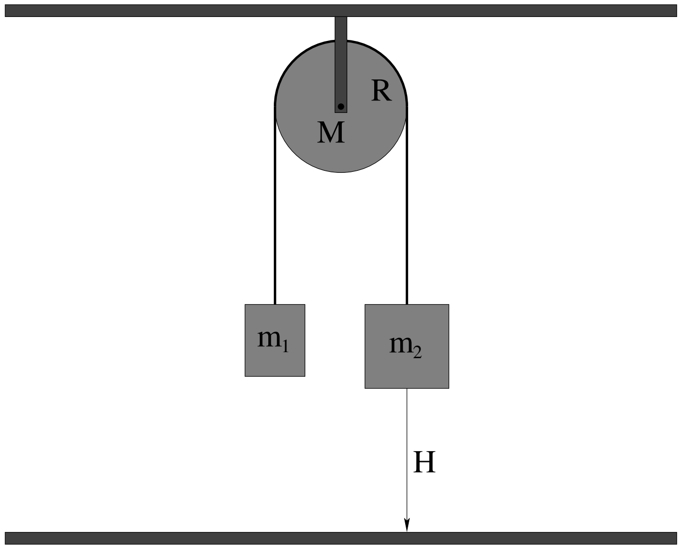
A cable spool of mass M, radius R and moment of inertia $I = \frac{1}{3}MR^2$ is wrapped around its OUTER disk with fishing line and set on a rough rope as shown. The fishing line is then pulled with a force F to the right as shown so that it rolls down the rope **without slipping**.

1) Which way does the spool roll (left or right)?

2) Find the magnitude of the acceleration of the spool.

3) Find the force the friction of the rope exerts on the spool.

Atwood's Machine with Massive Pulley



This problem will help you learn required concepts such as:

- Newton's Second Law
- Newton's Second Law for Rotating Systems (torque and angular acceleration)
- Moments of Inertia
- The Rolling Constraint
- Conservation of Mechanical Energy

so please review them before you begin.

In the figure above Atwood's machine is drawn – two masses m_1 and m_2 hanging over a *massive* pulley which you can model as a disk of mass M and radius R, connected by a massless unstretchable string. The string rolls on the pulley without slipping.

1) Draw three free body diagrams (isolated diagrams for each object showing just the forces acting on that object) for the three masses in the figure above.

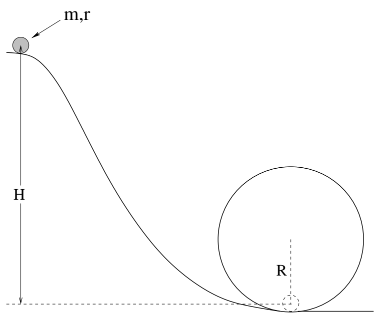
2) Convert each free body diagram into a statement of Newton's Second Law (linear or rotational) for that object.

3) Using the *rolling constraint* (that the pulley rolls without slipping as the masses move up or down) find the acceleration of the system and the tensions in the string on *both* sides of the pulley in terms of m_1, m_2, M, g, and R.

4) Suppose mass $m_2 > m_1$ and the system is released from rest with the masses at equal heights. When mass m_2 has descended a distance H, use conservation of

mechanical energy to find velocity of each mass and the angular velocity of the pulley.

Loop the Loop with a Rolling Ball



This problem will help you learn required concepts such as:

- Newton's Second Law
- Moments of Inertia
- Rotational Kinetic Energy
- The Rolling Constraint
- Conservation of Mechanical Energy
- Centripetal Acceleration
- Static Friction

so please review them before you begin. The moment of inertia of a solid ball is $I_{\rm ball} = \frac{2}{5}MR^2$.

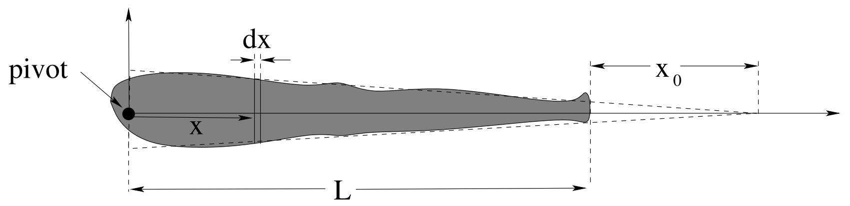
A solid ball of mass M and radius r sits at the top of a frictionless hill of height H leading to a circular loop-the-loop. The center of mass of the ball will move in a circle of radius R if it goes around the loop.

1) Find the minimum height $H_{\rm min}$ for which the ball *barely* goes around the loop staying on the track at the top, assuming that it rolls without slipping the entire time independent of the normal force.

2) How does your answer relate to the minimum height for the earlier homework problem where it was a block that slid around a frictionless track? Does this answer make sense? If it is higher, where did the extra potential energy go? If it is lower, where did the extra kinetic energy come from?

3) The assumption that the ball will roll around the track without slipping if released from this estimated *minimum* height is not a good one. Why not?

Find the Center of Mass and Moment of Inertia of a Leg



This problem will help you learn required concepts such as:

- Finding the Center of Mass using Integration
- Finding the Moment of Inertia using Integration

so please review them before you begin.

A *simple* model for the one-dimensional mass distribution of a human leg of length L and mass M is:

$$\lambda(x) = C \cdot (L + x_0 - x)$$

Note that this quantity is maximum at $x = 0$, varies linearly with x, and vanishes smoothly at $x = L + x_0$. That means that it doesn't *reach* $\lambda = 0$ when $x = L$, just as the mass per unit length of your leg doesn't reach zero at your ankles.

1) Find the constant C in terms of M, L, and x_0 by evaluating:

$$M = \int_0^L \lambda(x)\ dx$$

and solving for C.

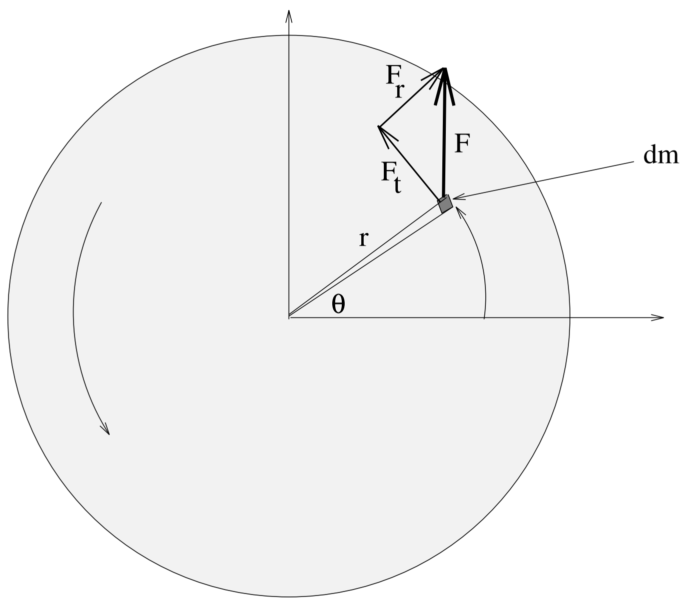
2) Find the center of mass of the leg (as a distance down the leg from the hip/pivot at the origin). You may leave your answer in terms of C (now that you know it) or you can express it in terms of L and x_0 only as you prefer.

3) Find the moment of inertia of the leg about the hip/pivot at the origin. Again, you may leave it in terms of C if you wish or express it in terms of M, L and x_0. Do your answers all have the right units?

4) How might one improve the estimate of the moment of inertia to take into account the foot (as a lump of "extra mass" m_f out there at $x = L$ that doesn't quite fit our linear model)?

This is, as you can see, something that an orthopedic specialist might well need to actually do with a much better model in order to e.g. outfit a patient with an artificial hip. True, they might use a computer to do the actual computations required, but is it plausible that they could possibly do what they need to do without knowing the physics involved in some detail?

Week 6: Vector Torque and Angular Momentum



We have just seen how Newton's Law can be applied to collections of particles as if the entire collection were a "particle" with the mass of the collection located at the center of mass of the collection. This trick works very well for solid objects made up of many parts, but isn't as useful for collections like a box containing a few loose marbles. This is because rigid objects retain their shape at all times – we only need to keep track of the position of the center of mass and the orientation of the entire object around the CM.

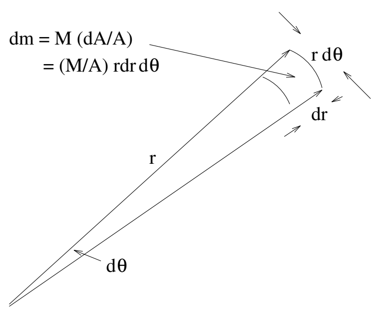
For this reason, we can break the general motion of rigid objects up into two parts: *translation* of the CM (the object as a whole) through space and *rotation* of the object about the CM. In the last section we learned how to treat the translation part in detail. In this section we will learn how to treat the dynamics of rotation in detail.

The figure above is a "generic" figure that we will use to think of rotation. You may imagine it to be a big, flat disk. If we color a dot on the disk with a marker pen, we can see that the mass underneath the mark is very small and localized – we can think of it as being a "particle" of mass dm and the whole disk consisting of the sum of all the little dots like this that we could draw that would eventually exactly cover the disk. Summing these little dots is done mathematically with integration. We say:

$$M_{\rm tot} = \int dm \tag{257}$$

Of course, written this way the integral is almost tautological (and hence useless). To make use of it, we need to find some way of expressing the integral in coordinates. Fortunately this integral is not too difficult to figure out for simple coordinate systems and figures.

In the case of the circular disk above, we note that the mass under our colored dot must be the total mass $M_{\rm tot}$ times the fraction of the total area of the disk A that happens to lie under our dot, dA. The only remaining chore is to figure out what dA (the area element of a disk) is in suitable coordinates. Let's look at it close up:



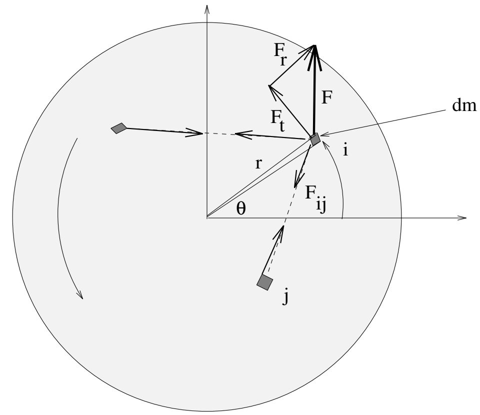
We see that a little box with differential sides in polar coordinates has area $dA = (dr) \times (rd\theta) = r\ dr\ d\theta$. If we add *all* the little boxes like this that exist in a disk with radius R,

$$A = \int_0^R \int_0^{2\pi} r\ dr\ d\theta = (2\pi)(\frac{R^2}{2}) = \pi R^2 \tag{258}$$

we indeed get the area of a circle! Similarly we could integrate over a circular annulus (with inner radius R_1 and outer radius R_2) or over a pie-shaped wedge and calculate their area. The *area element* dA looks useful!

From this we see that the differential mass $dm = \frac{M}{\pi R^2} r dr d\theta$ where we have now put in the area of the disk and its mass explicitly. We are now prepared to work out the dynamics of this little point of mass, subject to the *constraint* that the disk is rigid, so that all that it can do is translate and rotate.

Let us apply an external force to this point and write Newton's second law for this tiny chunk of mass (calling it the i-th chunk even though we're going to eventually integrate over all of them instead of doing a discrete sum. The other similar chunks of mass that make up the whole object we'll index with j.):



$$\vec{\boldsymbol{F}}_i = \vec{\boldsymbol{F}} + \sum_{j \neq i} \vec{\boldsymbol{F}}_{ij} \tag{259}$$

$$= dm\vec{\boldsymbol{a}} \tag{260}$$

As before (when we've seen internal forces), when we sum *all* of the force acting on the object due to the single external force $\vec{\boldsymbol{F}}$, the internal forces cancel due to Newton's Third law. We get:

$$\vec{\boldsymbol{F}} = \int dm\vec{\boldsymbol{a}}_{\rm cm} = M\vec{\boldsymbol{a}}_{\rm cm} \tag{261}$$

which describes the motion of the center of mass itself due to the application of the external force.

What about the motion of the little bit of mass with respect to the center of mass? Well, from the figure above we can see that the net (external) force on it has two components. One ($\vec{\boldsymbol{a}}_r$) acts along the line from the center of mass to the little chunk. If the object were not rigid, this force would pull the little chunk away from the center. However, the object is rigid, so this component just accelerates the chunk *and* the center of mass in together in such a way that their distance of separation remains constant.

The other, $\vec{\boldsymbol{a}}_t$, is of considerable interest and the subject of this chapter as the motion of the center of mass itself was the subject of the last chapter. First, we note that this component describes the *rotation* of this little chunk around the center of mass. Second, we note that the whole (rigid) mass rotates together, so that (once the axis of rotation is known) a single coordinate describes its orientation. This suggests that we find a clever way to write Newton's second law in this one-dimensional coordinate system:

$$\vec{\boldsymbol{F}}_t = dm\vec{\boldsymbol{a}}_t \tag{262}$$

$$|\vec{\boldsymbol{r}}||\vec{\boldsymbol{F}}_t| = |\vec{\boldsymbol{r}}|dm|\vec{\boldsymbol{a}}_t| \tag{263}$$

$$= r^2 dm\frac{|\vec{\boldsymbol{a}}_t|}{r} \tag{264}$$

$$= (r^2 dm)\alpha \tag{265}$$

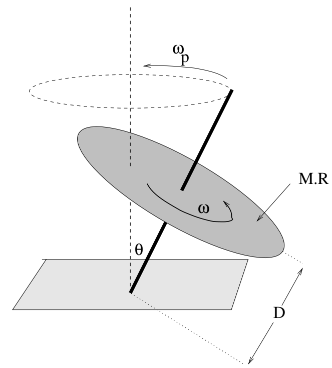
where we have used

$$\begin{aligned} s_t &= r\theta && (266) \\ v_t &= \frac{ds_t}{dt} = r\frac{d\theta}{dt} = r\omega && (267) \\ a_t &= \frac{dv_t}{dt} = \frac{d^2 s_t}{dt^2} = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2} = r\alpha && (268) \end{aligned}$$

to relate coordinates on the particular arc that contains dm to angular coordinates that apply to the whole object. We use Newton's second law rewritten in this way to *define* the *torque*:

$$\tau = (r^2 dm)\alpha \qquad (269)$$

Examples



Example 6.0.1: Precession of a Gyroscope

Precession is a very important example of torque in action. It is one of the most convincing demonstrations that torque is indeed a vector quantity.

Suppose one has a top, or a gyroscope, consisting of a spinning disk of mass M and radius R mounted on a frictionless axle resting on a tabletop. The axle is of length D. The disk is spinning at an angular velocity ω, which is presumed to be large so that the angular momentum of the disk is large.

It is straightforward to compute the precession frequency of this gyroscope – the rate (in radians/sec) at which it sweeps out a cone around the vertical.

First we compute the torque about the pivot point (where the axle rests on the table) exerted by gravity acting on the system:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} = mgD\sin(\theta) \tag{270}$$

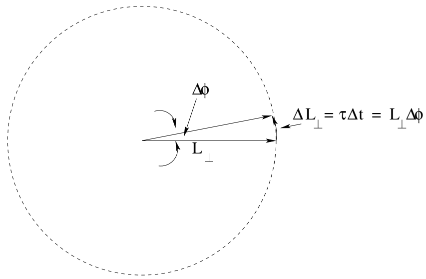
(into the page as drawn).

Second we compute the angular momentum of the system:

$$\vec{\boldsymbol{L}} = I\vec{\omega} = \frac{1}{2}MR^2\omega \tag{271}$$

(along the axis of rotation). Now note that the torque only changes the horizontal component of the angular momentum because it has no component parallel to the vertical component. This component is:

$$L_{\perp} = |\vec{\boldsymbol{L}}|\sin(\theta) \tag{272}$$



Now imagine looking down on the gyroscope from above so $L_\perp$ traces out a circle as the gyroscope precesses. In a time Δt the angular momentum change is:

$$\Delta L = \Delta L_\perp = \tau \Delta t = L_\perp \Delta\phi \tag{273}$$

where $\Delta\phi$ is the angle $L_\perp$ sweeps out in that time.

Dividing the latter relation out:

$$\begin{aligned} \omega_p &= \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt} = \frac{\tau}{L_\perp} \\ &= \frac{MgD\sin(\theta)}{\frac{1}{2}MR^2\omega\sin(\theta)} \\ &= \frac{2gD}{R^2\omega} \end{aligned} \tag{274}$$

where of course different spinning objects would have different relations from which τ or L are computed but the idea would be the same.

Precession of this sort is the general solution of the differential equation:

$$\frac{d\vec{A}}{dt} = \vec{A} \times \vec{B} \tag{275}$$

where the change in $\vec{A}$ is always perpendicular to the plane containing $\vec{A}$ (changing) and $\vec{B}$ (presumed fixed). In future courses you'll study the mathematics of the solution in more detail (for example, when $\vec{A}$ and $\vec{B}$ are expressed in a cartesian coordinate system, where this becomes three differential equations coupling the three components according to a tensor form) and will learn about an additional sort of wobble the motion can have called *nutation*.

You will *also* study this in the specific context of magnetic resonance. Magnetic fields exert a torque on a magnetic moment. Magnetic moments are created by spinning charge. Thus a spinning charged particle in a static magnetic field has a continuous torque acting on it that causes it to precess around the magnetic field precisely like a top! Much magic can be worked on these precessing charged particles.

Homework for Week 6

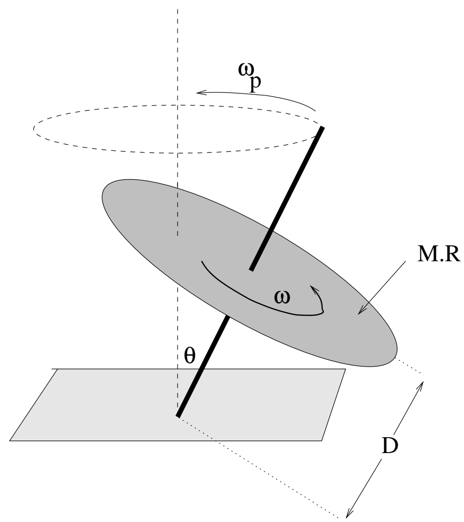
Problem 1.

Physics Concepts: As the first *required* step of solving the homework problems below, make a list of the following physics concepts or equations (that you *must know and understand* in order to solve them). Note that some of the necessary concepts will usually be from the previous weeks of the course. Also, one or two from previous weeks may be listed *just* for review and *not* be needed; you must not forget any of the earlier concepts as you learn new ones.

Note also that creating this list *is* part of your homework assignment and it must be handed in. As such, your homework (punched and added to e.g. a three ring binder) will gradually become a very effective study guide for hour exams and the final.

- Vector Torque
- Vector Angular Momentum
- Geometry of Precession
- Torque Due to Radial Forces
- Angular Momentum Conservation
- Centripetal Acceleration
- Work and Kinetic Energy
- Inelastic Collisions
- Conservation of Mechanical Energy
- Impulse
- Linear Momentum Conservation
- Non-equivalence of vL and $I\vec{\omega}$

Problem 2.



This problem will help you learn required concepts such as:

- Vector Torque
- Vector Angular Momentum
- Geometry of Precession

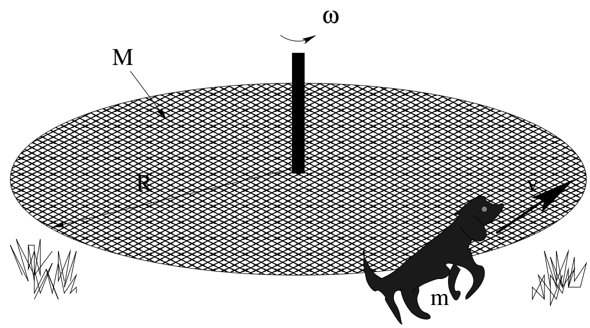
so please review them before you begin.

A top is made of a disk of radius R and mass M with a very thin, light nail ($r \ll R$ and $m \ll M$) for a spindle so that the disk is a distance D from the tip. The top is spun with a large angular velocity ω. When the top is spinning at a small angle θ with the vertical (as shown) what is the angular frequency ω_p of the top's precession?

Problem 3.

This problem will help you learn required concepts such as:

- Angular Momentum Conservation
- Inelastic Collisions



so please review them before you begin.

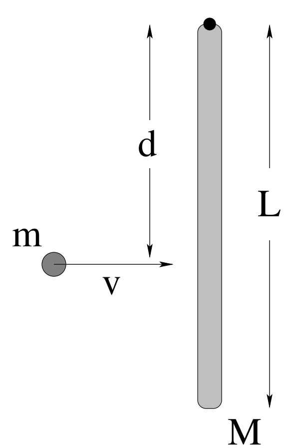
Satchmo the dog with mass m runs and jumps onto the edge of a merry-go-round initially at rest and then sits down there as it spins for a ride. The merry-go-round can be thought of as a disk of radius R and mass M, and has approximately frictionless bearings in its axle. At the time of this angular "collision" Satchmo is travelling at speed v perpendicular to the radius of the merry-go-round.

1) What is the angular velocity of the merry-go-round (and Satchmo) right after the collision?

2) How much of Satchmo's initial energy was lost in the collision?

(Believe me, it isn't enough – he's a border collie and he has plenty more!)

Problem 4.



This problem will help you learn required concepts such as:

- Angular Momentum Conservation
- Momentum Conservation
- Inelastic Collisions
- Impulse

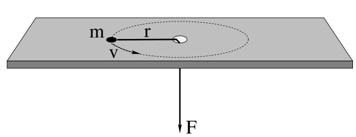
so please review them before you begin. You may also find it useful to read: Wikipedia: http://www.wikipedia.org/wiki/Center of Percussion.

A rod of mass M and length L rests on a frictionless table and is pivoted on a frictionless nail at one end as shown. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the end as shown, sticking to the rod.

- Find the angular velocity ω of the system about the nail after the collision.
- Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?
- Is there a value of d for which it *is* conserved? If there were such a value, it would be called the *center of percussion* for the rod for this sort of collision.

All answers should be in terms of M, m, L, v and d as needed. Note well that you should clearly indicate what physical principles you are using to solve this problem at the *beginning* of the work.

Problem 5.



This problem will help you learn required concepts such as:

- Torque Due to Radial Forces
- Angular Momentum Conservation
- Centripetal Acceleration
- Work and Kinetic Energy

so please review them before you begin.

A particle of mass M is tied to a string that passes through a hole in a frictionless table and held. The mass is given a push so that it moves in a circle of radius r at speed v. We will now analyze the physics of its motion in two stages.

1) What is the torque exerted on the particle by the string? Will angular momentum be conserved if the string pulls the particle into "orbits" with different radii?

2) What is the magnitude of the angular momentum L of the particle in the direction of the axis of rotation (as a function of m, r and v)?

3) Show that the magnitude of the force (the tension in the string) that must be exerted to keep the particle moving in a circle is:

$$F = \frac{L^2}{mr^3}$$

   This is a *general* result for a particle moving in a circle and in no way depends on the fact that the force is being exerted by a string in particular.

4) Show that the kinetic energy of the particle in terms of its angular momentum is:

$$K = \frac{L^2}{2mr^2}$$

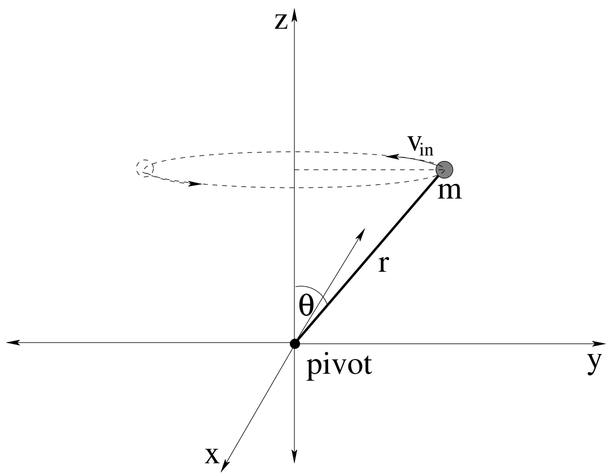
Now, suppose that the radius of the orbit and initial speed are r_i and v_i, respectively. From under the table, the string is *slowly* pulled down (so that the puck is always moving in an approximately circular trajectory and the tension in the string remains radial) to where the particle is moving in a circle of radius r_2.

5) Find its velocity v_2 using angular momentum conservation. This should be very easy.

6) Compute the work done by the force from part c) above and identify the answer *as* the work-kinetic energy theorem. Use this to to find the velocity v_2. You should get the same answer!

Note that the last two results are pretty amazing – they show that our torque and angular momentum theory so far is remarkably *consistent* since two *very* different approaches give the same answer. Solving this problem now will make it easy later to understand the *angular momentum barrier*, the angular kinetic energy term that appears in the radial part of conservation of mechanical energy in problems involving a central force (such as gravitation and Coulomb's Law). This in turn will make it easy for us to understand certain properties of orbits from their potential energy curves.

Problem 6.



This problem will help you learn required concepts such as:

- Vector Torque
- Non-equivalence of $\vec{L}$ and $I\vec{\omega}$.

so please review them before you begin.

In the figure above, a mass m is being spun in a circular path at a constant speed v around the z-axis on the end of a massless rigid rod of length r pivoted at the orgin (that itself is being pushed by forces not shown). All answers should be given in terms of m, r, θ and v. Ignore gravity and friction.

1) Find the angular momentum vector of this system *relative to the pivot* at the instant shown and draw it in on a copy of the figure.

2) Find the angular velocity vector of the mass m at the instant shown and draw *it* in on the figure above. Are the angular momentum vector and the angular velocity vectors the same? What *component* of the angular momentum is equal to $I\omega$?

3) What is the magnitude and direction of the torque exerted by the rod on the mass (relative to the pivot shown) in order to keep it moving in this circle at constant speed? (Hint: Consider the *precession* of the angular momentum around the z-axis where the precession frequency is ω.)

4) What is the *direction* of the force exerted by the rod in order to create this torque? (Hint: Shift gears and think about the actual trajectory of the particle. What must the direction of the force be?)

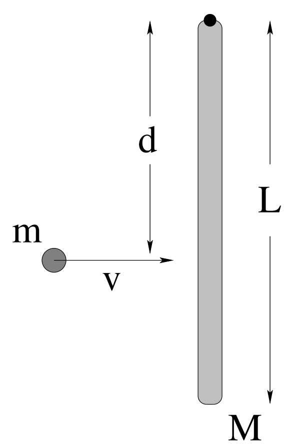
5) Set $\tau = rF_{\perp}$ or $\tau = r_{\perp}F$ and determine the magnitude of this force.

Gosh, could we have gotten this answer using a different argument without knowing anything at all about torque or angular momentum? Once again the physics appears to be amazingly consistent!

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

Another Blob of Putty Strikes a Rod



A rod of mass M and length L is hanging vertically from a frictionless pivot (where gravity is "down"). A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from its center of mass as shown, sticking to the rod.

1) Find the angular velocity ω_f of the system about the pivot (at the top of the rod) after the collision.

2) Find the distance $x_{\rm cm}$ from the pivot of the center of mass of the rod-putty system immediately after the collision.

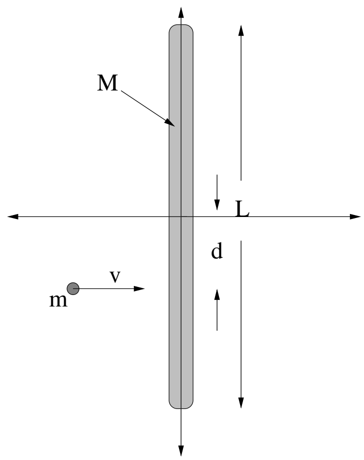
3) After the collision, the rod swings up to a maximum angle $\theta_{\rm max}$ and then comes momentarily to rest. Find $\theta_{\rm max}$.

All answers should be in terms of M, m, L, v, g and d as needed. The moment of inertia of a rod pivoted about one end is $I = \frac{1}{3}ML^2$, in case you need it.

Yet Another Blob of Putty Strikes a Rod

This problem will help you learn required concepts such as:

- Angular Momentum Conservation
- Momentum Conservation



- Inelastic Collisions
- Impulse

so please review them before you begin.

A rod of mass M and length L rests on a frictionless table. A blob of putty of mass m approaches with velocity v from the left and strikes the rod a distance d from the end as shown, sticking to the rod.

- Find the angular velocity ω of the system about the center of mass *of the system* after the collision. Note that the rod and putty will not be rotating about the center of mass of the rod!
- Is the linear momentum of the rod/blob system conserved in this collision for a general value of d? If not, why not?

All answers should be in terms of M, m, L, v and d as needed. Note well that you should clearly indicate what physical principles you are using to solve this problem at the *beginning* of the work.

Week 7: Statics

Statics Summary

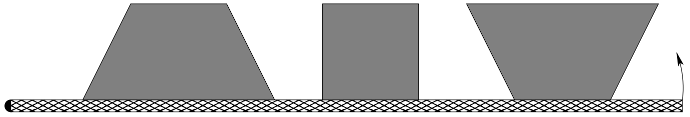
Particle Equilibrium

Object Equilibrium

Examples

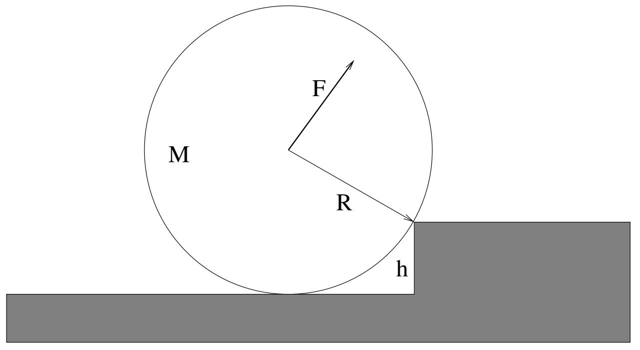
Homework for Week 7

Problem 1.



In the figure above, three shapes (with uniform mass distribution and thickness) are drawn sitting on a plane that can be tipped up gradually. Assuming that static friction is great enough that all of these shapes will tip over before they slide, rank them in the order they will tip over as the angle of the board they are sitting on is increased. Be sure to indicate any ties.

Problem 2.



This problem will help you learn required concepts such as:

- Static Equilibrium
- Torque (about selected pivots)
- Geometry of Right Triangles

so please review them before you begin.

A cylinder of mass M and radius R sits against a step of height $h = R/2$ as shown above. A force $\vec{F}$ is applied at right angles to the line connecting the corner of the step and the center of the cylinder. All answers should be in terms of M, R, g.

1) Find the minimum value of $|\vec{F}|$ that will roll the cylinder over the step if the cylinder does not slide on the corner.

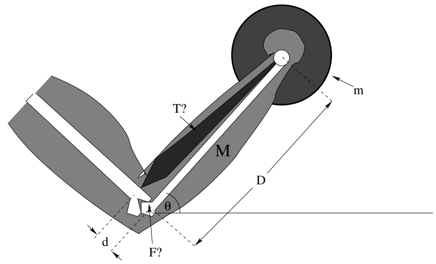
2) What is the force exerted by the corner (magnitude and direction) when that force $\vec{F}$ is being exerted on the center?

Problem 3.

This problem will help you learn required concepts such as:

- Static Equilibrium
- Force and Torque

so please review them before you begin.



An exercising human person holds their arm of mass M and a barbell of mass m at rest at an angle θ with respect to the horizontal in an isometric curl as shown. The muscle that supports the suspended weight is connected a short distance d up from the elbow joint. The bone that supports the weight has length D.

1) Find the tension T in the muscle, assuming for the moment that the center of mass of the forearm is in the middle at $D/2$. Note that it is *much larger* than the weight of the arm and barbell combined, assuming a reasonable ratio of $D/d \approx 25$ or thereabouts.

2) Find the force $\vec{F}$ (magnitude *and* direction) exerted on the supporting bone by the elbow joint. Again, note that it is much larger than "just" the weight being supported.

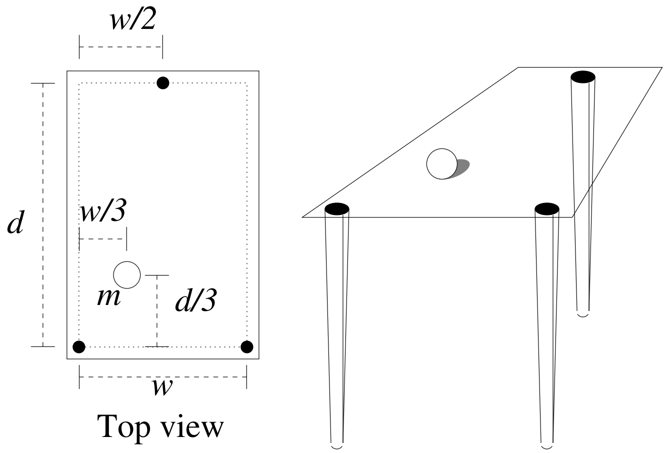
Please note that this is not an accurate description of musculature – in actual fact several muscles would work together to achieve the desired static equilibrium (or to do the dynamic work of actual barbell curls). However, all of these muscles work with *short* moment arms at the point of bone attachment to support weights with *long* moment arms and the *idea* and *methodology* of the calculation of the stresses on the muscles and joints is just more complicated versions of this same reasoning. Sports medicine and orthopedic medicine and prosthetic design all use this sort of physics every day.

Problem 4.

This problem will help you learn required concepts such as:

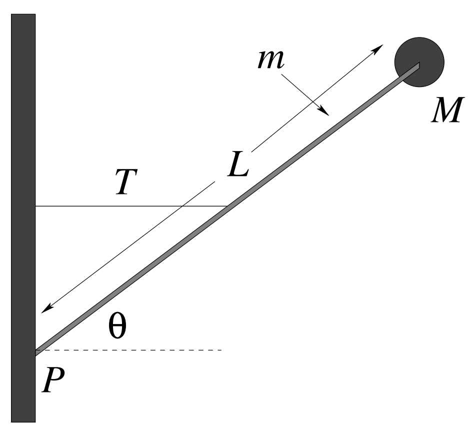
- Force Balance
- Torque Balance
- Static Equilibrium

so please review them before you begin.



The figure below shows a mass m placed on a table consisting of three narrow cylindrical legs at the positions shown with a light (presume massless) sheet of Plexiglas placed on top. What is the vertical force exerted by the Plexiglas on each leg when the mass is in the position shown?

Problem 5.

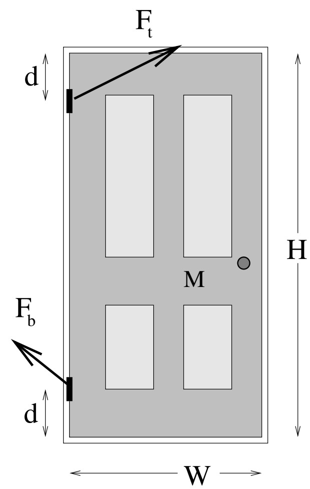


A small round mass M sits on the end of a rod of length L and mass m that is attached to a wall with a hinge at point P. The rod is kept from falling by a thin (massless) string attached horizontally between the midpoint of the rod and the wall. The rod makes an angle θ with the ground. Find:

1) the tension T in the string;

2) the force $\vec{\boldsymbol{F}}$ exerted by the hinge on the rod.

Problem 6.

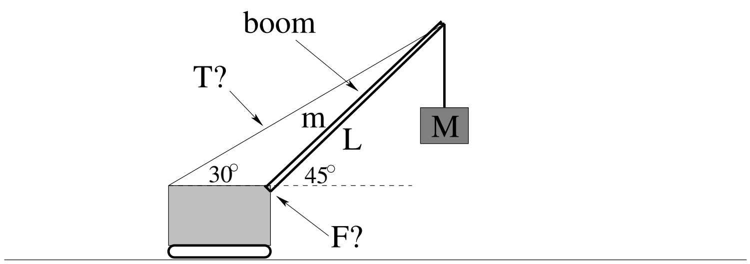


A door of mass M that has height H and width W is hung from two hinges located a distance d from the top and bottom, respectively. Assuming that the weight of the door is equally distributed between the two hinges, find the total force (magnitude and direction) exerted by each hinge. (Neglect the mass of the doorknob. The force directions drawn for you are **NOT** likely to be correct or even close.)

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

A Crane



A crane with a boom (the long support between the body and the load) of mass m and length L holds a mass M suspended as shown. Assume that the center of mass of the boom is at $L/2$. Note that the wire with the tension T is **fixed** to the top of the boom, not run over a pulley to the mass M.

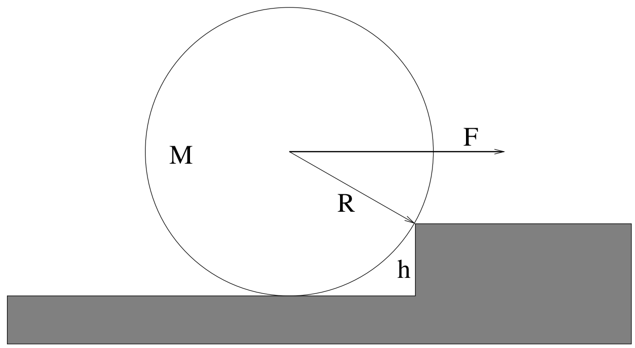
1) Find the tension in the wire.

2) Find the force exerted on the boom by the crane body.

Note:

$$\sin(30^\circ) = \cos(60^\circ) = \frac{1}{2}$$
$$\cos(30^\circ) = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$
$$\sin(45^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$$

Another Static Cylinder



A cylinder of mass M and radius R sits against a step of height $h = R/2$ as shown above. A force $\vec{F}$ is applied parallel to the ground as shown. All answers should be in terms of M, R, g.

1) Find the minimum value of $|\vec{F}|$ that will roll the cylinder over the step if the cylinder does not slide on the corner.

2) What is the force exerted by the corner (magnitude and direction) when that force $\vec{F}$ is being exerted on the center?

Week 8: Fluids

Fluids Summary

- **Fluids** are states of matter characterized by a lack of long range order. They are characterized by their **density** ρ and their **compressibility**. Liquids such as water are (typically) relatively incompressible; gases can be significantly compressed. Fluids have other characteristics, for example viscosity (how "sticky" the fluid is). We will ignore these in this course.

- **Pressure** is the force per unit area exerted by a fluid on its surroundings:

$$P = F/A \tag{276}$$

  Its SI units are *pascals* where 1 pascal = 1 newton/meter squared. Pressure is also measured in "atmospheres" (the pressure of air at or near sea level) where 1 atmosphere $\approx 10^5$ pascals. The pressure in an incompressible fluid varies with depth according to:

$$P = P_0 + \rho g D \tag{277}$$

  where P_0 is the pressure at the top and D is the depth.

- **Pascal's Principle** Pressure applied to a fluid is transmitted undiminished to all points of the fluid.

- **Conservation of Flow** We will study only steady/laminar flow in the absence of turbulence and viscosity.

$$A_1 v_1 = A_2 v_2 \tag{278}$$

- **Bernoulli's Equation**

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \tag{279}$$

- **Toricelli's Rule:** If a fluid is flowing through a very small hole (for example at the bottom of a large tank) then the velocity of the fluid at the large end can be neglected in Bernoulli's Equation.

- **Archimedes' Principle** The buoyant force on an object

$$F_b = \rho g V_{\text{disp}} \tag{280}$$

where frequency V_{disp} is the volume of fluid displaced by an object.

- **Venturi Effect** The pressure in a fluid *increases* as the velocity of the fluid *decreases*. This is responsible for e.g. the lift of an airplane wing.

Fluids

Fluids are the generic name given to two states of matter characterized by a lack of long range order and a high degree of mobility at the molecular scale. Fluids have the following properties:

- They usually assume the shape of any vessel they are placed in (exceptions are associated with surface effects such as surface tension and how well the fluid adheres to the surface in question).
- They are characterized by a mass per unit volume *density* ρ.
- They exert a *pressure* P (force per unit area) on themselves and any surfaces they are in contact with.
- The pressure can vary according to the dynamic and static properties of the fluid.
- The fluid has a measure of its "stickiness" and resistance to flow called *viscosity*. Viscosity is the internal friction of a fluid, more or less. We will treat fluids as being "ideal" and ignore viscosity in this course.
- Fluids are *compressible* – when the pressure in a fluid is increased, its volume descreases according to the relation:

$$\Delta P = -B\frac{\Delta V}{V} \tag{281}$$

where B is called the *bulk modulus* of the fluid (the equivalent of a spring constant).

- Fluids where B is a large number (so large changes in pressure create only tiny changes in fractional volume) are called *incompressible*. Water is an example of an incompressible fluid.
- Below a critical speed, the dynamic flow of a moving fluid tends to be laminar, where every bit of fluid moves parallel to its neighbors in response to pressure differentials and around obstacles. Above that speed it becomes turbulent flow. Turbulent flow is quite difficult to treat mathematically and is hence beyond the scope of this introductory course – we will restrict our attention to ideal fluids either static or in laminar flow.

Static Fluids

A fluid in static equilibrium must support its own weight. If one considers a small circular box of fluid with area ΔA perpendicular to gravity and sides of thickness dx, the force on the sides cancels due to symmetry. The force pushing down at the top of the box is $P_0 \Delta A$. The force pushing up from the bottom of the box is $(P_0 + dP)\Delta A$. The weight of the fluid in the box is $w = \rho g \Delta A dx$. Thus:

$$\{(P_0 + dP) - P_0\} \Delta A = \rho g \Delta A dx \tag{282}$$

or

$$dP = \rho g dx \tag{283}$$

If ρ is itself a function of pressure/depth (as occurs with e.g. air) this can be a complicated expression to evaluate. For an incompressible fluid such as water, ρ is constant over rather large variations in pressure. In that case, integration yields:

$$P = P_0 + \rho g D \tag{284}$$

Homework for Week 8

Problem 1.

A small boy is riding in a minivan with the windows closed, holding a helium balloon. The van goes around a corner to the left. Does the balloon swing to the left, still pull straight up, or swing to the right as the van swings around the corner?

Problem 2.

A boat carries large rocks out into the middle of a small pond to be used building a submerged reef. As the rocks are thrown overboard, the water level of the pond:

1) rises a little.

2) falls a little.

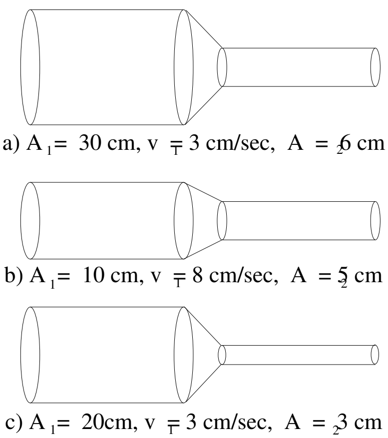
3) stays exactly the same.

4) We cannot tell what the water level of the pond does without more information.

Problem 3.

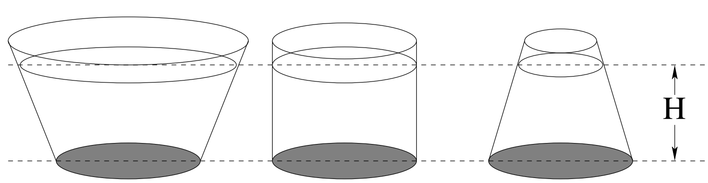
People with vascular disease or varicose veins are often told to walk in water 1-1.5 meters deep. Explain why.

Problem 4.



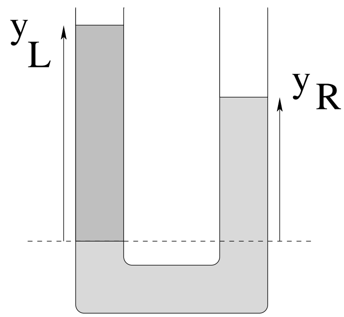
In the figure above three different pipes are shown, with cross-sectional areas and flow speeds as shown. Rank the three diagrams a, b, and c in the order of the speed of the outgoing flow.

Problem 5.



In the figure above three flasks are drawn that have the same (shaded) cross sectional area of the bottom. The depth of the water in all three flasks is H, and so the pressure at the bottom in all three cases is the same. Explain how the force exerted **by the fluid** on the circular bottom can be the same for all three flasks when all three flasks contain different weights of water.

Problem 6.



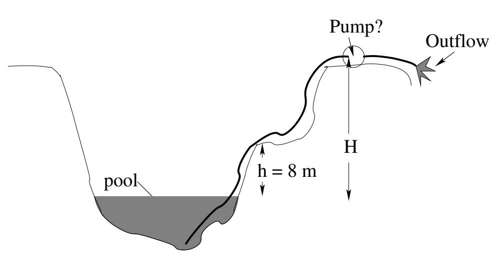
This problem will help you learn required concepts such as:

- Pascal's Principle
- Static Equilibrium

so please review them before you begin.

A vertical U-tube open to the air at the top is filled with oil (density ρ_o) on one side and water (density ρ_w) on the other, where $\rho_o < \rho_w$. Find y_L, the height of the column on the left, in terms of the densities, g, and y_R as needed. Clearly label the oil and the water in the diagram below and show all reasoning including the basic principle(s) upon which your answer is based.

Problem 7.



This problem will help you learn required concepts such as:

- Static Pressure

- Barometers

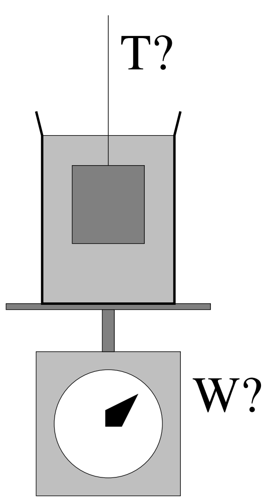
so please review them before you begin.

A pump is a machine that can maintain a pressure differential between its two sides. A particular pump that can maintain a pressure differential of as much as 10 atmospheres of pressure between the low pressure side and the high pressure side is being used on a construction site.

a) Your construction boss has just called you into her office to either explain why they aren't getting any water out of the pump on top of the $H = 25$ meter high cliff shown above. Examine the schematic above and show (algebraically) why it cannot possibly deliver water that high. Your explanation should include an invocation of the appropriate physical law(s) and an explicit calculation of the highest distance the a pump *could* lift water in this arrangement. Why is the notion that the pump "sucks water up" misleading? What really moves the water up?

b) If you answered a), you get to keep your job. If you answer b), you might even get a raise (or at least, get full credit on this problem)! Tell your boss where this single pump should be located to move water up to the top and show (draw a picture of) how it should be hooked up.

Problem 8.



This problem will help you learn required concepts such as:

- Archimedes Principle
- Weight

so please review them before you begin.

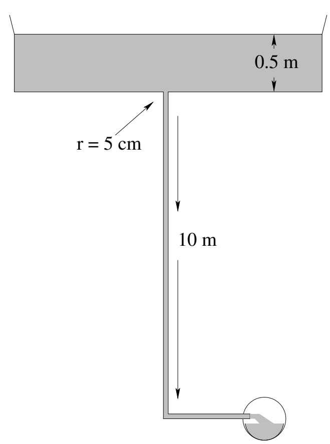
A block of density ρ and volume V is suspended by a thin thread and is immersed completely in a jar of oil (density $\rho_o < \rho$) that is resting on a scale as shown. The total mass of the oil and jar (alone) is M.

a) What is the buoyant force exerted by the oil on the block?

b) What is the tension T in the thread?

c) What does the scale read?

Problem 9.



This problem will help you learn required concepts such as:

- Pressure in a Static Fluid

so please review them before you begin.

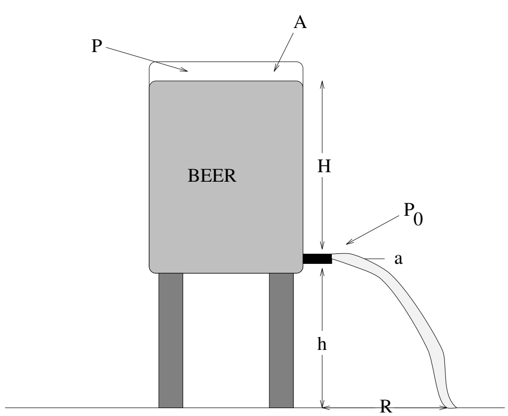
That it is dangerous to build a drain for a pool or tub consisting of a single narrow pipe that drops down a long ways before encountering air at atmospheric pressure was demonstrated tragically in an accident that occurred (no fooling!) within two miles from where you are sitting (a baby pool was built with just such a drain, and was being drained one day when a little girl sat down on the drain and was severely injured).

In this problem you will analyze why.

Suppose the mouth of a drain is a circle five centimeters in radius, and the pool has been draining long enough that its drain pipe is filled with water (and no bubbles) to a depth of ten meters below the top of the drain, where it exits in a sewer line open to atmospheric pressure. The pool is 50 cm deep. If a thin steel plate is dropped to suddenly cover the drain with a watertight seal, what is the force one would have to exert to remove it straight up?

Note carefully this force relative to the likely strength of mere flesh and bone (or even thin steel plates!) Ignorance of physics can be actively dangerous.

Problem 10.



This problem will help you learn required concepts such as:

- Bernoulli's Equation
- Toricelli's Law

so please review them before you begin.

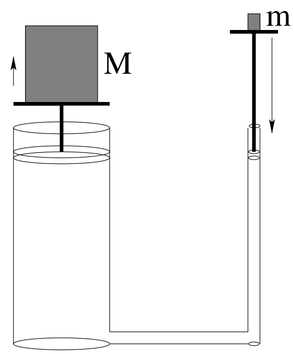
In the figure above, a CO_2 cartridge is used to maintain a pressure P on top of the beer in a beer keg, which is full up to a height H above the tap at the bottom (which is obviously open to normal air pressure). The keg has a cross-sectional area A at the top. Somebody has pulled the tube and valve off of the tap (which has a cross sectional area of a) at the bottom.

1) Find the speed with which the beer emerges from the tap. You may use the approximation $A \gg a$, but please do so only at the end. Assume laminar flow and no resistance.

2) Find the value of R at which you should place a pitcher (initially) to catch the beer.

3) Evaluate the answers to a) and b) for $A = 0.25$ m^2, $P = 2$ atmospheres, $a = 0.25$ cm^2, $H = 50$ cm.

Problem 11.



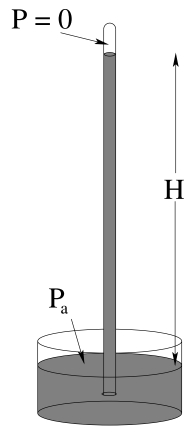
The figure above illustrates the principle of hydraulic lift. A pair of coupled cylinders are filled with an incompressible, very light fluid (assume that the mass of the fluid is zero compared to everything else).

1) If the mass M on the left is 1000 kilograms, the cross-sectional area of the left piston is 100 cm^2, and the cross sectional area of the right piston is 1 cm^2, what mass m should one place on the right for the two objects to be in balance?

2) Suppose one pushes the right piston down a distance of one meter. How much does the mass M rise?

Problem 12.

The idea of a barometer is a simple one. A tube filled with a suitable liquid is inverted into a reservoir. The tube empties (maintaining a seal so air bubbles cannot get into the tube) until the static pressure in the liquid is in balance with the *vacuum* that forms at



the top of the tube and the ambient pressure of the surrounding air on the fluid surface of the reservoir at the bottom.

1) Suppose the fluid is water, with $\rho_w = 1000$ kg/m^3. Approximately how high will the water column be? Note that water is not an ideal fluid to make a barometer with because of the height of the column necessary and because of its annoying tendency to boil at room temperature into a vacuum.

2) Suppose the fluid is mercury, with a specific gravity of 13.6. How high will the mercury column be? Mercury, as you can see, *is* nearly ideal for barometers except for the minor problem with its extreme toxicity and high vapor pressure.

Fortunately, there are many other ways of making good barometers.

Optional Problems: Start Review for Final!

At this point we are roughly four "weeks" out from our final exam[33]. I thus **strongly suggest** that you devote any extra time you have not to further reinforcement of fluid flow, but to a gradual slow review of all of the basic physics from the first half of the course. Make sure that you still remember and understand all of the basic principles of Newton's Laws, work and energy, momentum, rotation, torque and angular momentum. Look over your old homework and quiz and hour exam problems, review problems out of your notes, and look for help with any ideas that still aren't clear and easy.

[33]...which might be only *one and a half* weeks out in a summer session!

Week 9: Gravity

Cosmology and Kepler's Laws

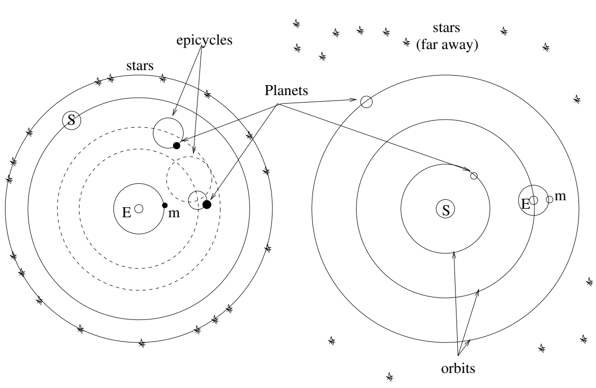
- Early western cosmology earth-centered.
- Ptolemy (140 A.D.) "explained" planets (which failed this model) with "epicycles". Church embraced this model as consistent with Genesis.
- Copernicus (1543 A.D.) solar-centered model.
- Tycho Brahe accumulated data and Johannes Kepler fit that data to specific orbits and deduced laws:

Kepler's Laws

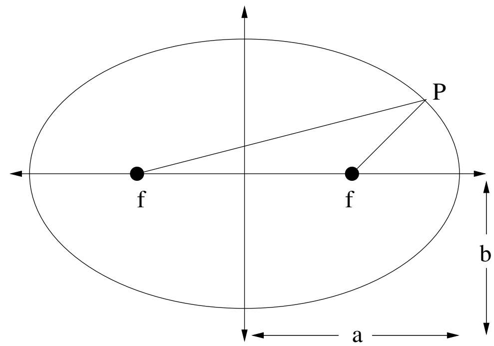
1) All planets move in elliptical orbits with the sun at one focus (see next section).

2) A line joining any planet to the sun sweeps out equal areas in equal times (dA/dt = constant).

3) The square of the period of any planet is proportional to the cube of the planet's mean distance from the sun ($T^2 = CR^3$). Note that the semimajor or semimi-



Ptolomeic (terricentric epicycles)   Copernican (heliocentric orbits) nor axis of the ellipse will serve as well as the mean, with different contants of proportionality.



Ellipses and Conic Sections

- An ellipse is one of the conic sections (intersections of a right circular cone with a crossing plane). The others are hyperbolas and parabolas (circles are special cases of ellipses). All conic sections are actually possible orbits, not just ellipses.

- One possible equation for an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{285}$$

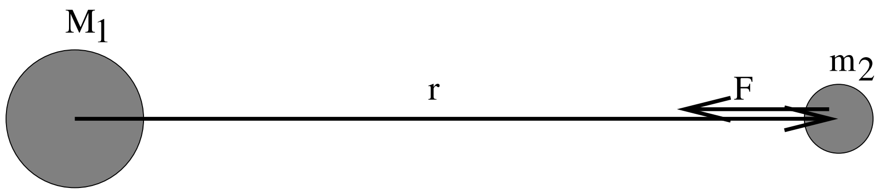
  The larger parameter (a above) is called the semimajor axis; the smaller (b) the semiminor; they lie on the similarly defined major and minor axes, where the foci of the ellipse lie on the major axis.

- Not all ellipses have major/minor axes that can be easily chosen to be x and y coordinates. Another general parameterization of an ellipse that is useful to us is a parametric cartesian representation:

$$x(t) = x_0 + a\cos(\omega t + \phi_x) \tag{286}$$
$$y(t) = y_0 + b\cos(\omega t + \phi_y) \tag{287}$$

  This equation will describe *any* ellipse centered on (x_0, y_0) by varying ωt from 0 to 2π. Adjusting the phase angles ϕ_x and ϕ_y and amplitudes a and b vary the orientation and eccentricity of the ellipse from a straight line at arbitrary angle to a circle.

- The foci of an ellipse are defined by the property that the sum of the distances from the foci to every point on an ellipse is a constant (so an ellipse can be drawn with a loop of string and two thumbtacks at the foci). If f is the distance of the foci from the origin, then the sum of the distances must be $2d = (f + a) + (a - f) = 2a$ (from the point $x = a$, $y = 0$. Also, $a^2 = f^2 + b^2$ (from the point $x = 0$, $y = b$). So $f = \sqrt{a^2 - b^2}$ where by convention $a \geq b$.



Newton's Law of Gravity

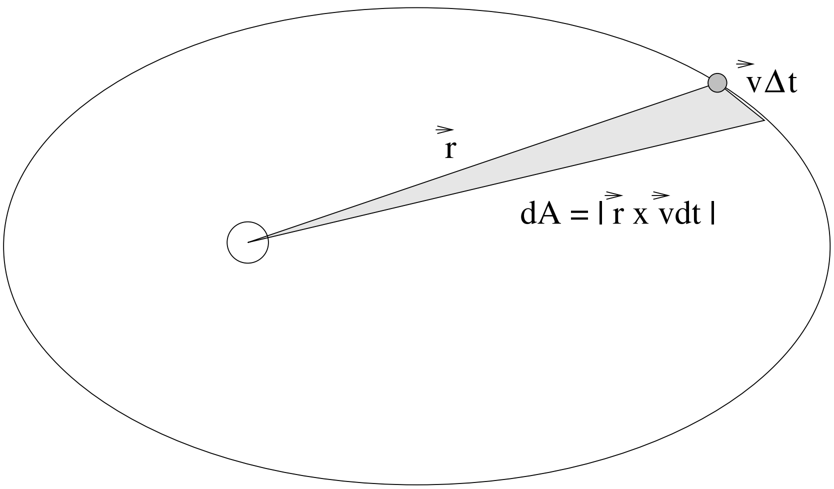
1) Force of gravity acts on a line joining centers of masses.
2) Force of gravity is attractive.
3) Force of gravity is proportional to each mass.
4) Force of gravity is **inversely proportional to the distance between the centers of the masses**.

or,

$$\vec{\boldsymbol{F}}_{12} = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}} \tag{288}$$

where $G = 6.67x10^{-11}$ N-m^2/kg^2 is the **universal gravitational constant**.

Kepler's first law follow from **solving** Newton's laws and the equations of motion for this particular force law. This is a bit difficult and beyond the scope of this course, although we *will* show that circular orbits are one special solution that easily satisfy Kepler's Laws.



Kepler's Second Law is proven by observing that this force is radial, and hence exerts no torque. Thus the angular momentum of a planet is constant! That is,

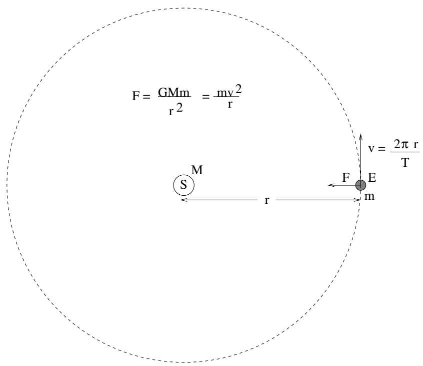
$$dA = \frac{1}{2}|\vec{\boldsymbol{r}} \times \vec{\mathbf{v}}dt| = \frac{1}{2}|\vec{\boldsymbol{r}}||\vec{\mathbf{v}}dt|\sin\theta \tag{289}$$

$$= \frac{1}{2m}|\vec{\boldsymbol{r}} \times m\vec{\mathbf{v}}dt| \tag{290}$$

or

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{\boldsymbol{L}}| = \text{constant} \tag{291}$$

(and Kepler's second law is proved for this force).



For a circular orbit, we can also prove Kepler's Third Law. The orbit is circular, so we have a relation between v and F_r.

$$\frac{GM_s m_p}{r^2} = m_p a_r = m_p \frac{v^2}{r} \tag{292}$$

so that

$$v^2 = \frac{GM_s}{r} \tag{293}$$

But, v is related to r and the period T by:

$$v = \frac{2\pi r}{T} \tag{294}$$

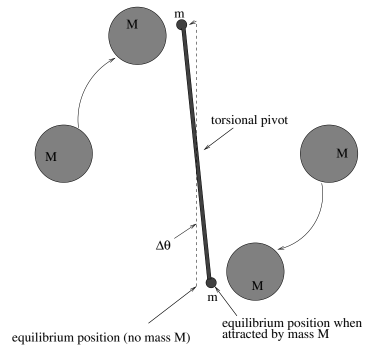
so that

$$v^2 = \frac{4\pi^2 r^2}{T^2} = \frac{GM_s}{r} \tag{295}$$

Finally,

$$r^3 = \frac{GM_s}{4\pi^2} T^2 \tag{296}$$

and Kepler's third law is proved for circular orbits (and the constant C evaluated for the solar system!).



The Gravitational Field

We define the gravitational field to be the **cause** of the gravitational force.

We define it conveniently to be the force per unit mass

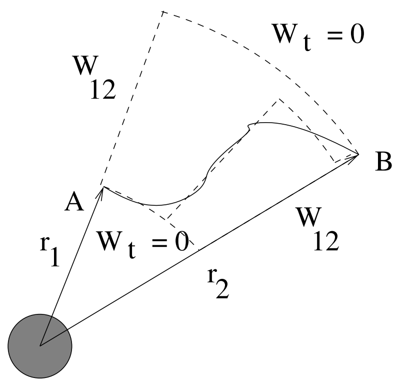
$$\vec{\mathbf{g}}(\vec{\boldsymbol{r}}) = -\frac{GM}{r^2}\hat{r} = \frac{\vec{\boldsymbol{F}}}{m} \tag{297}$$

The gravitational field at the surface of the earth is:

$$g(r) = \frac{F}{m} = \frac{GM_E}{R_E^2} \tag{298}$$

This equation can be used to find g, R_E, M_E, or G, from any of the other three, depending on which ones you think you know best. g is easy. R_E is actually also easy to measure independently and some classic methods were used to do so long before Columbus. M_E is hard! What about G?

Henry Cavendish made the first direct measurement of G using a torsional pendulum and some really massive balls. From this he was able to "weigh the earth" (find M_E). By measuring $\Delta\theta(r)$ (r measured between the centers) it was possible to directly measure G. He got 6.754 (vs 6.673 currently accepted) $\times 10^{-11}$ N-m^2/kg^2. Not bad!



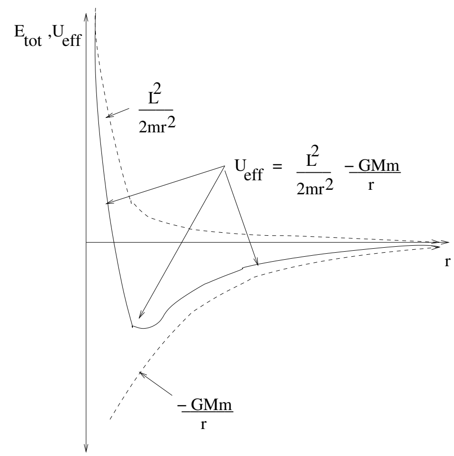
Gravitational Potential Energy

Gravitation is a conservative force, because the work done going "around" the attractor (perpendicular to the force) is zero, and the work done varying r is the same going out as in, so the work done is independent of path (see figure above). So:

$$\begin{aligned} U(r) &= -\int_{r_0}^{r} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} && (299) \\ &= -\int_{r_0}^{r} -\frac{GMm}{r^2} dr && (300) \\ &= -(\frac{GMm}{r} - \frac{GMm}{r_0}) && (301) \\ &= -\frac{GMm}{r} + \frac{GMm}{r_0} && (302) \end{aligned}$$

where r_0 is the radius of an arbitrary point where we define the potential energy to be zero. By convention, unless there is a good reason to choose otherwise, we require the zero of the potential energy to be at $r_0 = \infty$. Thus:

$$U(r) = -\frac{GMm}{r} \qquad (303)$$



Energy Diagrams and Orbits

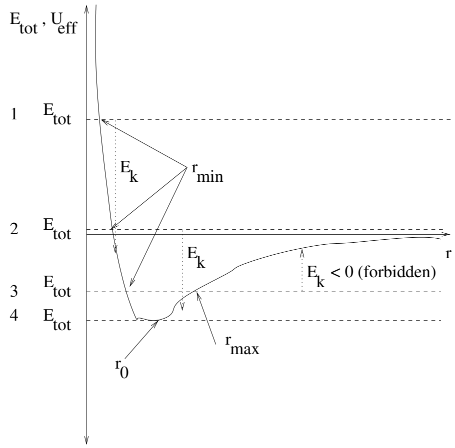
Let's write the total energy of a particle moving in a gravitational field in a clever way:

$$\begin{aligned} E_{\text{tot}} &= \frac{1}{2}mv^2 - \frac{GMm}{r} && (304) \\ &= \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\perp^2 - \frac{GMm}{r} && (305) \\ &= \frac{1}{2}mv_r^2 + \frac{1}{2mr^2}(mv_\perp r)^2 - \frac{GMm}{r} && (306) \\ &= \frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} && (307) \\ &= \frac{1}{2}mv_r^2 + U_{\text{eff}}(r) && (308) \end{aligned}$$

Where

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \quad (309)$$

is the (radial) potential energy plus the transverse kinetic energy (related to the constant angular momentum of the particle). If we plot the effective potential (and its pieces) we get a one-dimensional radial energy plot.



By drawing a constant total energy on this plot, the difference between $E_{\rm tot}$ and $U_{\rm eff}(r)$ is the radial kinetic energy, which must be positive. We can determine lots of interesting things from this diagram.

In this figure, we show orbits with a given angular momentum $\vec{\mathbf{L}} \neq 0$ and four generic total energies $E_{\rm tot}$. These orbits have the following characteristics and names:

1) $E_{\rm tot} > 0$. This is a **hyperbolic** orbit.

2) $E_{\rm tot} = 0$. This is a **parabolic** orbit. This orbit defines **escape velocity** as we shall see later.

3) $E_{\rm tot} < 0$. This is generally an **elliptical** orbit (consistent with Kepler's First Law).

4) $E_{\rm tot} = U_{\rm eff,min}$. This is a circular orbit. This is a special case of an elliptical orbit, but deserves special mention.

Note well that all of the orbits are **conic sections**. This interesting geometric connection between $1/r^2$ forces and conic orbits was a tremendous motivation for important mathematical work two or three hundred years ago.

Escape Velocity, Escape Energy

As we noted in the previous section, a particle has "escape energy" if and only if its total energy is greater than or equal to zero. We define the **escape velocity** (a misnomer!) of the particle as the minimum **speed** (!) that it must have to escape from its current gravitational field – typically that of a moon, or planet, or sun. Thus:

$$E_{\rm tot} = 0 = \frac{1}{2} m v_{\rm escape}^2 - \frac{GMm}{r} \tag{310}$$

so that

$$v_{\rm escape} = \sqrt{\frac{2GM}{r}} = \sqrt{2gr} \tag{311}$$

where in the last form $g = \frac{GM}{r^2}$ (the magnitude of the gravitational field – see next item). For earth:

$$v_{\rm escape} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2gR_E} = 11.2 \quad {\rm km/sec} \tag{312}$$

(Note: Recall the form derived by equating Newton's Law of Gravitation and mv^2/r in an earlier section for the velocity of a mass m in a circular orbit around a larger mass M:

$$v_{\rm circ}^2 = \frac{GM}{r} \tag{313}$$

from which we see that $v_{\rm escape} = \sqrt{2} v_{\rm circ}$.)

It is often interesting to contemplate this reasoning in reverse. If we drop a rock onto the earth from a state of rest "far away" (much farther than the radius of the earth, far enough away to be considered "infinity"), it will REACH the earth with escape (kinetic) energy. Since the earth is likely to be much larger than the rock, it will undergo an *inelastic* collision and release nearly *all* its kinetic energy as heat. If the rock is small, this is not a problem. If it is large (say, 1 km and up) it releases a *lot* of energy.

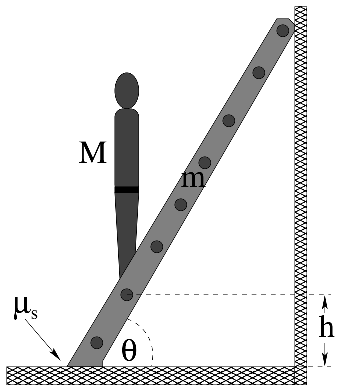
$$M = \frac{4\pi\rho}{3} r^3 \tag{314}$$

is a reasonable equation for the mass of a spherical rock. ρ can be estimated at 10^4 kg/m^3, so for $r \approx 1000$ meters, this is roughly $M \approx 4 \times 10^{13}$ kg, or around 10 billion metric tons of rock, about the mass of a small mountain.

This mass will land on earth with *escape velocity*, 11.2 km/sec, if it falls in from far away. Or more, of course – it may have started with velocity and energy from some other source – this is pretty much a minimum. As an exercise, compute the number of Joules this collision would release to toast the dinosaurs – or us!

Homework for Week 9

Problem 1.



This problem will help you learn required concepts such as:

- Torque Balance
- Force Balance
- Static Equilibrium
- Static Friction

so please review them before you begin.

In the figure above, a ladder of mass m and length L is leaning against a wall at an angle θ. A person of mass M begins to climb the ladder. The ladder sits on the ground with a coefficient of static friction μ_s between the ground and the ladder. The wall is frictionless – it exerts only a normal force on the ladder.

If the person climbs the ladder, find the height h where the ladder slips.

Problem 2.

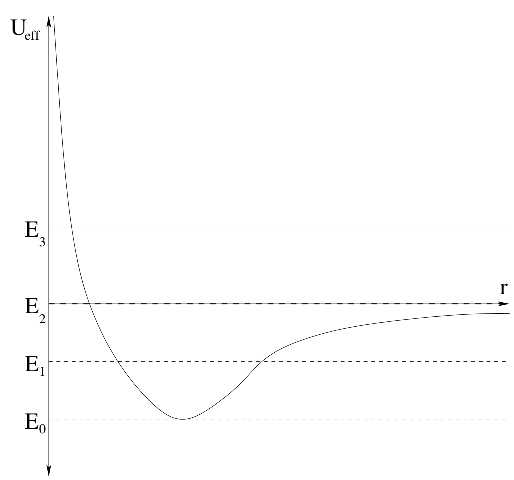
It is a horrible misconception that astronauts in orbit around the Earth are *weightless*, where weight (recall) is a measure of the actual gravitational force exerted on an object. Suppose you are in a space shuttle orbiting the Earth at a distance of two times the Earth's radius away from its center (a bit more than 6000 kilometers above the Earth's surface). What is your weight *relative* to your weight on the Earth's surface? Does your weight depend on whether or not you are moving? Why do you *feel* weightless inside an orbiting shuttle? Can you feel as "weightless" as an astronaut on the space shuttle (however briefly) in your own dorm room? How?

Problem 3.

Physicists are working to understand "dark matter", a phenomenological hypothesis invented to explain the fact that things such as the orbital periods around the centers of *galaxies* cannot be explained on the basis of estimates of Newton's Law of Gravitation using the total *visible* matter in the galaxy (which works well for the mass we can see in planetary or stellar context). By adding mass we cannot see until the orbital rates are explained, Newton's Law of Gravitation is preserved (and so are its general relativistic equivalents).

However, there are alternative hypotheses, one of which is that Newton's Law of Gravitation is *wrong*, deviating from a $1/r^2$ force law at very large distances (but remaining a central force). The orbits produced by such a $1/r^n$ force law (with $n \neq 2$) would not be elliptical any more, and $r^3 \neq CT^2$ – but would they still sweep out equal areas in equal times? Explain.

Problem 4.



The *effective radial potential* of a planetary object of mass m in an orbit around a star of mass M is:

$$U_{\text{eff}}(x) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

(a form you already explored in a previous homework problem). The total energy of four orbits are drawn as dashed lines on the figure above for some given value of L. Name the kind of orbit (circular, elliptical, parabolic, hyperbolic) each energy represents and mark its turning point(s).

Problem 5.

In a few lines prove Kepler's third law for *circular* orbits around a planet or star of mass M:

$$r^3 = CT^2$$

and determine the constant C and then answer the following questions:

1) Jupiter has a mean radius of orbit around the sun equal to 5.2 times the radius of Earth's orbit. How long does it take Jupiter to go around the sun (what is its orbital period or "year" T_J)?
2) Given the distance to the Moon of 3.84×10^8 meters and its (sidereal) orbital period of 27.3 days, find the mass of the Earth M_e.
3) Using the mass you just evaluated and your knowledge of g on the surface, estimate the radius of the Earth R_e.

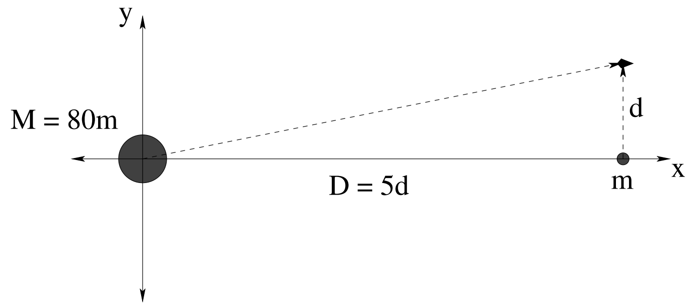
Check your answers using google/wikipedia. Think for just one short moment how much of the physics you have learned this semester is verified by the correspondance. Remember, I don't want you to believe anything I am teaching you because of my *authority* as a teacher but because it *works*.

Problem 6.

It is very costly (in energy) to lift a payload from the surface of the earth into a circular orbit, but it is generally only roughly twice as expensive to get from that circular orbit to anywhere you like if you are willing to wait a long time to get there. Prove this by comparing the total energy of a mass in a circular orbit to the total energy of a mass in a parabolic orbit that will arrive, at rest, "at infinity" after an infinite amount of time. (This is a special case of the *Virial Theorem* which is important in e.g. quantum chemistry and elsewhere.)

Problem 7.

This problem will help you learn required concepts such as:

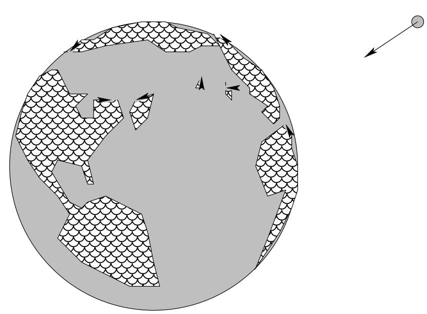


- Newton's Law of Gravitation
- The Vector Gravitational Field

so please review them before you begin.

The large mass above is the Earth, the smaller mass the Moon. Find the **vector gravitational field** acting on the spaceship on its way from Earth to Mars (swinging past the Moon at the instant drawn) in the picture above.

Problem 8.



This problem will help you learn required concepts such as:

- Gravitational Energy
- Fully Inelastic Collisions

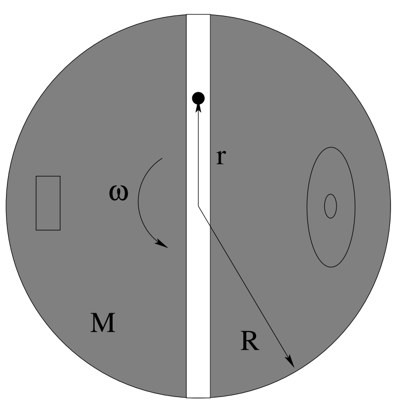
so please review them before you begin. For once you'll need a calculator, although you should still *do the problem algebraically first* and only substitute in numbers at the end. But they're "fun" numbers...

A bitter day comes: a (roughly spherical) asteroid is discovered that is approximately 10 km in radius that is falling in from far away so that it will strike the Earth. Make a sensible set of assumptions for things like its density ρ (and ignore the gravitational contribution of the sun to its energy) and estimate:

1) If it strikes the Earth (an *inelastic* collision if there ever was one) how much energy will be liberated as heat? Assume that the radius of the earth is $R = 6 \times 10^6$ meters and that $g = 10$ m/sec^2 at the surface of the earth or use values for G (known) or M_e (found elsewhere in this homework) as you prefer. Give the answer in both joules and in "tons" (of TNT) where 1 ton-of-TNT $= 4.2 \times 10^9$ joules. Compare the answer to (say) 30 Gigatons as a safe upper bound for the total combined explosive power of every weapon (including all the *nuclear* weapons) on earth.

2) An intrepid team of oil drillers led by Bruce Willis and Chuck Norris land on the surface of the asteroid to drill a deep hole blow it in two at the last second before a thrilling deadline. Not wanting to waste time drilling, Chuck jumps up from the surface of the asteroid at a speed of 5 m/sec to deliver a roundhouse kick that would surely break the asteroid in half (this *is* Chuck Norris, after all) and cause it to miss the earth – if it knows what's good for it. However, *does* Chuck ever come down (onto the asteroid, that is)?

Problem 9.



This problem will help you learn required concepts such as:

- Newton's Law of Gravitation
- Circular Orbits

- Centripetal Acceleration
- Kepler's Laws

so please review them before you begin.

A straight, smooth (frictionless) transit tunnel is dug through a spherical asteroid of radius R and mass M that has been converted into Darth Vader's *death star*. The tunnel is in the equatorial plane and passes through the center of the death star. The death star moves about in a hard vacuum, of course, and the tunnel is open so there are no drag forces acting on masses moving through it.

1) Find the force acting on a car of mass m a distance $r < R$ from the center of the death star.

2) You are commanded to find the precise rotational frequency of the death star ω such that objects in the tunnel will orbit *at* that frequency and hence will appear to *remain at rest* relative to the tunnel at any point along it. That way Darth can Use the Dark Side to move himself along it almost without straining his midichlorians. In the meantime, he is reaching his crooked fingers towards you and you feel a choking sensation, so better start to work.

3) Which of Kepler's laws does your orbit satisfy, and why?

Problem 10.

This problem will help you learn required concepts such as:

- Newton's Law of Gravitation
- Estimation

so please review them before you begin.

There is an old physics joke involving cows, and you will need to use its punchline to solve this problem.

A cow is standing in the middle of an open, flat field. A plumb bob with a mass of 1 kg is suspended via an unstretchable string 10 meters long so that it is hanging down roughly 2 meters away from the center of mass of the cow. Making any reasonable assumptions you like or need to, *estimate* the angle of deflection of the plumb bob from vertical due to the gravitational field of the cow.

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

No Optional Problems This Week (yet)

Week 10: Oscillations

Oscillation Summary

Oscillations occur whenever a force exists that pushes an object back towards a stable equilibrium position whenever it is displaced from it.

Such forces abound in nature – things are held together in structured form *because* they are in stable equilibrium positions and when they are disturbed in certain ways, they oscillate.

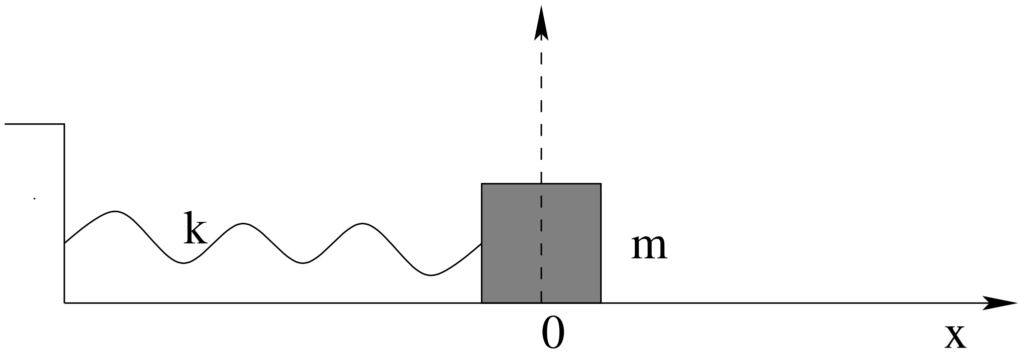
When the displacement from equilibrium is *small*, the restoring force is often *linearly* related to the displacement, at least to a good approximation. In that case the oscillations take on a special character – they are called *harmonic* oscillations as they are described by harmonic functions (sines and cosines) known from trigonometery.

In this course we will study **simple harmonic oscillators**, both with and without damping forces. The principle examples we will study will be masses on springs and various penduli.

Springs obey Hooke's Law: $\vec{F} = -k\vec{x}$ (where k is called the *spring constant*. A perfect spring produces perfect harmonic oscillation, so this will be our archetype.

A pendulum (as we shall see) has a restoring force or torque proportional to displacement for *small* displacements but is much too complicated to treat in this course for large displacements. It is a simple example of a problem that oscillates harmonically for small displacements but *not* harmonically for large ones.

An oscillator can be **damped** by dissipative forces such as friction and viscous drag. A damped oscillator can have exhibit a variety of behaviors depending on the relative strength and form of the damping force, but for one special form it can be easily described.



Springs

We will work in one dimension (call it x) and will for the time being place the spring equilibrium at the origin. Its equation of motion is thus:

$$F = -kx = ma = m\frac{d^2x}{dt^2} \tag{315}$$

Rearranging:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{316}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{317}$$

where $\omega^2 = k/m$ must have units of inverse time squared (why?).

This latter form is the standard *harmonic oscillator equation* (of motion). If we solve it *once and for all* now, whenever we can put an equation of motion into this form in the future we can just read off the solution by identifying similar quantities.

To solve it, we note that it basically tells us that $x(t)$ must be a function that has a second derivative proportional to the function itself. We know at least three functions whose second derivatives are proportional to themselves: cosine, sine and exponential. To learn something very important about the relationship between these functions, we'll assume the *exponential* form:

$$x(t) = x_0 e^{\alpha t} \tag{318}$$

(where α is an unknown parameter). Substituting this into the differential equation and differentiating, we get:

$$\begin{aligned} \frac{d^2 x_0 e^{\alpha t}}{dt^2} + \omega^2 x_0 e^{\alpha t} &= 0 \qquad (319) \\ \left(\alpha^2 + \omega^2\right) x_0 e^{\alpha t} &= 0 \qquad (320) \\ \left(\alpha^2 + \omega^2\right) &= 0 \qquad (321) \end{aligned}$$

where the last relation is called the *characteristic* equation for the differential equation. If we can find an α such that this equation is satisfied, then our assumed answer will indeed solve the D.E.

Clearly:

$$\alpha = \pm i\omega \qquad (322)$$

and we get two solutions! We will always get n independent solutions for an nth order differential equation, so this is good:

$$\begin{aligned} x_+(t) &= x_{0+} e^{+i\omega t} \qquad (323) \\ x_-(t) &= x_{0-} e^{-i\omega t} \qquad (324) \end{aligned}$$

and an arbitrary solution can be made up of a sum of these terms:

$$x(t) = x_{0+} e^{+i\omega t} + x_{0-} e^{-i\omega t} \qquad (325)$$

Simple Harmonic Oscillation

Solution

We generally are interested in *real* part of $x(t)$ when studying oscillating masses, so we'll stick to the following solution:

$$x(t) = X_0 \cos(\omega t + \phi) \tag{326}$$

where X_0 is called the *amplitude* of the oscillation and ϕ is called the *phase* of the oscillation. The amplitude tells you how big the oscillation is, the phase tells you when the oscillator was started relative to your clock (the one that reads t). Note that we could have used $\sin(\omega t + \phi)$ as well, or any of several other forms, since $\cos(\theta) = \sin(\theta + \pi/2)$. But you knew that.

X_0 and ϕ are two unknowns and have to be determined from the initial conditions, the givens of the problem. They are basically constants of integration *just like* x_0 and v_0 for the one-dimensional constant acceleration problem. From this we can easily see that:

$$v(t) = \frac{dx}{dt} = -\omega X_0 \sin(\omega t + \phi) \tag{327}$$

and

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 X_0 \cos(\omega t + \phi) = -\frac{k}{m}x(t) \tag{328}$$

(where the last relation proves the original differential equation).

Relations Involving ω

We remarked above that *omega* had to have units of t^{-1}. The following are some True Facts involving ω that You Should Know:

$$\omega = \frac{2\pi}{T} \tag{329}$$

$$= 2\pi f \tag{330}$$

where T is the *period* of the oscillator the time required for it to return to an identical position *and* velocity) and f is called the *frequency* of the oscillator. Know these relations *instantly*. They are easy to figure out but will cost you valuable time on a quiz or exam if you don't just take the time to completely embrace them now.

Note a very interesting thing. If we build a perfect simple harmonic oscillator, it oscillates at the same frequency *independent of its amplitude*. If we know the period and can count, we have just invented the *clock*. In fact, clocks are nearly *always* made out of various oscillators (why?); some of the earliest clocks were made using a pendulum as an oscillator and mechanical gears to count the oscillations, although now we use the much more precise oscillations of a bit of stressed crystalline quartz (for example) and electronic counters. The idea, however, remains the same.

Energy

The spring is a *conservative force*. Thus:

$$\begin{aligned} U &= -W(0 \to x) = -\int_0^x (-kx)dx = \frac{1}{2}kx^2 && (331) \\ &= \frac{1}{2}kX_0^2\cos^2(\omega t+\phi) && (332) \end{aligned}$$

where we have arbitrarily set the zero of potential energy to be the equilibrium position (what would it look like if the zero were at x_0?).

The kinetic energy is:

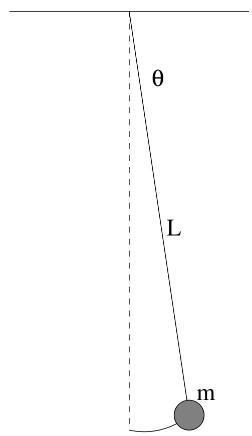
$$\begin{aligned} K &= \frac{1}{2}mv^2 && (333) \\ &= \frac{1}{2}m(\omega^2)X_0^2\sin^2(\omega t+\phi) && (334) \\ &= \frac{1}{2}m(\frac{k}{m})X_0^2\sin^2(\omega t+\phi) && (335) \\ &= \frac{1}{2}kX_0^2\sin^2(\omega t+\phi) && (336) \end{aligned}$$

The total energy is thus:

$$\begin{aligned} E &= \frac{1}{2}kX_0^2\sin^2(\omega t+\phi) + \frac{1}{2}kX_0^2\cos^2(\omega t+\phi) && (337) \\ &= \frac{1}{2}kX_0^2 && (338) \end{aligned}$$

and is *constant* in time! Kinda spooky how that works out...

Note that the energy oscillates between being all potential at the extreme ends of the swing (where the object comes to rest) and all kinetic at the equilibrium position (where the object experiences no force).



The Pendulum

The pendulum is another example of a simple harmonic oscillator, at least for small oscillations. Suppose we have a mass m attached to a string of length ℓ. We swing it up so that the stretched string makes a (small) angle θ_0 with the vertical and release it. What happens?

We write Newton's Second Law for the force component *tangent* to the arc of the circle of the swing as:

$$F_t = -mg\sin(\theta) = ma_t = m\ell\frac{d^2\theta}{dt^2} \tag{339}$$

where the latter follows from $a_t = \ell\alpha$ (the angular acceleration). Then we rearrange to get:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin(\theta) = 0 \tag{340}$$

This is *almost* a simple harmonic equation with $\omega^2 = \frac{g}{\ell}$. To make it one, we have to use the small angle approximation $\sin(\theta) \approx \theta$. Then

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \tag{341}$$

and we can just *read off the solution*:

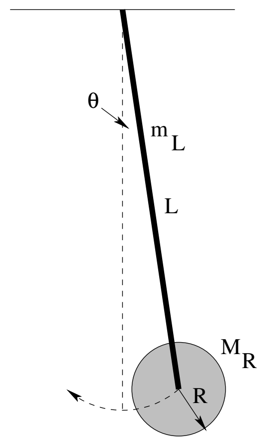
$$\theta(t) = \theta_0\cos(\omega t + \phi) \tag{342}$$

If you compute the gravitational potential energy for the pendulum for *arbitrary* angle θ, you get:

$$U(\theta) = mg\ell\,(1 - \cos(\theta)) \tag{343}$$

Somehow, this doesn't look like the form we might expect from blindly substituting into our solution for the SHO above:

$$U(t) = \frac{1}{2}mg\ell\theta_0^2\sin^2(\omega t + \phi) \tag{344}$$



As an interesting and fun exercise (that really isn't too difficult) see if you can prove that these two forms are really the same, *if* you make the small angle approximation for θ in the first form! This shows you pretty much where the approximation will break down as θ_0 is gradually increased. For large enough θ, the period of a pendulum clock *does* depend on the amplitude of the swing. This might explain grandfather clocks – clocks with very long penduli that can swing very slowly through very small angles – and why they were so accurate for their day.

The Physical Pendulum

In the treatment of the ordinary pendulum above, we just used Newton's Second Law directly to get the equation of motion. This was possible only because we could neglect the mass of the string and because we could treat the mass like a point mass at its end.

However, *real* grandfather clocks often have a large, massive pendulum like the one above – a long massive rod (of length L and mass m_L) with a large round disk (of radius R and mass M_R) at the end. The round weight rotates through an angle of $2\theta_0$ in each oscillation, so it has angular momemtum. Newton's Law for forces no longer suffices. We must use torque and the moment of inertia to obtain the frequency of the oscillator.

To do this we go through the *same steps* (more or less) that we did for the regular pendulum. First we compute the net gravitational torque on the system at an arbitrary (small) angle θ:

$$\tau = -\left(\frac{L}{2}m_L g + L M_R g\right)\sin(\theta) \tag{345}$$

(The - sign is there because the torque *opposes* the angular displacement from equilibrium.)

Next we set this equal to $I\alpha$, where I is the total moment of inertia for the *system* about the pivot of the pendulum and simplify:

$$\tau = -\left(\frac{L}{2}m_L g + LM_R g\right)\sin(\theta) = I\alpha = I\frac{d^2\theta}{dt^2} \tag{346}$$

$$I\frac{d^2\theta}{dt^2} + \left(\frac{L}{2}m_L g + LM_R g\right)\sin(\theta) = 0 \tag{347}$$

and make the small angle approximation to get:

$$\frac{d^2\theta}{dt^2} + \frac{\left(\frac{L}{2}m_L g + LM_R g\right)}{I}\ \theta = 0 \tag{348}$$

Note that for this problem:

$$I = \frac{1}{12}m_L L^2 + \frac{1}{2}M_R R^2 + M_R L^2 \tag{349}$$

(the moment of inertia of the rod plus the moment of inertial of the disk rotating about a parallel axis a distance L away from its center of mass). From this we can read off the angular frequency:

$$\omega^2 = \frac{4\pi^2}{T} = \frac{\left(\frac{L}{2}m_L g + LM_R g\right)}{I} \tag{350}$$

With ω in hand, we know everything. For example:

$$\theta(t) = \theta_0\cos(\omega t + \phi) \tag{351}$$

gives us the angular trajectory. We can easily solve for the period T, the frequency $f = 1/T$, the spatial or angular velocity, or whatever we like.

Note that the energy of this sort of pendulum can be tricky. Obviously its potential energy is easy enough – it depends on the elevation of the center of masses of the rod and the disk. The kinetic energy, however, is:

$$K = \frac{1}{2}I\left(\frac{d^2\theta}{dt^2}\right)^2 \tag{352}$$

where I do *not* write $\frac{1}{2}I\omega^2$ as usual because it confuses ω (the angular frequency of the oscillator, roughly contant) and $\omega(t)$ (the angular velocity of the pendulum bob, which varies between 0 and some maximum value in every cycle).

Damped Oscillation

So far, all the oscillators we've treated are *ideal.* There is no friction or damping. In the real world, of course, things *always* damp down. You have to keep pushing the kid on the swing or they slowly come to rest. Your car doesn't *keep* bouncing after going through a pothole in the road. Buildings and bridges, clocks and kids, real oscillators all have damping.

Damping forces can be very complicated. There is kinetic friction, which tends to be independent of speed. There are various fluid drag forces, which tend to depend on speed, but in a sometimes complicated way. There may be other forces that we haven't studied yet that contribute to damping. So in order to get beyond a very qualitative description of damping, we're going to have to specify a *form* for the damping force (ideally one we can work with, i.e. integrate).

We'll pick the simplest possible one:

$$F_d = -bv \tag{353}$$

(b is called the *damping constant* or *damping coefficient*) which is typical of an object being damped by a fluid at relatively low speeds. With this form we can get an exact solution to the differential equation easily (good), get a preview of a solution we'll need next semester to study LRC circuits (better), and get a very nice qualitative picture of damping besides (best).

We proceed with Newton's second law for a mass m on a spring with spring constant k and a damping force $-bv$:

$$F = -kx - bv = ma = m\frac{d^2x}{dt^2} \tag{354}$$

Again, simple manipulation leads to:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{355}$$

which is standard form.

Again, it looks like a function that is proportional to its own *first* derivative is called for (and in this case this excludes sine and cosine as possibilities). We guess $x(t) = x_0 e^{\alpha t}$ as before, substitute, cancel out the common $x(t) \neq 0$ and get the characteristic:

$$\alpha^2 + \frac{b}{m}\alpha + \frac{k}{m} = 0 \tag{356}$$

This is pretty easy, actually – each derivative just brings down an α.

To solve for α we have to use the dread *quadratic formula*:

$$\alpha = \frac{\frac{-b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{4k}{m}}}{2} \tag{357}$$

This isn't quite where we want it. We simplify the first term, factor a $-4k/m$ out from under the radical (where it becomes $i\omega_0$, where $\omega_0 = \sqrt{k/m}$ is the frequency of the *undamped* oscillator with the same mass and spring constant) and get:

$$\alpha = \frac{-b}{2m} \pm i\omega_0\sqrt{1 - \frac{b^2}{4km}} \tag{358}$$

Again, there are two solutions, for example:

$$x_\pm(t) = X_{0\pm} e^{\frac{-b}{2m}t} e^{\pm i\omega' t} \tag{359}$$

where

$$\omega' = \omega_0 \sqrt{1 - \frac{b^2}{4km}} \tag{360}$$

Again, we can take the real part of their sum and get:

$$x_\pm(t) = X_0 e^{\frac{-b}{2m}t} \cos(\omega' t + \phi) \tag{361}$$

where X_0 is the real initial amplitude and *phi* determines the relative phase of the oscillator.

Properties of the Damped Oscillator

There are several properties of the damped oscillator that are important to know.

- The amplitude damps *exponentially* as time advances. After a certain amount of time, the amplitude is halved. After the *same* amount of time, it is halved again.
- The frequency is shifted.
- The oscillator can be (under)damped, critically damped, or overdamped.

The oscillator is *underdamped* if ω' is real, which will be true if $4km > b^2$. It will undergo true oscillations, eventually approaching zero amplitude due to damping.

The oscillator is *critically* dampled if ω' is zero, when $4km = b^2$. The oscillator will *not oscillate* – it will go to zero *exponentially* in the shortest possible time.

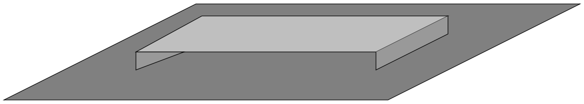
The oscillator is *overdamped* if ω' is imaginary, which will be true if $4km < b^2$. In this case α is entirely real and has a component that damps very slowly. The amplitude goes to zero *exponentially* as before, but over a longer (possibly much longer) time and does not oscillate through zero at all.

A car's shock absorbers should be barely underdamped. If the car "bounces" once and then damps to zero when you push down on a fender and suddenly release it, the shocks are good. If it bounces three of four time the shocks are too underdamped and dangerous as you could lose control after a big bump. If it doesn't bounce up and back down at all at all and instead slowly oozes back up to level from below, it is overdamped and dangerous, as a succession of sharp bumps could leave your shocks still compressed and unable to absorb the impack.

Tall buildings also have dampers to keep them from swaying in a strong wind. Houses are build with lots of dampers in them to keep them quiet. Fully understanding damped (and eventually driven) oscillation is essential to many sciences as well as both mechanical and electrical engineering.

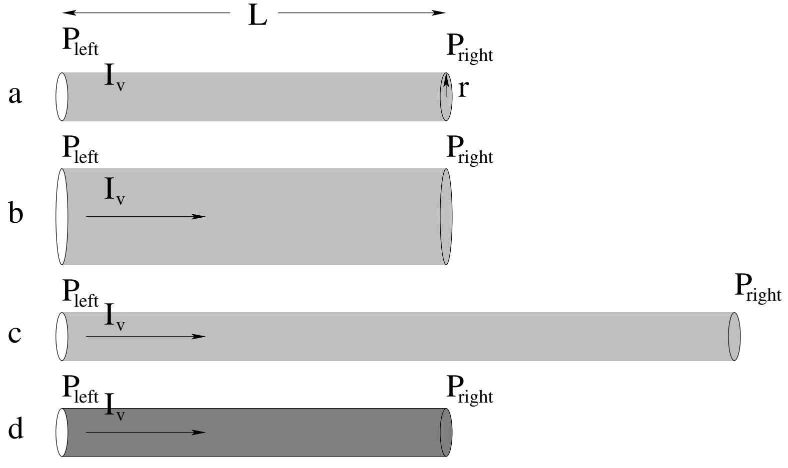
Homework for Week 10

Problem 1.



A card bent at the ends is placed on a table as shown above. Can you blow the card over by blowing underneath it? (Hint – you can try this at home if you have a piece of thin cardboard in your dorm.)

Problem 2.



In this class we usually idealize fluid flow by neglecting resistance (drag) and the viscosity of the fluid as it passes through cylindrical pipes so we can use the Bernoulli equation. As discussed in class, however, it is important to have at sound conceptual understanding of at least the qualitative effect of including the effects of resistance and viscosity. Use *Poiseuille's Law* (equations 13.24 and 13.25 in Tipler and Mosca) to answer and discuss the following simple questions to make sure your conceptual understanding of these concepts is qualitatively sound.

1) Is $\Delta P_a = P_{\text{left}} - P_{\text{right}}$ greater than, less than, or equal to zero in figure a) above, where blood flows at a rate I_v horizontally through a blood vessel with constant radius r and some length L against the **resistance** of that vessel?

2) If the radius r **increases** (while flow I_v and length L remain the same as in a), does the pressure difference ΔP_b increase, decrease, or remain the same compared to ΔP_a?

3) If the length **increases** (while flow I_v and radius r remains the same as in a), does the pressure difference ΔP_c increase, decrease, or remain the same compared to ΔP_a?

4) If the viscosity η of the blood **increases** (where flow I_v, radius r, and length L are all unchanged compared to a) do you expect the pressure difference ΔP_d difference across a blood vessel to increase, decrease, or remain the same compared to ΔP_a?

Blood viscosity is chemically related to things like the "stickiness" of platelets and their abundance in the blood. Viscosity and the radius r of blood vessels can be regulated or altered (within limits) by drugs, disease, diet and exercise – which have a completely understandable effect upon blood pressure.

Problem 3.

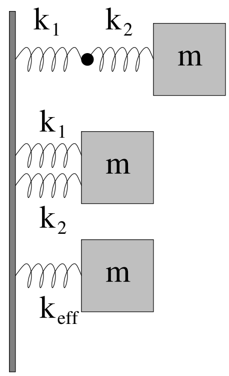
Harmonic oscillation is conceptually very important because (as has been remarked in class) many things that are *stable* will oscillate more or less harmonically if perturbed a small distance away from their stable equilibrium point. Draw a graph of a "generic" interaction potential energy with a stable equilibrium point and explain in words and equations where, and why, this should be.

Problem 4.

A pendulum with a string of length L supporting a mass m on the earth has a certain period T. A physicist on the moon, where the acceleration near the surface is around $g/6$, wants to make a pendulum with the same period. What mass and length of string could be used to accomplish this?

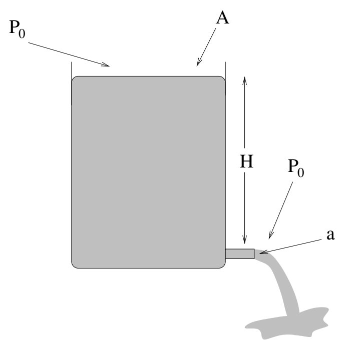
Problem 5.

In the figure above a mass M is attached to two springs k_1 and k_2 in "series" – one connected to the other end-to-end – and in "parallel" – the two springs side by side. In



the third picture the same mass is shown attached to a single spring with constant k_{eff}. What should k_{eff} be so that the force on the mass is the same for any given displacement Δx from equilibrium?

Problem 6.



This problem will help you learn required concepts such as:

- Bernoulli's Equation
- Torricelli's Law

so please review them before you begin.

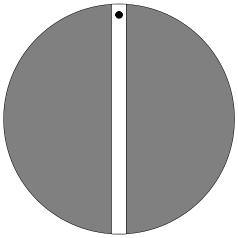
In the figure above, a large drum of water is open at the top and filled up to a height H above a tap at the bottom (which is also open to normal air pressure). The drum has a cross-sectional area A at the top and the tap has a cross sectional area of a at the bottom.

1) Find the speed with which the water emerges from the tap. Assume laminar flow without resistance. Compare your answer to the speed a mass has after falling a height H in a uniform gravitational field (after using $A \gg a$ to simplify your final answer, Torricelli's Law).

2) How long does it take for all of the water to flow out of the tap? (Hint: Start by *guessing* a reasonable answer using dimensional analysis and insight gained from a). That is, think about how you expect the time to vary with each quantity and form a simple expression with the relevant parameters that has the right units. Next, find an expression for the velocity of the top. Integrate to find the time it takes for the top to reach the bottom.) Compare your answer(s) to each other and the time it takes a mass to fall a height H in a uniform gravitational field. Does the correct answer make dimensional and physical sense?

3) Evaluate the answers to a) and b) for $A = 0.50$ m^2, $a = 0.5$ cm^2, $H = 100$ cm.

Problem 7.



This problem will help you learn required concepts such as:

- Newton's Law of Gravitation
- Harmonic Oscillation

so please review them before you begin.

A straight, smooth (frictionless) transit tunnel is dug through a planet of radius R whose mass density ρ_0 is constant. The tunnel passes through the center of the planet and is lined up with its axis of rotation (so that the planet's rotation is **irrelevant** to this problem). All the air is evacuated from the tunnel to eliminate drag forces.

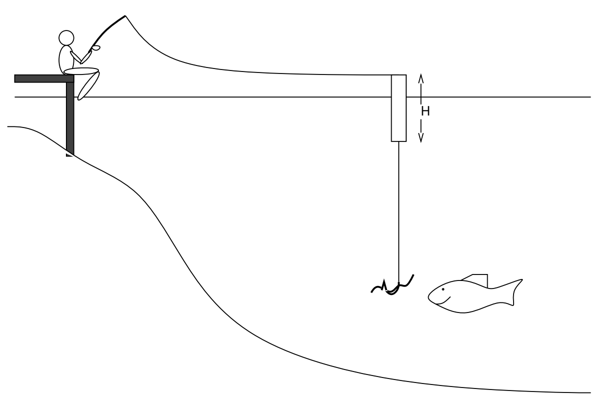
a) Find the force acting on a car of mass m a distance $r < R$ from the center of the planet.

b) Write Newton's second law for the car, and extract the differential equation of motion. From this find $r(t)$ for the car, assuming that it starts at $r_0 = R$ on the North Pole at time $t = 0$.

c) How long does it take the car to get to the South Pole starting from rest at the North Pole? Compare this to the period of a circular orbit inside a planet (a few homeworks ago).

All answers should be given in terms of G, ρ_0, R and m (or in terms of quantities you've already defined in terms of these quantities, such as ω).

Problem 8.



This problem will help you learn required concepts such as:

- Static equilibrium
- Archimedes Principle
- Simple Harmonic Oscillation

so please review them before you begin.

Jane is fishing in still water off of the old dock. She is using a cylindrical bobber as shown. The bobber has a cross sectional area of A and a length of H, and is balanced so that it remains vertical. When it is floating at equilibrium (supporting the weight of the hook and worm dangling underneath) $2H/3$ of its length is submerged in the water. You can neglect the volume of the line, hook and worm.

1) What is the combined mass M of the bobber, hook and worm from the data? You may give your answer in terms of ρ_w, the density of water.

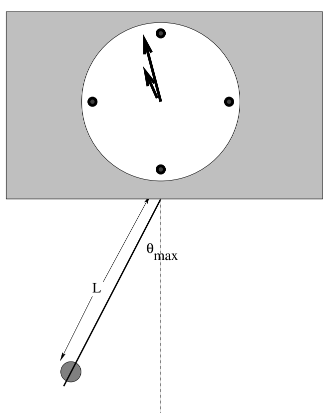
2) A fish gives a tug on the worm and pulls the bobber straight down an additional distance $y_0 = H/3$ (so the bobber's top is precisely at the surface of the water but doesn't go completely under). At $t = 0$ it lets go. What is the **net** restoring force on the bobber as a function of y, the displacement from equilibrium?

3) Use this force and the calculated mass M from a) to write Newton's 2nd Law for the motion of the bobber up and down (in y) and turn it into an equation of motion in standard form for a harmonic oscillator.

4) Neglecting the damping effects of the water, write an equation for the displacement of the bobber from its equilibrium depth as a function of time, $y(t)$ (the solution to the equation of motion).

5) With what frequency does the bobber bob? Evaluate your answer for $H = 10$ cm, $A = 1$ cm^2 (reasonable values for a fishing bobber).

Problem 9.



This problem will help you learn required concepts such as:

- Simple Harmonic Oscillation
- Force or Torque
- Small Angle Approximation

so please review them before you begin.

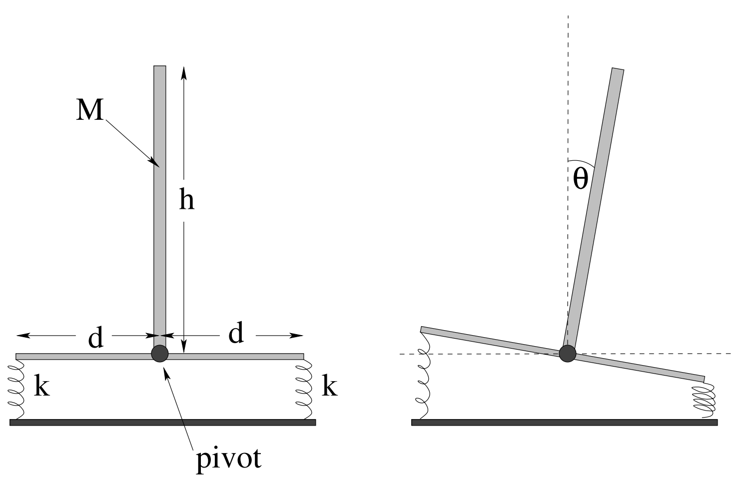
A grandfather clock is constructed with a pendulum that consists of a long, light (assume massless) rod and a small, heavy (assume point-like) mass that can be slid up and down the rod changing L to "tune" the clock. The clock keeps perfect time when the period of oscillation of the pendulum is $T = 2$ seconds. When the clock is running, the maximum angle the rod makes with the vertical is $\theta_{\rm max} = 0.05$ radians (a "small angle").

1) *Derive* the equation of motion for the rod when it freely swings and solve for $\theta(t)$ assuming it starts at $\theta_{\rm max}$ at $t = 0$. You may use either force or torque.

2) At what distance L from the pivot should the mass be set so that the clock keeps correct time? As always, solve the problem algebraically *first* and only then worry about numbers.

3) In your answer above, you idealized by assuming the rod to be massless and the ball to be pointlike. In the real world, of course, the rod has a small mass and the ball is a ball, not a point mass. *Without doing any additional algebra*, will the clock run slow or fast if you set the mass *exactly* where you computed it in part b)? Should you reduce L or increase L a bit to compensate and get better time? Explain.

Problem 10.



This problem will help you learn required concepts such as:

- Simple Harmonic Oscillation
- Springs (Hooke's Law)
- Small Angle Approximation

so please review them before you begin.

The pole in the figure, of mass $M = 50$ kg and length $h = 1.0$ m is supported by two identical springs with spring constant k connected to a platform as shown. The springs attach to the platform a distance $d = 0.5$ m from the base of the pole, which is pivoted on a frictionless hinge. Gravity acts down.

1) On the right-hand figure, indicate the forces acting on the pole and platform when the pole tips, deviating from the vertical by a small angle θ. You may neglect the mass of the platform.

2) Using your answer above, find the total torque on the system (pole and platform) about the hinged bottom of the pole, and note its direction. Use the approximations $\sin(\theta) \approx \theta$; $\cos(\theta) \approx 1$.

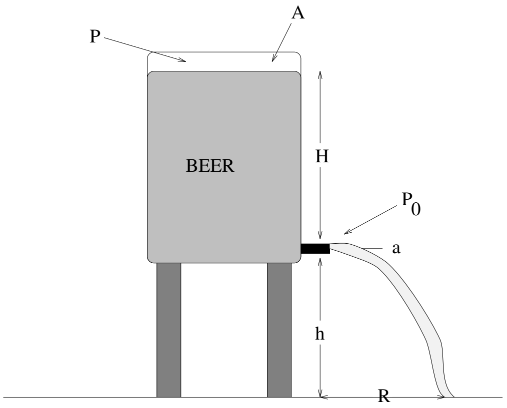
3) Find a minimal value for the spring constant k of the two springs such that the pole is stable in the vertical position. This means that a small deviation as above produces a torque that restores the pole to the vertical.

4) For springs with spring constant $k = 9600$ N/m, find the angular frequency with which the pole oscillates about the vertical.

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

Beer Keg



This problem will help you learn required concepts such as:

- Bernoulli's Equation
- Toricelli's Law

so please review them before you begin.

In the figure above, a CO_2 cartridge is used to maintain a pressure P on top of the beer in a beer keg, which is full up to a height H above the tap at the bottom (which is obviously open to normal air pressure). The keg has a cross-sectional area A at the top. Somebody has pulled the tube and valve off of the tap (which has a cross sectional area of a) at the bottom.

1) Find the speed with which the beer emerges from the tap. You may use the approximation $A \gg a$, but please do so only at the end. Assume laminar flow and no resistance.
2) Find the value of R at which you should place a pitcher (initially) to catch the beer.
3) Evaluate the answers to a) and b) for $A = 0.25$ m^2, $P = 2$ atmospheres, $a = 0.25$ cm^2, $H = 50$ cm.

Week 11: The Wave Equation

Wave Summary

- **Wave Equation**

$$\frac{d^2y}{dt^2} - v^2\frac{d^2y}{dx^2} = 0 \tag{362}$$

where for waves on a string:

$$v = \pm\sqrt{\frac{T}{\mu}} \tag{363}$$

- **Superposition Principle**

$$y(x,t) = Ay_1(x,t) + By_2(x,t) \tag{364}$$

(sum of solutions is solution). Leads to interference, standing waves.

- **Travelling Wave Pulse**

$$y(x,t) = f(x \pm vt) \tag{365}$$

where $f(u)$ is an arbitrary functional shape or pulse

- **Harmonic Travelling Waves**

$$y(x,t) = y_0 \sin(kx \pm \omega t). \tag{366}$$

where frequency f, wavelength λ, wave number $k = 2\pi/\lambda$ and angular frequency ω are related to v by:

$$v = f\lambda = \frac{\omega}{k} \tag{367}$$

- **Stationary Harmonic Waves**

$$y(x,t) = y_0 \cos(kx + \delta)\cos(\omega t + \phi) \tag{368}$$

where one can select k and ω so that waves on a string of length L satisfy fixed or free boundary conditions.

- **Energy (of wave on string)**

$$E_{\rm tot} = \frac{1}{2}\mu\omega^2 A^2 \lambda \tag{369}$$

is the total energy in a wavelength of a travelling harmonic wave. The wave transports the power

$$P = \frac{E}{T} = \frac{1}{2}\mu\omega^2 A^2 \lambda f = \frac{1}{2}\mu\omega^2 A^2 v \tag{370}$$

past any point on the string.

- **Reflection/Transmission of Waves**

  1) Light string (medium) to heavy string (medium): Transmitted pulse right side up, reflected pulse inverted. (A fixed string boundary is the limit of attaching to an "infinitely heavy string").
  2) Heavy string to light string: Transmitted pulse right side up, reflected pulse right side up. (a free string boundary is the limit of attaching to a "massless string").

Waves

We have seen how a particle on a spring that creates a restoring force proportional to its displacement from an equilibrium position oscillates harmonically in time about that equilibrium. What happens if there are *many* particles, *all* connected by tiny "springs" to one another in an extended way? This is a good metaphor for many, many physical systems. Particles in a solid, a liquid, or a gas both attract and repel one another with forces that maintain an average particle spacing. Extended objects under tension or pressure such as strings have components that can exert forces on one another. Even fields (as we shall learn next semester) can interact so that changes in one tiny element of space create changes in a neighboring element of space.

What we observe in all of these cases is that changes in any part of the medium "propagate" to other parts of the medium in a very systematic way. The motion observed in this propagation is called a *wave*. We have all observed waves in our daily lives in many contexts. We have watched water waves propagate away from boats and raindrops. We listen to sound waves (music) generated by waves created on stretched strings or from tubes driven by air and transmitted invisibly through space by means of radio waves. We read these words by means of light, an electromagnetic wave. In advanced physics classes one learns that all matter is a sort of quantum wave, that indeed *everything* is really a manifestation of waves.

It therefore seems sensible to make a first pass at understanding waves and how they work in general, so that we can learn and understand more in future classes that go into detail.

The concept of a wave is simple – it is an *extended structure* that oscillates in both *space* and in *time*. We will study two kinds of waves at this point in the course:

- **Transverse Waves** (e.g. waves on a string). The displacement of particles in a transverse wave is *perpendicular* to the direction of the wave itself.
- **Longitudinal Waves** (e.g. sound waves). The displacement of particles in a longitudinal wave is in the same direction that the wave propagates in.

Some waves, for example water waves, are simultaneously longitudinal and transverse. Transverse waves are probably the most important waves to understand for the future; light is a transverse wave. We will therefore start by studying transverse waves in a simple context: waves on a stretched string.

Waves on a String

Suppose we have a uniform string (such as a guitar string) that is stretched so that it is under tension T. The string is characterized by its mass per unit length μ – thick guitar strings have more mass per unit length than thin ones. It is fairly harmless at this point to imagine that the string is fixed to pegs at the ends that maintain the tension.

Now imagine that we have plucked the string somewhere between the end points so that it is displaced in the y-direction from its equilibrium (straight) stretched position and has some curved shape. If we examine a short segment of the string of length Δx, we can write Newton's 2nd law for that segment. If $\theta issmall$, in the x-direction the components of the tension nearly perfectly cancel. Each bit of string therefore moves more or less straight up and down, and a useful solution is described by $y(x, t)$, the y **displacement** of the string at position x and time t. The permitted solutions must be *continuous* if the string does not break.

In the y-direction, we find write the force law by considering the *difference* between the y-components at the ends:

$$F_y = T\sin(\theta_2) - T\sin(\theta_1) = \mu\Delta x a_y \tag{371}$$

We make the small angle approximation: $\sin(\theta) \approx \tan(\theta) \approx \theta$, divide out the $\mu\Delta x$, and note that $\tan(\theta) = \frac{dy}{dx}$ (the slope of the string is the tangent of the angle the string makes with the horizon). Then:

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{\Delta(\frac{dy}{dx})}{\Delta x} \tag{372}$$

In the limit that $\Delta x \to 0$, this becomes:

$$\frac{d^2y}{dt^2} - \frac{T}{\mu}\frac{d^2y}{dx^2} = 0 \tag{373}$$

The quantity $\frac{T}{\mu}$ has to have units of $\frac{L^2}{t^2}$ which is a velocity squared.

We therefore formulate this as the one dimensional *wave equation*:

$$\frac{d^2y}{dt^2} - v^2\frac{d^2y}{dx^2} = 0 \tag{374}$$

where

$$v = \pm\sqrt{\frac{T}{\mu}} \tag{375}$$

are the velocity(s) of the wave on the string. This is a second order linear homogeneous differential equation and has (as one might imagine) well known and well understood solutions.

Note well: At the tension in the string increases, so does the wave velocity. As the mass density of the string increases, the wave velocity decreases. This makes *physical sense*. As tension goes up the restoring force is greater. As mass density goes up one accelerates less for a given tension.

Solutions to the Wave Equation

In one dimension there are at least three distinct solutions to the wave equation that we are interested in. Two of these solutions *propagate* along the string – energy is *transported* from one place to another by the wave. The third is a *stationary* solution, in the sense that the wave doesn't propagate in one direction or the other (not in the sense that the string doesn't move). But first:

An Important Property of Waves: Superposition

The wave equation is *linear*, and hence it is easy to show that if $y_1(x,t)$ solves the wave equation and $y_2(x,t)$ (independent of y_1) *also* solves the wave equation, then:

$$y(x,t) = Ay_1(x,t) + By_2(x,t) \tag{376}$$

solves the wave equation for arbitrary (complex) A and B.

This property of waves is most powerful and sublime.

Arbitrary Waveforms Propagating to the Left or Right

The first solution we can discern by noting that the wave equation equates a second derivative in time to a second derivative in space. Suppose we write the solution as $f(u)$ where u is an unknown function of x and t and substitute it into the differential equation and use the chain rule:

$$\frac{d^2 f}{du^2}(\frac{du}{dt})^2 - v^2\frac{d^2 f}{du^2}(\frac{du}{dx})^2 = 0 \tag{377}$$

or

$$\frac{d^2 f}{du^2}\left\{(\frac{du}{dt})^2 - v^2(\frac{du}{dx})^2\right\} = 0 \tag{378}$$

$$\frac{du}{dt} = \pm v\frac{du}{dx} \tag{379}$$

with a simple solution:

$$u = x \pm vt \tag{380}$$

What this tells us is that *any* function

$$y(x,t) = f(x \pm vt) \tag{381}$$

satisfies the wave equation. *Any* shape of wave created on the string and propagating to the right or left is a solution to the wave equation, although not all of these waves will vanish at the ends of a string.

Harmonic Waveforms Propagating to the Left or Right

An interesting special case of this solution is the case of *harmonic* waves propagating to the left or right. Harmonic waves are simply waves that oscillate with a given harmonic frequency. For example:

$$y(x,t) = y_0 \sin(x - vt) \tag{382}$$

is one such wave. y_0 is called the *amplitude* of the harmonic wave. But what sorts of parameters describe the wave itself? Are there more than one harmonic waves?

This particular wave looks like a sinusoidal wave propagating to the right (positive x direction). But this is not a very convenient parameterization. To better describe a general harmonic wave, we need to introduce the following quantities:

- The **frequency** f. This is the number of cycles per second that pass a point or that a point on the string moves up and down.
- The **wavelength** λ. This the distance one has to travel down the string to return to the same point in the wave cycle at any given instant in time.

To convert x (a distance) into an angle in radians, we need to multiply it by 2π radians per wavelength. We therefore define the *wave number*:

$$k = \frac{2\pi}{\lambda} \tag{383}$$

and write our harmonic solution as:

$$\begin{aligned} y(x,t) &= y_0 \sin(k(x - vt)) && (384)\\ &= y_0 \sin(kx - kvt) && (385)\\ &= y_0 \sin(kx - \omega t) && (386) \end{aligned}$$

where we have used the following train of algebra in the last step:

$$kv = \frac{2\pi}{\lambda} v = 2\pi f = \frac{2\pi}{T} = \omega \tag{387}$$

and where we see that we have two ways to write v:

$$v = f\lambda = \frac{\omega}{k} \tag{388}$$

As before, you should simply *know* every relation in this set of algebraic relations between λ, k, f, ω, v to save time on tests and quizzes. Of course there is also the harmonic wave travelling to the left as well:

$$y(x,t) = y_0 \sin(kx + \omega t). \tag{389}$$

A final observation about these harmonic waves is that because arbitrary functions can be *expanded* in terms of harmonic functions (e.g. Fourier Series, Fourier Transforms) and because the wave equation is linear and its solutions are superposable, the two solution forms above are not really distinct. One can expand the "arbitrary" $f(x-vt)$ in a sum of $\sin(kx-\omega t)$'s for special frequencies and wavelengths. In one dimension this doesn't give you much, but in two or more dimensions this process helps one compute the *dispersion* of the wave caused by the wave "spreading out" in multiple dimensions and reducing its amplitude.

Stationary Waves

The third special case of solutions to the wave equation is that of *standing waves.* They are especially apropos to waves on a string fixed at one or both ends. There are two ways to find these solutions from the solutions above. A harmonic wave travelling to the right and hitting the end of the string (which is fixed), it has no choice but to reflect. This is because the *energy* in the string cannot just disappear, and if the end point is fixed no work can be done by it on the peg to which it is attached. The reflected wave has to have the same amplitude and frequency as the incoming wave. What does the *sum* of the incoming and reflected wave look like in this special case?

Suppose one adds two harmonic waves with equal amplitudes, the same wavelengths and frequencies, but that are travelling in *opposite* directions:

$$\begin{aligned} y(x,t) &= y_0\left(\sin(kx-\omega t)+\sin(kx+\omega t)\right) && (390)\\ &= 2y_0\sin(kx)\cos(\omega t) && (391)\\ &= A\sin(kx)\cos(\omega t) && (392)\end{aligned}$$

(where we give the standing wave the arbitrary amplitude A). Since all the solutions above are independent of the phase, a second useful way to write stationary waves is:

$$y(x,t) = A\cos(kx)\cos(\omega t) \tag{393}$$

Which of these one uses depends on the details of the boundary conditions on the string.

In this solution a sinusoidal form oscillates harmonically up and down, but the solution has some very important new properties. For one, it is always *zero* when $x=0$ for all possible *lambda*:

$$y(0,t) = 0 \tag{394}$$

For a given λ there are certain other x positions where the wave vanishes at all times. These positions are called *nodes* of the wave. We see that there are nodes for any L such that:

$$y(L,t) = A\sin(kL)\cos(\omega t) = 0 \tag{395}$$

which implies that:

$$kL = \frac{2\pi L}{\lambda} = \pi, 2\pi, 3\pi, \ldots \tag{396}$$

or

$$\lambda = \frac{2L}{n} \tag{397}$$

for $n = 1, 2, 3, ...$

Only waves with these wavelengths and their associated frequencies can persist on a string of length L fixed at both ends so that

$$y(0,t) = y(L,t) = 0 \tag{398}$$

(such as a guitar string or harp string). Superpositions of these waves are what give guitar strings their particular tone.

It is also possible to stretch a string so that it is fixed at one end but so that the *other* end is *free to move* – to slide up and down without friction on a greased rod, for example. In this case, instead of having a node at the free end (where the wave itself vanishes), it is pretty easy to see that the *slope* of the wave at the end has to vanish. This is because if the slope were not zero, the terminating rod would be pulling up or down on the string, contradicting our requirement that the rod be frictionless and not *able* to pull the string up or down, only directly to the left or right due to tension.

The slope of a sine wave is zero only when the sine wave itself is a maximum or minimum, so that the wave on a string free at an end must have an *antinode* (maximum magnitude of its amplitude) at the free end. Using the same standing wave form we derived above, we see that:

$$kL = \frac{2\pi L}{\lambda} = \pi/2, 3\pi/2, 5\pi/2 \ldots \tag{399}$$

for a string fixed at $x = 0$ and free at $x = L$, or:

$$\lambda = \frac{4L}{2n-1} \tag{400}$$

for $n = 1, 2, 3, ...$

There is a second way to obtain the standing wave solutions that particularly exhibits the relationship between waves and harmonic oscillators. One assumes that the solution $y(x,t)$ can be written as the *product* of a fuction in x alone and a second function in t alone:

$$y(x,t) = X(x)T(t) \tag{401}$$

If we substitute this into the differential equation and divide by $y(x,t)$ we get:

$$\begin{aligned}
\frac{d^2y}{dt^2} = X(x)\frac{d^2T}{dt^2} &= v^2\frac{d^2y}{dx^2} = v^2T(t)\frac{d^2X}{dx^2} && (402)\\
\frac{1}{T(t)}\frac{d^2T}{dt^2} &= v^2\frac{1}{X(x)}\frac{d^2X}{dx^2} && (403)\\
&= -\omega^2 && (404)
\end{aligned}$$

where the last line follows because the second line equations a function of t (only) to a function of x (only) so that both terms must equal a constant. This is then the two equations:

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{405}$$

and

$$\frac{d^2X}{dt^2} + k^2 X = 0 \tag{406}$$

(where we use $k = \omega/v$).

From this we see that:

$$T(t) = T_0 \cos(\omega t + \phi) \tag{407}$$

and

$$X(x) = X_0 \cos(kx + \delta) \tag{408}$$

so that the most general stationary solution can be written:

$$y(x,t) = y_0 \cos(kx + \delta)\cos(\omega t + \phi) \tag{409}$$

Reflection of Waves

We argued above that waves have to reflect of of the ends of stretched strings because of energy conservation. This is true independent of whether the end is fixed or free – in neither case can the string do work on the wall or rod to which it is affixed. However, the behavior of the reflected wave is different in the two cases.

Suppose a wave *pulse* is incident on the fixed end of a string. One way to "discover" a wave solution that apparently conserves energy is to imagine that the string *continues* through the barrier. At the same time the pulse hits the barrier, an *identical* pulse hits the barrier from the other, "imaginary" side.

Since the two pulses are identical, energy will clearly be conserved. The one going from left to right will transmit its energy onto the imaginary string beyond at the same rate energy appears going from right to left from the imaginary string.

However, we still have two choices to consider. The wave from the imaginary string could be *right side up* the same as the incident wave or *upside down*. Energy is conserved either way!

If the right side up wave (left to right) encounters an upside down wave (right to left) they will always be *opposite* at the barrier, and when superposed they will *cancel* at the barrier. This corresponds to a *fixed string*. On the other hand, if a right side up wave encounters a right side up wave, they will *add* at the barrier with opposite *slope*. There will be a maximum at the barrier with zero slope – just what is needed for a free string.

From this we deduce the general rule that wave pulses *invert* when reflected from a fixed boundary (string fixed at one end) and reflect right side up from a free boundary.

When two strings of different weight (mass density) are connected, wave pulses on one string are both transmitted onto the other and are generally partially reflected from the boundary. Computing the transmitted and reflected waves is straightforward but beyond the scope of this class (it starts to involve real math and studies of boundary conditions). However, the following qualitative properties of the transmitted and reflected waves should be learned:

- Light string (medium) to heavy string (medium): Transmitted pulse right side up, reflected pulse inverted. (A fixed string boundary is the limit of attaching to an "infinitely heavy string").
- Heavy string to light string: Transmitted pulse right side up, reflected pulse right side up. (a free string boundary is the limit of attaching to a "massless string").

Energy

Clearly a wave can carry energy from one place to another. A cable we are coiling is hung up on a piece of wood. We flip a pulse onto the wire, it runs down to the piece of wood and knocks the wire free. Our lungs and larnyx create sound waves, and those waves trigger neurons in ears far away. The sun releases nuclear energy, and a few minutes later that energy, propagated to earth as a light wave, creates sugar energy stored inside a plant that is still later released while we play basketball[34]. Since moving energy around seems to be important, perhaps we should figure out how a wave manages it.

Let us restrict our attention to a harmonic wave of known angular frequency ω. Our results will still be quite general, because arbitrary wave pulses can be fourier decomposed as noted above. Consider a small piece of the string of length dx and mass $dm = \mu dx$. This piece of string, displaced to its position $y(x,t)$, will have potential energy:

$$\begin{aligned} dU &= \frac{1}{2} dm\omega^2 y^2(x,t) && (410) \\ &= \frac{1}{2}\mu dx\omega^2 y^2(x,t) && (411) \\ &= \frac{1}{2}A^2\mu\omega^2\sin^2(kx-\omega t)dx && (412) \end{aligned}$$

We can easily integrate this over any specific interval. Let us pick a particular time $t = 0$ and integrate it over a single wavelength:

$$\begin{aligned} U &= \int_0^\lambda \frac{1}{2}A^2\mu\omega^2\sin^2(kx)dx && (413) \\ &= \frac{1}{2k}A^2\mu\omega^2\int_0^\lambda \sin^2(kx)kdx && (414) \\ &= \frac{1}{2k}A^2\mu\omega^2\int_0^{2\pi}\sin^2(\theta)d\theta && (415) \\ &= \frac{1}{4}A^2\mu\omega^2\lambda && (416) \end{aligned}$$

(where we have used the easily proven relation:

$$\int_0^{2\pi}\sin^2(\theta)d\theta = \pi \quad (417)$$

[34]Painfully and badly, in my case. As I'm typing this, my ribs and ankles hurt from participating in the Great Beaufort 3 on 3 Physics Basketball Tournament, Summer Session 1, 2011! But what the heck, we won and are in the finals. And it was energy propagated by a wave, stored (in my case) as fat, that helped get us there...

to do the final form of the integral.)

Now we need to compute the kinetic energy in a wavelength at the same instant.

$$\begin{aligned} dK &= \frac{1}{2}dm\left(\frac{dy}{dt}\right)^2 && (418)\\ &= \frac{1}{2}A^2\mu\omega^2\cos^2(kx-\omega t)dx && (419)\end{aligned}$$

which has exactly the same integral:

$$K = \frac{1}{4}A^2\mu\omega^2\lambda \tag{420}$$

so that the total energy in a wavelength of the wave is:

$$E_{\rm tot} = \frac{1}{2}\mu\omega^2 A^2\lambda \tag{421}$$

Study the dependences in this relation. Energy depends on the *amplitude squared!* (Emphasis to convince you to remember this! It is important!) It depends on the mass per unit length times the length (the mass of the segment). It depends on the frequency squared.

This energy *moves* as the wave propagates down the string. If you are sitting at some point on the string, all the energy in one wavelength passes you in one period of oscillation. This lets us compute the *power* carried by the string – the energy per unit time that passes us going from left to right:

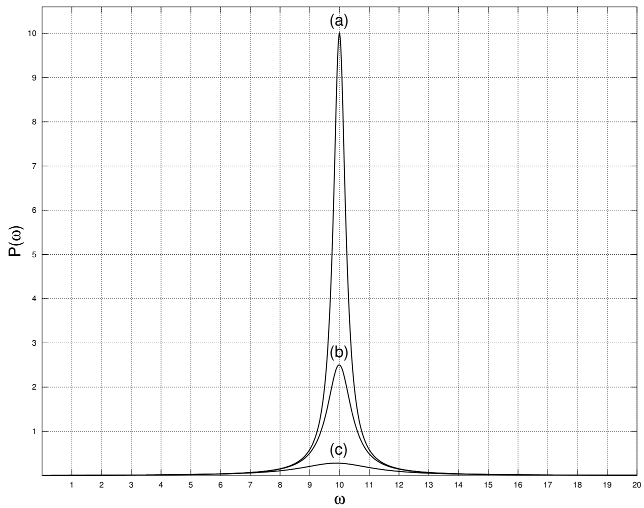
$$P = \frac{E}{T} = \frac{1}{2}\mu\omega^2 A^2\lambda f = \frac{1}{2}\mu\omega^2 A^2 v \tag{422}$$

We can think of this as being the *energy per unit length* (the total energy per wavelength divided by the wavelength) times the *velocity of the wave.* This is a very *good* way to think of it as we prepare to study light waves, where a very similar relation will hold.

Homework for Week 11

Problem 1.

In the figure above, three resonance curves showing the amplitude of steady-state driven oscillation $A(\omega)$ as functions of ω. In all three cases the resonance frequency ω_0 is the same. Put down an *estimate* of the Q-value of each oscillator by looking at the graph. It may help for you to put down the definition of Q most relevant to the process of estimation on the page.



1)

2)

3)

Problem 2.

Roman soldiers (like soldiers the world over even today) marched in step at a constant frequency – except when crossing wooden bridges, when they broke their march and walked over with random pacing. Why? What might have happened (and originally did sometimes happen) if they marched across with a collective periodic step?

Problem 3.

You have to take a long hike on level ground, and are in a hurry to finish it. On the other hand, you don't want to waste energy and arrive more tired than you have to be.

Your *stride* is the length of your steps. Your *pace* is the frequency of your steps, basically the number of steps you take per minute. Your *average speed* is the product of your pace and your stride: the distance travelled per minute is the number of steps you take per minute times the distance you cover per step.

Your best strategy to cover the distance faster but with minimum additional energy consumed is to:

1) Increase your stride but keep your pace about the same.

2) Increase your pace, but keep your stride about the same.

3) Increase your pace and your stride.

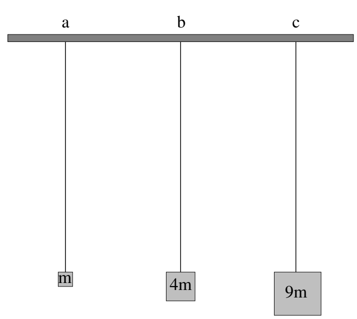
4) Increase your stride but decrease your pace.

5) Increase your pace but decrease your stride.

(in all cases so that your average speed increases).

Note well that this is a physics problem, so be sure to *justify your answer* with a physical argument. You might want to think about *why* one answer will probably accomplish your goal within the constraints and the others will not.

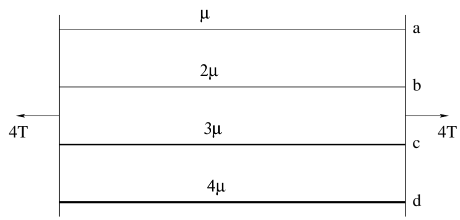
Problem 4.



Three strings of length L (not shown) with the same mass per unit length μ are suspended vertically and blocks of mass m, $4m$ and $9m$ are hung from them. The total mass of each string $\mu L \ll m$ (the strings are *much* lighter than the masses hanging from them). If the speed of a wave pulse on the first string (a) is v_0, what is the speed of the same wave pulse on the second (b) and third (c) strings?

Problem 5.

In the figure above, the neck of a stringed instrument is schmatized. Four strings of different *thickness* and the same length are stretched in such a way that the tension in each is about the same (T) for a total of $4T$ between the end bridges – if this were not so, the neck of the guitar or ukelele or violin would tend to bow towards the side with the greater tension. If the velocity of sound on the first (lightest) string is v, what is the speed of sound in terms of v of the other three?



Problem 6.

The Doppler shift of *light waves* is an important tool (probably *the* most important tool) used in the search for **extrasolar planets**, which are currently being discovered at the rate of better than one a day by special telescopes and projects such as the Kepler project. Explain how this works using the idea of the Doppler shift itself plus pictures and physics concepts that are familiar to you. You are encouraged to use the Internet and talk about this with your peers to help get a handle on the idea if it isn't immediately obvious to you. At the moment, with well over 500 extrasolar planets catalogued and more being discovered every week, it is becoming pretty clear that planets are *common* in our galaxy about stars that contain elements heavier than hydrogen and helium (which are usually "second generation stars") rather than being in any sense the *exception*.

This, in turn, makes it more plausible that life is not unique to our planet or even our solar system, as there are very probably many stars with planets the right size, orbital distance from the star, and general chemical environment required to sustain life in the over 100 billion stars of our Milky Way galaxy – and then there are the 500 billion or so other *galaxies* visible to the Hubble Space Telescope (with no boundary in sight and no reason to think that there aren't many trillions of galaxies, each with perhaps a hundred billion stars on average, containing quintillions of planets). That's a whole lot of rolls of the cosmic dice that can lead to an Earth-like planet and life.

Problem 7.

Fred is standing on the ground and Jane is blowing past him at a closest distance of approach of a few meters at twice the speed of sound in air. Both Fred and Jane are holding a loudspeaker that has been emitting sound at the *frequency* f_0 for some time.

1) Who hears the sound produced by the *other* person's speaker as *single frequency sound* when they are approaching one another and what frequency do they hear?

2) What does the *other* person hear (when they hear anything at all)?

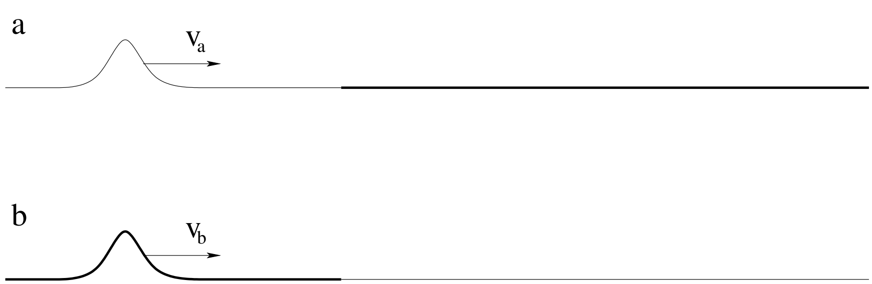
3) What frequenc(ies) do each of them hear *after* Jane has passed and is receding into the distance?

Problem 8.

Sunlight reaches the surface of the earth with *roughly* 1000 Watts/meter^2 of intensity. What is the "intensity level" of a *sound* wave that carries as much energy per square meter, in **decibels**? In table 15-1 in Tipler and Mosca, what kind of sound sources produce this sort of intensity? Bear in mind that the Sun is *150 million kilometers away* where sound sources capable of reaching the same intensity are typically only a few meters away. Hmmm, seems as though the Sun produces a *lot* of (electromagnetic) energy compared to terrestrial sources of (sound) energy.

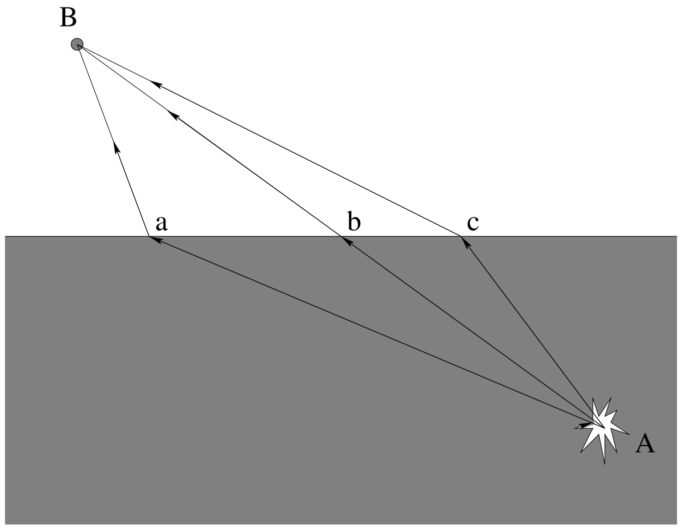
While you are at it, the human body produces energy at the rate of roughly 100 Watts. *Estimate* the fraction of this energy that goes into my lecture when I am speaking in a loud voice in front of the class (loud enough to be heard as loudly as normal conversation ten meters away).

Problem 9.



Two combinations of two strings with different mass densities are drawn above that are connected in the middle. In both cases the string with the greatest mass density is drawn darker and thicker than the lighter one, and the strings have the same tension T in both a and b. A wave pulse is generated on the string pairs that is travelling from left to right as shown. The wave pulse will arrive at the junction between the strings at time t_a (for a) and t_b (for b). Sketch reasonable *estimates* for the transmitted and reflected wave pulses onto the a and b figures at time $2t_a$ and $2t_b$ respectively. Your sketch should correctly represent things like the relative speed of the reflected and transmitted wave and any changes you might reasonably expect for the amplitude and appearance of the pulses.

Problem 10.



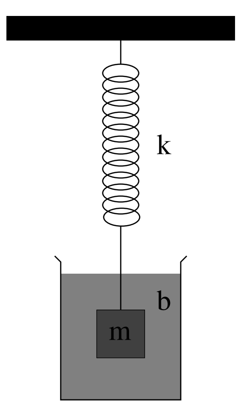
Sound waves travel faster in water than they do in air. Light waves travel faster in air than they do in water. Based on this, which of the three paths pictured above are more likely to *minimize* the time required for the

1) Sound

2) Light

produced by an underwater explosion to travel from the explosion at A to the pickup at B? Why (explain your answer)?

Problem 1.



A mass m is attached to a spring with spring constant k and immersed in a medium with damping coefficient b. (Gravity, if present at all, is irrelevant as shown in class). The net force on the mass when displaced by x from equilibrium and moving with velocity v_x is thus:

$$F_x = ma_x = -kx - bv_x$$

(in one dimension).

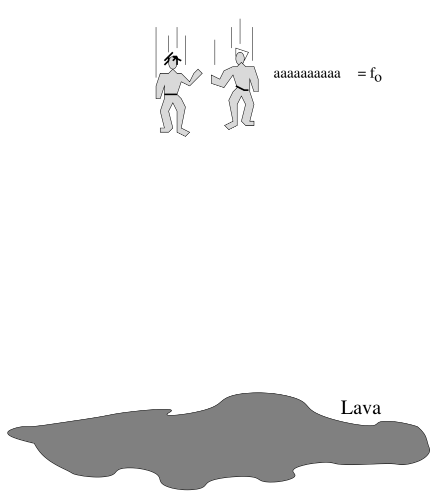
1) Convert this equation (Newton's second law for the mass/spring/damping fluid arrangement) into the equation of motion for the system, a "second order linear homogeneous differential equation" as done in class.

2) Optionally *solve* this equation, finding in particular the exponential damping rate of the solution (the real part of the exponential time constant) and the shifted frequency ω', assuming that the motion is underdamped. You can put down any form you like for the answer; the easiest is probably a sum of exponential forms. However, you may also simply **put down the solution** derived in class if you plan to *just* memorize this solution instead of learn to derive and understand it.

3) Using your answer for ω' from part b), write down the criteria for damped, underdamped, and critically damped oscillation.

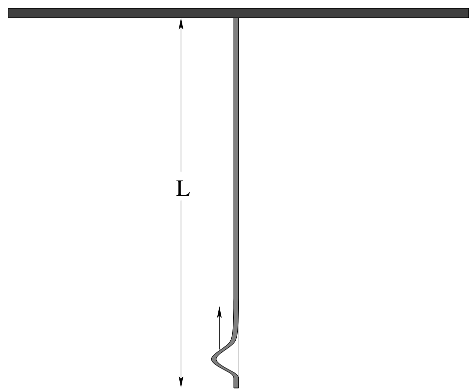
4) Draw *three* qualitatively correct graphs of $x(t)$ if the oscillator is pulled to a position x_0 and released at rest at time $t = 0$, one for each damping. Note that you should be able to do this part even if you cannot derive the curves that you draw or ω'. Clearly label each curve.

Problem 2.



Bill and Ted are falling at a constant speed (terminal velocity) into hell, and are screaming at a frequency f_0. They hear their own voices reflecting back to them from the puddle of molten rock that lies below at a frequency of $2f_0$. How fast are they falling relative to the speed of sound?

Problem 3.



This problem will help you learn required concepts such as:

- Speed of Wave on String
- Static Equilibrium
- Relationship betweeen Distance, Velocity, and Time

so please review them before you begin.

A string of total length L with a mass density μ is shown hanging from the ceiling above.

1) Find the tension in the string as a function of y, the distance *up* from its bottom end. Note that the string is not massless, so each small bit of string must be in static equilibrium.

2) Find the velocity $v(y)$ of a small wave pulse cast into the string at the bottom that is travelling upward.

3) Find the amount of time it will take this pulse to reach the top of the string, reflect, and return to the bottom. Neglect the size (width in y) of the pulse relative to the length of the string.

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

None yet.

Week 12: Sound

Sound Summary

- **Speed of Sound** in a fluid
$$v = \sqrt{\frac{B}{\rho}} \tag{423}$$
where B is the bulk modulus of the fluid and ρ is the density. These quantities vary with pressure and temperature.
- **Speed of Sound** in air is $v_a \approx 340$ m/sec.
- **Doppler Shift: Moving Source**
$$f' = \frac{f_0}{(1 \mp \frac{v_s}{v_a})} \tag{424}$$
where f_0 is the unshifted frequency of the sound wave for receding (+) and approaching (-) source.
- **Doppler Shift: Moving Receiver**
$$f' = f_0(1 \pm \frac{v_s}{v_a}) \tag{425}$$
where f_0 is the unshifted frequency of the sound wave for receding (-) and approaching (+) receiver.
- **Stationary Harmonic Waves**
$$y(x,t) = y_0 \sin(kx)\cos(\omega t) \tag{426}$$
for displacement waves in a pipe of length L closed at one or both ends. This solution has a node at $x = 0$ (the closed end). The permitted resonant frequencies are determined by:
$$kL = n\pi \tag{427}$$
for $n = 1, 2...$ (both ends closed, nodes at both ends) or:
$$kL = \frac{2n-1}{2}\pi \tag{428}$$
for $n = 1, 2, ...$ (one end closed, nodes at the closed end).

- **Beats** If two sound waves of equal amplitude and slightly different frequency are added:

$$s(x,t) = s_0 \sin(k_0 x - \omega_0 t) + s_0 \sin(k_1 x - \omega_1 t) \quad (429)$$
$$= 2s_0 \sin(\frac{k_0 + k_1}{2}x - \frac{\omega_0 + \omega_1}{2}t) \cos(\frac{k_0 - k_1}{2}x - \frac{\omega_0 - \omega_1}{2}t) \quad (430)$$

  which describes a wave with the average frequency and twice the amplitude modulated so that it "beats" (goes to zero) at the difference of the frequencies $\delta f = |f_1 - f_0|$.

Sound Waves in a Fluid

Waves propagate in a fluid much in the same way that a disturbance propagates down a closed hall crowded with people. If one shoves a person so that they knock into their neighbor, the neighbor falls against *their* neighbor (and shoves back), and their neighbor shoves against their still further neighbor and so on.

Such a wave differs from the transverse waves we studied on a string in that the displacement of the medium (the air molecules) is *in the same direction* as the direction of propagation of the wave. This kind of wave is called a *longitudinal* wave.

Although different, sound waves can be related to waves on a string in many ways. Most of the similarities and differences can be traced to one thing: a string is a one dimensional medium and is characterized only by length; a fluid is typically a three dimensional medium and is characterized by a volume.

Air (a typical fluid that supports sound waves) does not support "tension", it is under pressure. When air is compressed its molecules are shoved closer together, altering its density and occupied volume. For small changes in volume the pressure alters approximately *linearly* with a coefficient called the "bulk modulus" B describing the way the pressure increases as the fractional volume decreases. Air does not have a mass per unit length μ, rather it has a mass per unit volume, ρ.

The velocity of waves in air is given by

$$v_a = \sqrt{\frac{B}{\rho}} \approx 343\mathrm{m/sec} \quad (431)$$

The "approximately" here is fairly serious. The actual speed varies according to things like the air pressure (which varies significantly with altitude and with the weather at any given altitude as low and high pressure areas move around on the earth's surface) and the temperature (hotter molecules push each other apart more strongly at any given density). The speed of sound can vary by a few percent from the approximate value given above.

Sound Wave Solutions

Sound waves can be characterized one of two ways: as organized fluctuations in the *position* of the molecules of the fluid as they oscillate around an equilibrium displacement or as organized fluctuations in the *pressure* of the fluid as molecules are crammed closer together or are diven farther apart than they are on average in the quiescent fluid.

Sound waves propagate in one direction (out of three) at any given point in space. This means that in the direction *perpendicular* to propagation, the wave is spread out to form a "wave front". The wave front can be nearly arbitrary in shape initially; thereafter it evolves according to the mathematics of the wave equation in three dimensions (which is similar to but a bit more complicated than the wave equation in one dimension).

To avoid this complication and focus on general properties that are commonly encountered, we will concentrate on two particular kinds of solutions:

1) *Plane Wave* solutions. In these solutions, the entire wave moves in one direction (say the x direction) and the wave front is a 2-D plane perpendicular to the direction of propagation. These (displacement) solutions can be written as (e.g.):

$$s(x, t) = s_0 \sin(kx - \omega t) \tag{432}$$

where s_0 is the maximum displacement in the travelling wave (which moves in the x direction) and where all molecules in the entire *plane* at position x are displaced by the same amount.

Waves far away from the sources that created them are best described as plane waves. So are waves propagating down a constrained environment such as a tube that permits waves to only travel in "one direction".

2) *Spherical Wave* solutions. Sound is often emitted from a source that is highly localized (such as a hammer hitting a nail, or a loudspeaker). If the sound is emitted equally in all directions from the source, a spherical wavefront is formed. Even if it is not emitted equally in all directions, sound from a localized source will generally form a spherically curved wavefront as it travels away from the point with constant speed. The displacement of a spherical wavefront *decreases* as one moves further away from the source because the *energy* in the wavefront is spread out on larger and larger surfaces. Its form is given by:

$$s(r, t) = \frac{s_0}{r} \sin(kr - \omega t) \tag{433}$$

where r is the radial distance away from the point-like source.

Sound Wave Intensity

The energy density of sound waves is given by:

$$\frac{dE}{dV} = \frac{1}{2} \rho \omega^2 s^2 \tag{434}$$

(again, very similar in form to the energy density of a wave on a string). However, this energy per unit *volume* is propagated in a single direction. It is therefore spread out so that it crosses an *area*, not a single point. Just how much energy an object receives therefore depends on how much *area* it intersects in the incoming sound wave, not just on the energy density of the sound wave itself.

For this reason the energy carried by sound waves is best measured by *intensity*: the energy per unit time per unit area perpendicular to the direction of wave propagation. Imagine a box with sides given by ΔA (perpendicular to the direction of the wave's propagation) and $v\Delta t$ (in the direction of the wave's propagation. All the energy in this box crosses through ΔA in time Δt. That is:

$$\Delta E = (\frac{1}{2}\rho\omega^2 s^2)\Delta A v \Delta t \tag{435}$$

or

$$I = \frac{\Delta E}{\Delta A \Delta t} = \frac{1}{2}\rho\omega^2 s^2 v \tag{436}$$

which looks very much like the *power* carried by a wave on a string. In the case of a plane wave propagating down a narrow tube, it is very similar – the power of the wave is the intensity times the tube's cross section.

However, consider a spherical wave. For a spherical wave, the intensity looks something like:

$$I(r,t) = \frac{1}{2}\rho\omega^2 \frac{s_0^2 \sin^2(kr - \omega t)}{r^2} v \tag{437}$$

which can be written as:

$$I(r,t) = \frac{P}{r^2} \tag{438}$$

where P is the total power in the wave.

This makes sense from the point of view of energy conservation and symmetry. If a source emits a power P, that energy has to cross each successive spherical surface that surrounds the source. Those surfaces have an area that varies like $A = 4\pi r^2$. A surface at $r = 2r_0$ has 4 times the area of one at $r = r_0$, but the *same* total power has to go through both surfaces. Consequently, the intensity at the $r = 2r_0$ surface has to be 1/4 the intensity at the $r = r_0$ surface.

It is important to remember this argument, simple as it is. Think back to Newton's law of gravitation. Remember that gravitational field diminishes as $1/r^2$ with the distance from the source. Electrostatic field also diminishes as $1/r^2$. There seems to be a shared connection between symmetric propagation and spherical geometry; this will form the basis for *Gauss's Law* in electrostatics and much beautiful math.

Doppler Shift

Everybody has heard the doppler shift in action. It is the rise (or fall) in frequency observed when a source/receiver pair approach (or recede) from one another. In this section we will derive expressions for the doppler shift for moving source and moving receiver.

Moving Source

Suppose your receiver (ear) is stationary, while a source of harmonic sound waves at fixed frequency f_0 is approaching you. As the waves are emitted by the source they have a fixed wavelength $\lambda_0 = v_a/f_0 = v_a T$ and expand spherically from the point where the source was at the time the wavefront was emitted.

However, that point moves in the direction of the receiver. In the time between wavefronts (one period T) the source moves a distance $v_s T$. The distance between successive wavefronts in the direction of motion is thus:

$$\lambda' = \lambda_0 - v_s T \tag{439}$$

and (factoring and freely using e.g. $f = 1/T$):

$$\frac{v_a}{f'} = \frac{v_a - v_s}{f_0} \tag{440}$$

or

$$f' = \frac{f_0}{1 - \frac{v_s}{v_a}} \tag{441}$$

If the source is moving away from the receiver, everything is the same except now the wavelength is shifted to be bigger and the frequency smaller (as one would expect from changing the sign on the velocity):

$$f' = \frac{f_0}{1 + \frac{v_s}{v_a}} \tag{442}$$

Moving Receiver

Now imagine that the source of waves at frequency f_0 is stationary but the receiver is moving towards the source. The source is thus surrounded by spherical wavefronts a distance $\lambda_0 = v_a T$ apart. At $t = 0$ the receiver crosses one of them. At a time T' later, it has moved a distance $d = v_r T'$ in the direction of the source, and the wave from the source has moved a distance $D = v_a T'$ toward the receiver, and the receiver encounters the next wave front. That is:

$$\begin{aligned}
\lambda_0 &= d + D && (443)\\
&= v_r T' + v_a T' && (444)\\
&= (v_r + v_a) T' && (445)\\
v_a T &= (v_r + v_a) T' && (446)
\end{aligned}$$

We use $f_0 = 1/T$, $f' = 1/T'$ (where T' is the apparent time between wavefronts to the receiver) and rearrange this into:

$$f' = f_0(1 + \frac{v_r}{v_a}) \tag{447}$$

Again, if the receiver is moving away from the source, everything is the same but the sign of v_r, so one gets:

$$f' = f_0(1 - \frac{v_r}{v_a}) \tag{448}$$

Moving Source and Moving Receiver

This result is just the product of the two above – moving source causes one shift and moving receiver causes another to get:

$$f' = f_0 \frac{1 \mp \frac{v_r}{v_a}}{1 \pm \frac{v_s}{v_a}} \tag{449}$$

where in both cases *relative* approach shifts the frequency up and *relative* recession shifts the frequency down.

I do *not* recommend memorizing these equations – I don't have them memorized myself. It is *very* easy to confuse the forms for source and receiver, and the derivations take a few seconds and are likely worth points in and of themselves. If you're going to memorize anything, memorize the *derivation* (a process I call "learning", as opposed to "memorizing"). In fact, this is excellent advice for 90% of the material you learn in this course!

Standing Waves in Pipes

Everybody has created a stationary resonant harmonic sound wave by whistling or blowing over a beer bottle or by swinging a garden hose or by playing the organ. In this section we will see how to compute the harmonics of a given (simple) pipe geometry for an imaginary organ pipe that is open or closed at one or both ends.

The way we proceed is straightforward. Air cannot penetrate a closed pipe end. The air molecules at the very end are therefore "fixed" – they cannot displace into the closed end. The *closed* end of the pipe is thus a *displacement node*. In order *not* to displace air the closed pipe end has to exert a force on the molecules by means of pressure, so that the closed end is a pressure antinode.

At an open pipe end the argument is inverted. The pipe is open to the air (at fixed background/equilibrium pressure) so that there must be a pressure node at the open end. Pressure and displacement are $\pi/2$ out of phase, so that the *open* end is also a *displacement antinode*.

Actually, the air pressure in the standing wave doesn't instantly equalize with the background pressure at an open end – it sort of "bulges" out of the pipe a bit. The displacement antinode is therefore just *outside* the pipe end, not *at* the pipe end. You may still draw a displacement antinode (or pressure node) as if they occur at the open pipe end; just remember that the distance from the open end to the first displacement node is *not* a very accurate measure of a quarter wavelength and that open organ pipes are a bit "longer" than they appear from the point of view of computing their resonant harmonics.

Once we understand the boundary conditions at the ends of the pipes, it is pretty easy to write down expressions for the standing waves and to deduce their harmonic frequencies.

Pipe Closed at Both Ends

There are displacement nodes at both ends. This is just like a string fixed at both ends:

$$s(x,t) = s_0 \sin(k_n x)\cos(\omega_n t) \tag{450}$$

which has a node at $x = 0$ for all k. To get a node at the other end, we require:

$$\sin(k_n L) = 0 \tag{451}$$

or

$$k_n L = n\pi \tag{452}$$

for $n = 1, 2, 3....$ This converts to:

$$\lambda_n = \frac{2L}{n} \tag{453}$$

and

$$f_n = \frac{v_a}{\lambda_n} = \frac{v_a n}{2L} \tag{454}$$

Pipe Closed at One End

There is a displacement node at the closed end, and an antinode at the open end. This is just like a string fixed at one end and free at the other. Let's arbitrarily make $x = 0$ the closed end. Then:

$$s(x,t) = s_0 \sin(k_n x)\cos(\omega_n t) \tag{455}$$

has a node at $x = 0$ for all k. To get an antinode at the other end, we require:

$$\sin(k_n L) = \pm 1 \tag{456}$$

or

$$k_n L = \frac{2n-2}{2}\pi \tag{457}$$

for $n = 1, 2, 3...$ (odd half-integral multiples of π. This converts to:

$$\lambda_n = \frac{2L}{2n-1} \tag{458}$$

and

$$f_n = \frac{v_a}{\lambda_n} = \frac{v_a(2n-1)}{4L} \tag{459}$$

Pipe Open at Both Ends

There are displacement antinodes at both ends. This is just like a string free at both ends. We could therefore proceed from

$$s(x,t) = s_0 \cos(k_n x)\cos(\omega_n t) \tag{460}$$

and

$$\cos(k_n L) = \pm 1 \tag{461}$$

but we could *also* remember that there are *pressure* nodes at both ends, which makes them like a string fixed at both ends again. Either way one will get the same frequencies one gets for the pipe closed at both ends above (as the cosine is ± 1 for $k_n L = n\pi$) but the picture of the nodes is still different – be sure to draw displacement antinodes at the open ends!

Beats

If you have ever played around with a guitar, you've probably noticed that if two strings are fingered to be the "same note" but are really slightly out of tune and are struck together, the resulting sound "beats" – it modulates up and down in intensity at a low frequency often in the ballpark of a few cycles per second.

Beats occur because of the superposition principle. We can add any two (or more) solutions to the wave equation and still get a solution to the wave equation, even if the solutions have different frequencies. Recall the identity:

$$\sin(A) + \sin(B) = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}) \tag{462}$$

If one adds two waves with different wave numbers/frequencies and uses this rule, one gets

$$\begin{aligned} s(x,t) &= s_0 \sin(k_0 x - \omega_0 t) + s_0 \sin(k_1 x - \omega_1 t) & (463) \\ &= 2 s_0 \sin(\frac{k_0 + k_1}{2} x - \frac{\omega_0 + \omega_1}{2} t)\cos(\frac{k_0 - k_1}{2} x - \frac{\omega_0 - \omega_1}{2} t) & (464) \end{aligned}$$

This describes a wave that has twice the maximum amplitude, the *average* frequency (the first term), and a second term that (at any point x) oscillates like $\cos(\frac{\Delta\omega t}{2})$.

The "frequency" of this second modulating term is $\frac{f_0 - f_1}{2}$, but the ear *cannot hear* the inversion of phase that occurs when it is negative and the difference is small. It just hears maximum amplitude in the rapidly oscillating *average* frequency part, which goes to *zero* when the slowing varying cosine does, twice per cycle. The ear then *hears* two beats per cycle, making the "beat frequency":

$$f_{\text{beat}} = \Delta f = |f_0 - f_1| \tag{465}$$

Interference and Sound Waves

We will not cover interference and diffraction of harmonic sound waves in this course. Beats are a common experience in sound as is the doppler shift, but sound wave interference is not so common an experience (although it can definitely and annoyingly occur if you hook up speakers in your stereo out of phase). Interference *will* be treated next semester in the context of coherent light waves. *Just* to give you a head start on that, we'll indicate the basic ideas underlying interference here.

Suppose you have two sources that are at the *same* frequency and have the *same* amplitude and phase but are at different locations. One source might be a distance x away from you and the other a distance $x + \Delta x$ away from you. The waves from these two sources add like:

$$\begin{aligned} s(x,t) &= s_0 \sin(kx - \omega t) + s_0 \sin(k(x + \Delta x) - \omega t) && (466) \\ &= 2s_0 \sin(k(x + \frac{\Delta x}{2} - \omega t) \cos(k\frac{\Delta x}{2}) && (467) \end{aligned}$$

The sine part describes a wave with twice the amplitude, the same frequency, but shifted slightly in phase by $k\Delta x/2$. The cosine part is *time independent* and *modulates* the first part. For some values of Δx it can *vanish*. For others it can have magnitude one.

The intensity of the wave is what our ears hear – they are insensitive to the phase (although certain echolocating species such as bats may be sensitive to phase information as well as frequency). The average intensity is proportional to the wave amplitude *squared*:

$$I_0 = \frac{1}{2}\rho\omega^2 s_0^2 v \tag{468}$$

With two sources (and a maximum amplitude of two) we get:

$$\begin{aligned} I &= \frac{1}{2}\rho\omega^2 (2^2 s_0^2 \cos^2(k\frac{\Delta x}{2}) v && (469) \\ &= 4I_0 \cos^2(k\frac{\Delta x}{2}) && (470) \end{aligned}$$

There are two cases of particular interest in this expression. When

$$\cos^2(k\frac{\Delta x}{2}) = 1 \tag{471}$$

one has *four times* the intensity of one source at peak. This occurs when:

$$k\frac{\Delta x}{2} = n\pi \tag{472}$$

(for $n = 0, 1, 2...$) or

$$\Delta x = n\lambda \tag{473}$$

If the *path difference* contains an *integral number of wavelengths* the waves from the two sources arrive *in phase*, add, and produce sound that has twice the amplitude and four times the intensity. This is called *complete constructive interference.*

On the other hand, when

$$\cos^2(k\frac{\Delta x}{2}) = 0 \tag{474}$$

the sound intensity *vanishes.* This is called *destructive* interference. This occurs when

$$k\frac{\Delta x}{2} = \frac{2n+1}{2}\pi \tag{475}$$

(for $n = 0, 1, 2...$) or

$$\Delta x = \frac{2n+1}{2}\lambda \tag{476}$$

If the path difference contains a *half integral* number of wavelengths, the waves from two sources arrive exactly out of phase, and *cancel.* The sound intensity vanishes.

You can see why this would make hooking your speakers up out of phase a bad idea. If you hook them up out of phase the waves *start* with a phase difference of π – one speaker is pushing out while the other is pulling in. If you sit equidistant from the two speakers and then harmonic waves with the same frequency from a single source coming from the two speakers *cancel* as they reach you (usually not perfectly) and the music sounds very odd indeed, because other parts of the music are not being played equally from the two speakers and don't cancel.

You can also see that there are many other situations where constructive or destructive interference can occur, both for sound waves and for other waves including water waves, light waves, even waves on strings. Our "standing wave solution" can be rederived as the superposition of a left- and right-travelling harmonic wave, for example. You can have interference from more than one source, it doesn't have to be just two.

This leads to some really excellent engineering. Ultrasonic probe arrays, radiotelescope arrays, sonar arrays, diffraction gratings, holograms, are all examples of interference being put to work. So it is worth it to learn the general idea as early as possible, even if it isn't assigned.

Homework for Week 12

Problem 1.

String one has mass μ and is at tension T and has a the travelling harmonic wave on it:

$$y_1(x, t) = A\sin(kx - \omega t)$$

Identical string two has the superposition of two harmonic travelling waves on it:

$$y_2(x,t) = A\sin(kx - \omega t) + 3A\sin(kx + \omega t)$$

If the energy density (total mechanical energy per unity length) is E_1, what is E_2 in terms of E_1?

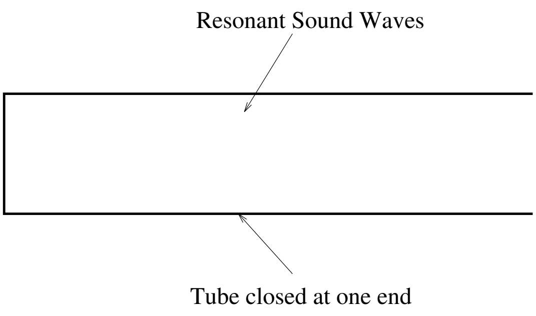
Problem 2.

Two identical strings of length L have mass μ and are fixed at both ends. One string has tension T. The other has tension $1.21T$. When plucked, the first string produces a tone at frequency f_1. What is the *beat* frequency produced if the second string is plucked at the same time, producing a tone f_2? Are the beats likely to be audible if f_1 is 500 Hz?

Problem 3.

You measure the intensity level of a single frequency sound wave produced by a loudspeaker with a calibrated microphone to be 80 dB. At that intensity, the *peak* pressure in the sound wave at the microphone is p_0. The loudspeaker's amplitude is turned up until the intensity level is 100 dB. What is the peak pressure of the sound wave now?

Problem 4.

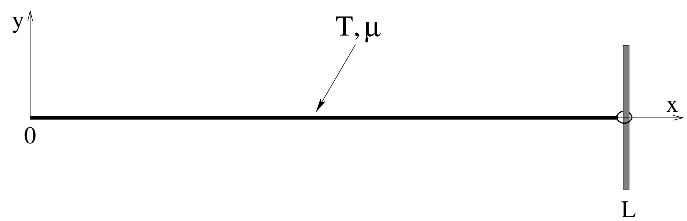


An organ pipe is made from a brass tube closed at one end as shown. The pipe is 3.4 meters long. (Assume that the speed of sound is 340 meters/second). When played it produces a sound that is a mixture of the first, third and sixth harmonic (mode).

a) What are the frequencies of these modes?

b) Qualitatively sketch the wave amplitudes for the first and the third harmonic modes (only) in on the figure, indicating the nodes and antinodes. Be sure to indicate whether the nodes or antinodes drawn are for **pressure/density** waves or **displacement** waves!

Problem 5.



A string of mass density μ is stretched to a tension T, fixed at $x = 0$, and free (frictionless loop) at $x = L$. The transverse string displacement is measured in the y direction. All answers should be given in terms of these quantities or new quantities you define in terms of these quantities.

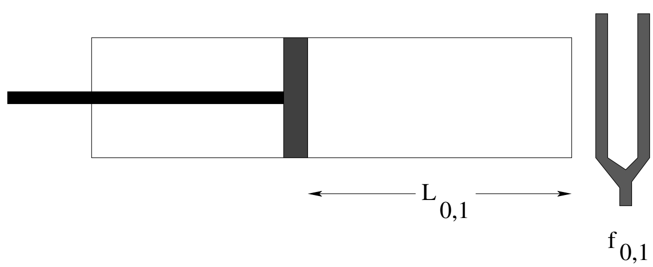
a) Derive the wave equation (the ordinary differential equation of motion) for waves on a string and identify the wave velocity squared in terms of T and μ.

b) Write down the equation $y_n(x, t)$ for a generic standing wave on this string with mode index n, assuming that the string is maximally displaced at $t = 0$. Verify that it is a solution to the ODE in a).

c) Find $k_n, \omega_n, f_n, \lambda_n$ for the first four modes supported by the string. Sketch them in on the axes below, labelling nodes and antinodes. Note that you should be able to draw the modes and find at least the wavelengths from the pictures alone.

Problem 6.

You crash land on a strange planet and all your apparatus for determining if the planet's



atmosphere is like Earth's is wrecked. In desperation you decide to measure the speed of sound in the atmosphere before taking off your helmet. You do have a barometer handy and can see that the air pressure outside is approximately one atmosphere and the temperature seems to be about 300 °K, so if the speed of sound is the same as on Earth the air *might* be breathable.

You jury rig a piston and cylinder arrangement like the one shown above and take out your two handy tuning forks, one at $f_0 = 3400$ Hz and one at $f_1 = 6800$ Hz (two frequencies that I'm sure you will note are "handy").

a) Using the 3400 Hz fork as shown, what do you *expect* (or rather, hope) to hear as you move the piston in and out (varying L_0). In particular, what are the shortest few values of L_0 for which you expect to hear a maximum resonant intensity from the tube if the speed of sound in the unknown atmosphere is indeed the same as in air (which you will cleverly note I'm *not* telling you:-)?

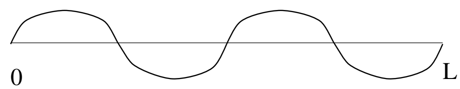
b) Using the 6800 Hz fork you hear your first maximum (for the smallest value of L_1) at $L_1 = 5$ cm. Should you sigh with relief and rip off your helmet?

c) What is the *next* value for L_1 for which you should hear a maximum (given the measurement in b) and what should the difference between the two equal in terms of the wavelength of the 6800 Hz wave in the unknown gas? Draw the *displacement* wave for this case only schematically in on the diagram above (assuming that the L shown is this second-smallest value of L_1 for the f_1 tuning fork), and indicate where the nodes and antinodes are.

Optional Problems

The following problems are **not required or to be handed in**, but are provided to give you some extra things to work on or test yourself with *after* mastering the required problems and concepts above and to prepare for quizzes and exams.

Energy in Standing Wave Mode



A string of total mass M and total length L is fixed at both ends, stretched so that the speed of waves on the string is v. It is plucked so that it harmonically vibrates in its $n = 4$ mode:

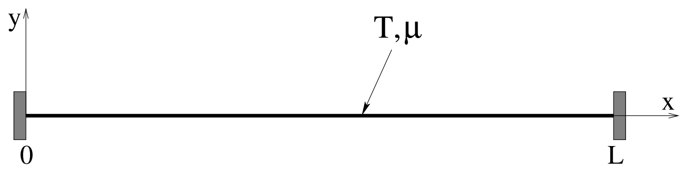
$$y(x,t) = A_4 sin(k_4 x) cos(\omega_4 t).$$

Find (derive) the instantaneous total kinetic energy in the string in terms of M, L, $n = 4$, v and A_4 (although it will simplify matters to use k_4 and ω_4 **once you define them**).

Remember (FYI):

$$\int_0^{n\pi} \sin^2(u) du = \int_0^{n\pi} \cos^2(u) du = \frac{n\pi}{2}$$

Standing Waves on String Fixed at Both Ends



A string of mass density μ is stretched to a tension T and fixed at $x = 0$ and $x = L$. The transverse string displacement is measured in the y direction. All answers should be given in terms of these quantities or new quantities you define in terms of these quantities.

1) Derive the wave equation (the ordinary differential equation of motion) for waves on a string and identify the wave velocity squared in terms of T and μ.

2) Write down the equation $y_n(x, t)$ for a generic standing wave on this string with mode index n, assuming that the string is maximally displaced at $t = 0$. Verify that it is a solution to the ODE in a).

3) Find $k_n, \omega_n, f_n, \lambda_n$ for the first four modes supported by the string. Sketch them in on the axes below, labelling nodes and antinodes. Note that you should be able to draw the modes and find at least the wavelengths from the pictures alone.


y
x
0
L