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1 Autumn tournament - 2023

Problem 1.1. *(8.1) Let a be the largest value of the expression $24y-9y^2$, where y is a rational number and b is the smallest integer satisfying the inequality*

$$(t+3)^3-(6t-7)^2-(t-9)^3<3.$$

Factor into irreducible factors with integer coefficients the expression

$$a(x-1)x^3+bx-2x-1.$$

(*Ivaylo Kortezov*)

Solution. We have $24y-9y^2=16-(3y-4)^2$ whose largest value $a=16$ is reached for $y=\frac{4}{3}$. The given inequality is equivalent to

$$t^3+9t^2+27t+27-36t^2+84t-49-t^3+27t^2-243t+729<3$$

$$-132t+704<0,$$

i.e. $t>\frac{16}{3}$ and $b=6$. Substituting $a=16$, $b=6$ in the given expression, we get

$$16(x-1)x^3+6x-2x-1=16x^4-16x^3+4x-1$$

$$=(16x^4-1)-4x(4x^2-1)=(4x^2-1)(4x^2+1)-4x(4x^2-1)$$
$$=(4x^2-4x+1)(2x-1)(2x+1)=(2x-1)^3(2x+1).$$

□

Problem 1.2. *(8.2) We will call a convex quadrilateral innovative if its diagonals divide it into four triangles with the same angle measures. For example, a square is an innovative quadrilateral because the four triangles have angles $90°$, $45°$, $45°$. To find the measures of the angles of an innovative quadrilateral if one of them is $13°$.*

(*Miroslav Marinov*)

Solution. Let the quadrilateral be $ABCD$ with $\angle BAD=13°$ and the diagonals AC and BD intersect at O. If we assume that the diagonals are not perpendicular, then for $\angle AOB>90°$ (the case $\angle AOD>90°$ is analogous) we have $\angle AOB>90\ ^{circ}>\angle AOD$ and $\angle AOB>\angle OAD$, $\angle AOB>\angle ODA$ (since $\angle AOB$ is an exterior angle for triangle AOD), i.e. the measure of $\angle AOB$ does not occur in triangle AOD, a contradiction. Thus AC and BD are perpendicular.

Furthermore, if $\angle BAO=\angle ADO$, then $\angle BAD=\angle BAO+\angle OAD=\angle BAO+90°-\angle ADO=90°$, a contradiction with $\angle BAD=13°$. This leaves only the possibility $\angle BAO=\angle DAO$, in which case AC bisects $\angle BAD$, i.e. AC is the bisector of BD. Now from the triangles AOD and DOC it follows either $\angle ADO=\angle CDO$ (in which case BD is the bisector of AC and $ABCD$ is a rhombus), or $\angle ADO=\angle DCO=90°-\angle CDO$, i.e. $\angle ADC=90°$, analogously $\angle ABC=90°$ and the fourth angle is $\angle BCD=180°-\angle BAD$. □

Problem 1.3. *(8.3) Find all pairs (a, b) of co-prime naturals such that $a < b$ and b divide*

$$(n+2)a^{n+1002} - (n+1)a^{n+1001} - na^{n+1000}$$

for every natural number n.

(*Miroslav Marinov*)

Solution. Since a and b are co-prime, so are b and a^{n+1000}, respectively, what is requested is equivalent to b dividing $(n+2)a^2 - (n+1)a - n$ for each n. From $n = 1$ and $n = 2$ we get that necessarily b divides $3a^2 - 2a - 1$ and $4a^2 - 3a - 2$. Hence b divides

$$4(3a^2 - 2a - 1) - 3(4a^2 - 3a - 2) = a + 2$$

and since $3a^2 - 2a - 1 = 3(a-2)(a+2) - 2(a+2) + 15$, then necessarily b divides 15.

Clearly $b > a \geq 1$, i.e. $b \geq 2$. If $b = 15$, then $b \mid a + 2$ gives $a = 13$, but $4 \cdot 13^2 - 3 \cdot 13 - 2 = 635$ is not divisible by 3. If $b = 3$, then $a = 1$, but $4 \cdot 1^2 - 3 \cdot 1 - 2 = -1$ is not divisible by 3.

It remains $b = 5$, respectively $a = 3$. Indeed, $(n+2) \cdot 3^2 - (n+1) \cdot 3^1 - n = 5(n+3)$ is divisible by 5. $\square$

Problem 1.4. *(8.4) An integer is written in each of the fields of a 9×9 square table. For every k numbers in the same row (column), their sum is in the same row (column). Find the smallest possible number of zeros in the table if:*
a) $k = 5$;
b) $k = 8$.

(*Ivaylo Kortezov*)

Solution. a) Example: we number the rows and columns from 1 to 9. We write 1 in the fields $(i; i)$ $(i = 1, \ldots, 9)$; -1 in field $(1; 9)$ and in fields $(i; i-1)$ $(i = 2, \ldots, 9)$; 0 in other fields. Possible sums are 1, 0, and -1.

Evaluation: Suppose there are at least 19 non-zero numbers. From Dirichlet's principle, there will be at least three non-zero numbers on any row, and therefore at least two non-zero numbers with the same sign, let's say positive ones (the situation with negative ones is analogous). Let's arrange the numbers in this order by size: $a_1 \leq a_2 \leq \ldots \leq a_9$, where $a_9 \geq a_8 > 0$. If $a_5 \geq 0$, then $a_5 + a_6 + a_7 + a_8 + a_9 \geq a_8 + a_9 > a_9$ must be of the same order, a contradiction. If $a_5 < 0$, then $a_1 + a_2 + a_3 + a_4 + a_5 < a_1$ must be of the same order, a contradiction.

b) A possible example without zeros is as follows (works because $5.3 + 3.(-4) = 3$ and $4.3 + 4.(-4) = -4$):

3	3	3	3	3	-4	-4	-4	-4
-4	3	3	3	3	3	-4	-4	-4
-4	-4	3	3	3	3	3	-4	-4
-4	-4	-4	3	3	3	3	3	-4
-4	-4	-4	-4	3	3	3	3	3
3	-4	-4	-4	-4	3	3	3	3
3	3	-4	-4	-4	-4	3	3	3
3	3	3	-4	-4	-4	-4	3	3
3	3	3	3	-4	-4	-4	-4	3
3	3	3	3	3	-4	-4	-4	-4

$\square$

Problem 1.5. *(9.1) Given the functions $f(x) = |x - 2| - |x - 4|$ and $g(x) = |x - 8| - 2$. Find the area of the figure with vertices, the intersection points of the graphs of the functions $f(x)$ and $g(x)$ and the intersection points of the graph of $g(x)$ with the x-axis.*

(*Nedyalka Dimitrova*)

Solution. The graph of $g(x)$ consists of two rays with a common vertex at $x = 8$. We remove the module and easily calculate its intersection points with the x-axis through the equations $6 - x = 0$ and $x - 10 = 0$ - $A(6, 0)$ and $B(10, 0)$.

After removing the modules in $f(x)$ we see that in the interval $(-\infty, 2)$ $f(x) = -2$, in the interval $[2, 4]$ $f(x) = 2x - 6$ and in the interval $(4, \infty)$ $f(x) = 2$. We solve the equations $f(x) = g(x)$ for each of the three intervals. We get the following intersection points - $D(4, 2)$ and $C(12, 2)$. It follows that the figure $ABCD$ is a trapezoid with bases 8 and 4 and height 2. Therefore, its area is $\frac{8+4}{2} = 12$. $\square$

Problem 1.6. *(9.2) Given is an obtuse isosceles triangle ABC with $CA = CB$ and circumcenter O. The point P on AB is such that $AP < \frac{AB}{2}$ and Q on AB is such that $BQ = AP$. The circle with diameter CQ meets (ABC) at E and the lines CE, AB meet at F. If N is the midpoint of CP and ON, AB meet at D, show that $ODCF$ is cyclic.*

(*Alexander Ivanov*)

Solution. Let T be the midpoint of CQ (it is anyway the center of the important circle with diameter CQ). Then OC is the perpendicular bisector of AB (as $AC = BC$), triangle ONT is isosceles by symmetry (as P and Q are symmetric with respect to the midpoint M of AB and hence with respect to CO, by the problem condition) and OT is the perpendicular bisector of CE, so $\angle OCF = 90^\circ - \angle OCT = 90^\circ - \angle OCD = \angle ODM = \angle ODF$, done! $\square$

Problem 1.7. *(9.3) In the house of the wealthy Lady Gilmore, one of her most precious possessions has been stolen: her pearl necklace. The task of unraveling the mystery fell on the shoulders of Inspector Goodenough. He had the following* *information: on the day of the theft, 7 of Lady Gilmore's servants, whom we will refer to as A, B, C, D, E, F, G for the confidentiality of the investigation, entered the room with the necklace. Each claimed to have been in the room only once for an unspecified period of time. Additionally, A claims to have met B, C, F, G in the room; B claims to have met A, C, D, E, F; C claims to have met A, B, E; E claims to have met B, C, F; F claims to have met A, B, D, E; G claims to have met A, D and D claims to have met B, F, G. Inspector Goodenough concluded that one of the servants was lying. Who is he?*

(*Lyuben Lichev*)

Solution. We will first prove the following lemma.

Lemma. Let X, Y, Z and T be four of the servants. If it is known that the pairs X, Y; Y, Z; Z, T and T, X were in the room together at some point, then one of the pairs X, Z and Y, T also detected each other.

Proof of Lemma. Let us assume, without community restriction, that Y and T were not in the room together, and Y left the room before T (the other cases are analogous). Then, X and Z were in the room together in the period between Y's departure and T's arrival.

Notice that A, C, E, F satisfy the condition of the lemma, but none of A, E and C, F intersect. The same goes for A, B, D, G. The only common element of these pairs is A. It remains to be ascertained that it is possible that all the other pairs met, as they claim, by entering exactly once. This is possible with the following sequence of entries and exits: enter G, enter D, exit G, enter B, enter F, exit D, enter E, exit F , C enters, B exits, E exits, C exits. $\square$

Problem 1.8. *(9.4) Let p, q be coprime integers, such that $|\frac{p}{q}| \leq 1$. For which p, q, there exist even integers $b_1, b_2, \ldots, b_n$, such that*

$$\frac{p}{q} = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \ldots}}}?$$

(*Mihail Shkolnikov*)

Solution. Set $b_i = 2\ell_i$, all of them are even. At the first step we get the pairs

$$A := \{(p, q) : (p, q) = (1, 2\ell_1), \ell_1 \in \mathbb{Z}. \qquad (1).$$

At each subsequent step we expand the set A by adding additional pairs (p', q') defined as

$$\left\{(p', q') : \frac{p'}{q'} = \frac{1}{2\ell + \frac{p}{q}}, (p, q) \in A, \ell \in \mathbb{Z}\right\} \qquad (2).$$

By (2) we obtain

$$p' = q\,;\; q' = 2\ell q + p \quad (3)$$

So, if (p, q) is obtained in the process of expanding, we also add (p', q') defined as in (3), We want to characterize all pairs (p, q) that can be obtained in this way. Note that if $(p, q) = 1$, then from (3) follows $(p', q') = 1$. Let us prove that the set of pairs (p, q) in question is

$$A := \{(p, q) : p, q \in \mathbb{Z} \setminus \{0\}, |p| < |q|, (p, q) = 1, \text{ one of } p, q \text{ is even, the other is odd.}\}$$

Note that we ruled out $|p| = |q|$ because $\frac{p}{q} = \pm 1$ cannot be represented as wanted. The transformation described in (3) shows that we cannot step outside A. To prove that all pairs in A can be generated, we follow a standard procedure - take $(p', q') \in A$ and search for $(p, q) \in A$ that generates (p', q') via (3), but we want (p, q) be "less"than (p', q'). That's how induction works. Solving (3) with respect to (p, q) yields

$$p = q' - 2\ell p' \, ; \, q = p' \qquad (4)$$

Clearly, we can choose $\ell \in \mathbb{Z}$ such that $|q' - 2\ell p'| < |p'|$ (since $p' \neq 0$). Indeed, consider the points $q' - 2\ell p'$ where ℓ runs through all integers. These points are at a distance $2p'$ apart, and none of them hits p' because $(p', q') = 1$. Hence, the point closest to 0 has magnitude less than p'. So, we started from $(p', q') \in A$, and found $(p, q) \in A$ that generates (p', q') and moreover, $q = p', |p| < |p'| < |q'|$. Apparently (p, q) is "less"than (p', q'), that is, $|p| < |p'|, |q| < |q'|$. Now, induction does the job. Obviously, $(1, 2), (-1, 2), (1, -2), (-1, -2)$ can be represented as continued fractions satisfying the statement. Assume that all $(p, q) \in A$ with $|p| \leq N, |q| \leq N$ can be represented like that. Take $(p', q') \in A, |q'| = N + 1$. As just shown, there exists $(p, q) \in A, |p| < |q| \leq N$ such that

$$\frac{p'}{q'} = \frac{1}{2\ell + \frac{p}{q}}.$$

The induction step is complete and the result follows. Note that it also follows that the representation of numbers in A as continued fractions like that is unique. Indeed, take a pair $(p', q') \in A$. The pair $(p, q) \in A$ that satisfies (4) and $|p| < |p'|$ is determined uniquely. Uniqueness follows easily.

Remark. The main difficulty in my opinion is that the the answer wasn't told to the students. The result could have been much better otherwise. Some motivation about guessing the answer can be found in this blog $\square$

Problem 1.9. *(10.1) Solve the equation*

$$(x + 1)\sqrt{x^2 + 2x + 2} + x\sqrt{x^2 + 1} = 0.$$

(*Nedyalka Dimitrova*)

Solution. Let $a = \sqrt{x^2 + 2x + 2} > 0$ and $b = \sqrt{x^2 + 1} > 0$. Therefore

$$x = \frac{(x^2 + 2x + 2) - (x^2 + 1) - 1}{2} = \frac{a^2 - b^2 - 1}{2}$$

$$x + 1 = \frac{a^2 - b^2 + 1}{2}.$$

The equation is equivalent to:

$$\begin{aligned}\frac{a^2-b^2+1}{2}\cdot a+\frac{a^2-b^2-1}{2}\cdot b=0\\(a^2-b^2)a+a+(a^2-b^2)b-b=0\\(a-b)\left((a+b)^2+1\right)=0.\end{aligned}$$

Hence, $a=b$ and $x=-\frac{1}{2}$. $\square$

Problem 1.10. *(10.2) Given is an acute triangle ABC with circumcenter O. The point P on BC such that $BP<\frac{BC}{2}$ and the point Q is on BC, such that $CQ=BP$. The line AO meets BC at D and N is the midpoint of AP. The circumcircle of (ODQ) meets (BOC) at E. The lines NO,OE meet BC at K,F. Show that $AOKF$ is cyclic.*

(*Alexander Ivanov*)

Solution. Let A' be the antipode of A and let $AO\cap(BOC)=R$. Since $NO\parallel PA'$, by Reim's theorem we need $APA'F$ being cyclic, or $DP\cdot DF=DA\cdot DA'=DB\cdot DC=DO\cdot DR$, so we need $OPRF$ being cyclic, or that $\angle OFP=\angle ORP$. By shooting lemma, $RDEF$ is cyclic, so $\angle OFP=\angle ERD$, so we need AR being the angle bisector of $\angle PRE$. Since AR bisects $\angle BRC$, it is sufficient to show that $\angle BRP=\angle CRE$.

To use that $EQDO$ is cyclic, let $EQ\cap(BOC)=S$; Reim's implies $RS\parallel BC$, which together with the length condition gives that $RSQP$ is an isosceles trapezoid. Hence, $\angle CRE=\angle CSQ=\angle BRP$, which finishes the problem.

Remark. One can show that AE and NO still meet on (BOC) even when P is not the foot of the perpendicular from A - just invert about (ABC) and if $NO\cap(BOC)=T$, then $(AOKF)$ goes to the line A,E,T! So, the current problem is the main component of the proof for the generalization of the IGO problem. $\square$

Problem 1.11. *(10.3) Find all positive integers k, so that there exists a polynomial $f(x)$ with rational coefficients, such that for all sufficiently large n,*

$$f(n)=\operatorname{lcm}(n+1,n+2,\dots,n+k).$$

(*Alexander Ivanov*)

Solution. For $k=1$ and $k=2$, the required polynomials are $f(x)=x+1$ and $f(x)=(x+1)(x+2)$, respectively. Let $k\geq 3$ and assume that such a polynomial $f(x)$ exists. For any prime number p, its degree in $\operatorname{lcm}(n+1,n+2,\dots,n+k)$ is $\max\{\alpha_1,\alpha_2,\dots,\ alpha_k\}$, where α_i is the power of p in the canonical representation of $n+i$, $i=1,\dots,k$. If this is, for example, α_s, then it would be obtained if we take

$$\frac{(n+1)(n+2)\cdots(n+k)}{p^{\alpha_1}p^{\alpha_2}\cdots p^{\alpha_{s-1}}p^{\alpha_{s+1}}\cdots p^{\alpha_k}},$$

it being clear that the powers of p in the denominator are divisors of $\prod\limits_{1\le i\ne s\le k}(s-i)$. Therefore

$$\operatorname{lcm}(n+1,n+2,\dots,n+k)=\frac{(n+1)(n+2)\cdots(n+k)}{C_n}, \qquad (1)$$

where C_n is a divisor of $\prod\limits_{1\le i<j\le k}(j-i)$. Since C_n can take a finite number of possible values, there will be a natural number C such that for infinitely many n, $f(n)=\frac{(n+1)(n+2)\cdots(n+k)}{C}$. So for infinitely many x, $f(x)=\frac{(x+1)(x+2)\cdots(x+k)}{C}$, whence

$$f(x)\equiv\frac{(x+1)(x+2)\cdots(x+k)}{C}, \quad \forall x\in\mathbb{R}.$$

Therefore

$$\operatorname{lcm}(n+1,n+2,\dots,n+k)=\frac{(n+1)(n+2)\cdots(n+k)}{C}, \quad \text{for all } n\in\mathbb{N}.$$

Let's assume this is possible. Let's choose a prime $p<k$ such that p does not divide k. Let $n+k+1=p^m$ for sufficiently large m. From the above formula we have

$$\frac{\operatorname{lcm}(n+2,n+3,\dots,n+k+1)}{\operatorname{lcm}(n+1,n+2,\dots,n+k)}=\frac{n+k+1}{n+1}. \qquad (2)$$

The degree of p in the numerator of the left side is m and in the denominator – at least 1, while the degree of p in the numerator of the right side is m and in the denominator – 0. We derive a contradiction! Therefore, the assumption is wrong and for $k\ge 3$ there does not exist a polynomial with the desired property. □

Problem 1.12. *(10.4) In every cell of a board* 101×101 *is written a positive integer. For any choice of* 101 *cells from different rows and columns, their sum is divisible by* 101*. Show that the number of ways to choose a cell from each row of the board, so that the total sum of the numbers in the chosen cells is divisible by* 101*, is divisible by* 101*.*

(*Borislav Kirilov*)

Solution. (Vlad Spataru) Index the rows and columns from 0 to 100. We shall only work modulo 101 in what follows. Observe that we may let $(0,0)=0$ by adding some constant to all the terms of the table. Next, because of the condition in the statement,

$$(u,v)+(i,j)=(u,j)+(i,v),$$

for any u,v,i,j. Thus, there exist $a_0=0,a_1,\dots,a_{100}$ and $b_0=0,b_1,\dots,b_{100}$ so that $(i,j)=a_i+b_j$. Now, letting Σ_a and Σ_b be the sum of the a_i and the b_i resepctively, the condition in the statement then yields $\Sigma_a+\Sigma_b=0$. Note that

$$\sum_{i=0}^{100}x^{101b_i}=\left(\sum_{i=1}^{100}x^{b_i}\right)^{101}:=\sum_{i=0}^{\infty}x^i\alpha(i).$$

Observe that the number of ways of adequately choosing one cell from each row corresponds to the number of 101-tuples with values from $\{b_0, b_1, \ldots, b_{100}\}$ with sum equal to Σ_b that is $\alpha(\Sigma_b) + \alpha(101 + \Sigma_b) + \cdots$.

Using the fact that 101 is prime and computing the latter expression for $x = \exp(2\pi i/101)$ we get the desired

$$0 = \sum_{i=0}^{\infty} \alpha\left(101i + \Sigma_b\right).$$

□

Problem 1.13. *(11.1) A quadruplet of distinct positive integers (a, b, c, d) is called k-good if the following conditions hold:*

1. *Among a, b, c, d, no three form an arithmetic progression.*
2. *Among $a+b, a+c, a+d, b+c, b+d, c+d$, there are k of them, forming an arithmetic progression.*

a) Find a 4-good quadruplet.
b) What is the maximal k, such that there is a k-good quadruplet?

(*Emil Kolev*)

Solution. a) The quadruple $(7, 6, 4, 3)$ is nice because there are no three numbers in it to form an arithmetic progression, but of the six numbers

$$7 + 6 = 13, 7 + 4 = 11, 7 + 3 = 10, 6 + 4 = 10, 6 + 3 = 9, 4 + 3 = 7$$

the numbers $7, 9, 11, 13$ form an arithmetic progression.

b) Without restriction let $a > b > c > d$. Then

$$a + b > a + c > \max(a + d, b + c) > \min(a + d, b + c) > b + d > c + d.$$

Note that if:

1) $a + b$, $a + c$ and $a + d$ form an arithmetic progression, then $2(a + c) = (a + b) + (a + d) \iff 2c = b + d$;

2) $a + b$, $a + c$ and $b + c$ form an arithmetic progression, then $2(a + c) = (a + b) + (b + c) \iff 2b = a + c$;

3) $a + d$, $b + d$ and $c + d$ form an arithmetic progression, then $2(b + d) = (a + d) + (c + d) \iff 2b = a + c$;

4) $b + c$, $b + d$ and $c + d$ form an arithmetic progression, then $2(b + d) = (b + c) + (c + d) \iff 2c = b + d$.

In all four cases we get a contradiction with the given condition.

From the above it is clear that all 6 numbers cannot form an arithmetic progression.

Let's assume that 5 of them form an arithmetic progression. Notice that whichever of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$ and $c+d$ we delete, there is always some from progressions 1., 2., 3., or 4., contradiction.

It follows from a) that the required k is 4. $\square$

Problem 1.14. *(11.2) The points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC so that $BA_1 = BC_1$ and $CA_1 = CB_1$. The lines C_1A_1 and A_1B_1 meet the line through A, parallel to BC, at P, Q. Let the circumcircles of the triangles APC_1 and AQB_1 meet at R. Given that R lies on AA_1, show that R lies on the incircle of ABC.*

(*Emil Kolev*)

Solution. Observe that (A_1, C_1, R, B_1) are concyclic. Let I be the incenter of triangle ABC. Then BI and CI perpendicularly bisects C_1A_1 and A_1B_1 respectively. Hence, I is the circumcenter of $\triangle A_1B_1C_1$. Let $\angle ACB = 2\gamma$. $\angle A_1IB_1 = 2\angle B_1C_1A_1 = 2\angle A_1RB_1 = 2\angle AQB_1 = 2\angle B_1A_1C = 180-2\gamma$. Therefore (I, A_1, B_1, C) concyclic$\rightarrow$ $IA_1 \perp BC$, $IB_1 \perp AC$ Similarly $IC_1 \perp AB$, so circle (A_1, C_1, R, B_1) is the incircle of ABC. Thus, R lies on the incircle of ABC. $\square$

Problem 1.15. *(11.3) Find the smallest possible number of divisors a positive integer n may have, which satisfies the following conditions:*

1. $24 \mid n+1$;
2. *The sum of the squares of all divisors of n is divisible by 48 (1 and n are included).*

(*Adelina Chopanova*)

Problem 1.16. *(11.4) Let G be a complete bipartite graph with partition sets A and B of sizes km and kn, respectively. The edges of G are colored in k colors. Prove that there exists a monochromatic connected component with at least $m+n$ vertices (which means that there exists a color and a set of vertices, such that between any two of them, there is a path consisting of edges only in that color).*

(Dragomir Grozev)

Solution. There are at least kmn edges colored in the most used color. We delete the remaining edges and prove that there exists a connected component with at least $m+n$ vertices in the remaining graph. The idea is to consider all the connected components. If each one has less than $m+n$ vertices, then the graph is too fragmented and it's impossible to have too many (at least kmn) edges even if all edges are present in every connected component. It boils down to prove a certain inequality.

Lemma. Let x, y be two positive real numbers and $\ell, k; \ell \geq k$ be natural numbers. Let $x_i, y_i, i = 1, 2, \dots, \ell$ satisfy the following conditions.

$$\begin{aligned} &x_i \geq 0, y_i \geq 0, x_i + y_i \leq (x+y)/k, i = 1, 2, \dots, \ell; \\ &\sum_{i=1}^{\ell} x_i = x\,, \sum_{i=1}^{\ell} y_i = y. \qquad (1) \end{aligned}$$

Then it holds

$$\sum_{i=1}^{\ell} x_i y_i \leq \frac{xy}{k}.$$

The equality is reached only if $x_i = x/k, y_i = y/k, i = 1, 2, \dots, k; x_i = y_i = 0, i > k$.

Proof. Let us denote $f(x, y) := \sum_{i=1}^{\ell} x_i y_i$, where $x = (x_1, \dots, x_\ell), y = (y_1, \dots, y_\ell)$. The conditions in (1) determine a compact set, hence f attains its maximum value on it, say, at the points $x'_i, y'_i, i = 1, 2, \dots, \ell$. We can assume $x'_1 \geq x'_2 \geq \dots \geq x'_\ell$. We shall prove that y'_i are also in decreasing order. Let us assume on the contrary that $y'_i < y'_{i+1}$. Set

$$x_i = x_{i+1} := (x'_i + x'_{i+1})/2; y_i = y_{i+1} := (y'_i + y'_{i+1})/2.$$

Then (Chebyshev inequality)

$$x_i y_i + x_{i+1} y_{i+1} > x'_i y'_i + x'_{i+1} y'_{i+1}.$$

But it contradicts the maximality of $x'_i, y'_i, i = 1, \dots, \ell$. Next, if $x'_1 + y'_1 < (x+y)/k$ we can similarly set $x_1 := x'_1 + \varepsilon, x_2 := x'_2 - \varepsilon; y_1 := y'_1 + \delta, x_2 := x'_2 - \delta$ for sufficiently small $\varepsilon, \delta \geq 0$ and get a larger value of f. Therefore, $x'_1 + y'_1 = (x+y)/k$.

Let k' be the largest index for which $x_{k'} > 0$ and $y_{k'} > 0$. In the same way we can see that $x'_i + y'_i = (x+y)/k, i = 1, 2, \dots, k'$. Hence, $k' \leq k$. Assume that $k' < k$ and $y_i = 0, i > k'$. We modify x', y' as follows. Set

$$\begin{aligned} &x_i := x'_i, y_i := y'_i, i = 1, 2, \dots, k' - 1; \\ &x_{k'} := x'_{k'}, y_{k'} := y'_{k'} - \varepsilon, x_{k'+1} := (x+y)/k, y_{k'+1} := \varepsilon. \end{aligned}$$

For $i > k' + 1$ we set $y_i = 0$, and the values of $x_i, i > k' + 1$ are irrelevant providing they comply with (1). Since $x'_k < (x+y)/k$, it can be seen that $f(x, y) > f(x', y')$ which contradicts the maximality of x', y'.

Thus, $k' = k$. We prove that $x'_i = x/k, y'_i = y/k, i = 1, 2, \dots, k$. Assume on the contrary it doesn't hold and let j be the first index for which $x'_j \neq x/k$. WLOG let $x'_j < x/k$. Then there exists $i > j$ for which $x'_i > x/k$. This means $x'_i > x'_j$ and thus the sequence $x'_1, x'_2, \dots, x'_k$ is not decreasing, contradiction. To recap, we established that $x'_i = x/k, y'_i = y/k, i = 1, 2, \dots, k$. In this case $f(x', y') = kxy$, and Lemma 1 is proved. $\square$

Back to the problem. The number of all edges of K is k^2mn, hence there is a color, say, white such that at least kmn edges are colored white. We delete all edges colored in a color other than white. We'll prove that in the remaining graph K', there exists a connected component with at least $m+n$ vertices. Assume on the contrary it is not true. Denote the connected components of K' by $G(A_i, B_i), i = 1, 2, \ldots, \ell$ and let $|A_i| = m_i, |B_i| = n_i, i = 1, 2, \ldots, \ell$. We have

$$\sum i = 1^{\ell} m_i = km, \sum_{i=1}^{\ell} n_i = kn, m_i + n_i < m + n.$$

According to Lemma 1,

$$\sum_{i=1}^{\ell} m_i n_i < kmn$$

which contradicts the choice of the white color. This means that for at least one index i it holds $m_i + n_i \geq m + n$.

Remark. More comments can be found in this blog. This problem allows a generalization as described here. □

Problem 1.17. *(12.1) Let $x_0, x_1, \ldots$ be a sequence of real numbers such that $x_0 = 1$ and $x_{n+1} = \sin(x_n) + \frac{\pi}{2} - 1$ for all $n \geq 0$. Show that the sequence converges and find its limit.*

(*Kristyan Vasilev*)

Solution. We will first prove that the sequence is strictly increasing. Note that for each n we have $x_n \leq \frac{\pi}{2}$, since $\sin x \leq 1$ for each $x \in \mathbb{R}$. Besides we have that the function $f(x) = \sin x - x$ is decreasing for $x \in \mathbb{R}$, because $f'(x) = \cos x - 1 \leq 0$ for each $x \in \mathbb{R}$.

Therefore, for $x \in (-\infty, \frac{\pi}{2})$ we have $f(x) \geq \sin\frac{\pi}{2} - \frac{\pi}{2}$. So we get that $\sin x_n \geq x_n + 1 - \frac{\pi}{2}$, which is equivalent to $x_{n+1} \geq x_n$. Therefore, the series $(x_n)_{n=1}^{\infty}$ is increasing and since it is bounded it follows that it is convergent. If l is its limit, then for l it is true that $l = \sin l + \frac{\pi}{2} - 1$, i.e. $f(l) = f(\frac{\pi}{2})$. The function f is decreasing, which means that $l = \frac{\pi}{2}$. □

Problem 1.18. *(12.2) Given is an acute triangle ABC with incenter I and the incircle touches BC, CA, AB at D, E, F. The circle with center C and radius CE meets EF for the second time at K. If X is the C-excircle touchpoint with AB, show that CX, KD, IF concur.*

(*Kristyan Vasilev*)

Solution. We claim the concurrency point is the F-antipode F'. It is well-known that this is $\overline{CX} \cap \overline{IF}$.

Let $P = \overline{EF} \cap \overline{DF'}$ and $Q = \overline{DF} \cap \overline{EF'}$. Then since $\angle PEQ = \angle PDQ = 90°$, $DEPQ$ is cyclic.

Now, we have

$$\begin{aligned}\angle EFD &= \angle EFF' = 90^\circ - \angle PF'E = 90^\circ - \angle DFE \\ &= -90^\circ + \left(90^\circ - \frac{\angle A}{2}\right) + \left(90^\circ - \frac{\angle B}{2}\right) = \frac{\angle C}{2},\end{aligned}$$

so the center of $(DEPQ)$ lies on (CDE) (and is on the same side of $\overline{DE}$ as C). On the other hand, it also lies on the perpendicular bisector of $\overline{DE}$, so it must be C itself. This gives us $P = K$, and the conclusion follows. $\square$

Problem 1.19. *(12.3) Solve in positive integers the equation*

$$m^{\frac{1}{n}} + n^{\frac{1}{m}} = 2 + \frac{2}{mn(m+n)^{\frac{1}{m}+\frac{1}{n}}}.$$

(Kristyan Vasilev)

Solution. (Marin Hristov) There are no solutions in positive integers. Without loss of generality, assume that $m \geq n$. If $n \geq 3$, then

$$\left(1 + \frac{1}{m}\right)^m = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{m^k} < \sum_{k=0}^{m} \frac{1}{k!} < \sum_{k=0}^{\infty} \frac{1}{k!} = e < n \Longrightarrow n^{\frac{1}{m}} > 1 + \frac{1}{m}$$

Similarly, $m^{\frac{1}{n}} > 1 + \frac{1}{n}$, whence

$$m^{\frac{1}{n}} + n^{\frac{1}{m}} > 2 + \frac{1}{m} + \frac{1}{n} \geq 2 + \frac{2}{mn} > 2 + \frac{2}{mn(m+n)^{\frac{1}{m}+\frac{1}{n}}}.$$

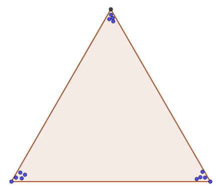
The cases $n = 1, 2$ can be solved directly. $\square$

Problem 1.20. *(12.4) A set of points in the plane is called good if the distance between any two points in it is at most 1. Let $f(n, d)$ be the largest positive integer such that in any good set of $3n$ points, there is a circle of diameter d, which contains at least $f(n, d)$ points. Prove that there exists a positive real ϵ, such that for all $d \in (1 - \epsilon, 1)$, the value of $f(n, d)$ does not depend on d and find that value as a function of n.*

(Kristiyan Vasilev, Konstantin Garov)

Solution. Since $m(n, d)$ is an increasing function of d and it cannot be larger than $3n$, clearly it hits a constant when d approaches 1. Fix some $d, d < 1$, it may be very close to 1. Take an equilateral triangle with side length 1 and put inside it $3n$ points - n points near each of its vertices so that they be at distance less than $(1 - d)/2$ from the corresponding vertex - fig 1. This set of points apparently is a good set. A disk with radius d cannot cover a point that is near to one vertex and a point near another vertex. That is, the maximum number of points that could be covered with such a disk is n. This means that $m(n, d) \leq n$ no matter how close to 1 the value of d is.

Фигура 1



We will prove that $m(n,d) = n$ when d is close to 1. To prove this we need to study the good set of points. How close together are its points? Can we claim that all of them lie in disk with some not too large radius? Of course, they lie in a disk with radius 1 since if you take one of the points, the rest are at distance at most 1 from this point. But can we claim that all of them lie in a disk with a smaller radius? Clearly, they may not lie in a disk with radius 1/2 - the equilateral triangle with side length 1 is an example that disproves this. Crucial to this problem is the following observation.

Lemma. Any good set can be put inside a disk with radius $\dfrac{1}{\sqrt{3}}$. That is, any good set can be put inside a circle circumscribed about an equilateral triangle with a side length 1.

The official solution proves it by using Helly's theorem which is instructive. Another method is used here.

Proof. Let X be a good set. We first show that X lies in a circle circumscribed about an acute triangle with side lengths less or equal to 1. Take the convex hull of X and put it inside a circle with some radius. We begin to tighten this circle until it touches 1 and then 2 points of the convex hull. We can tighten the circle further unless these two points lie on its diameter. In this case A lies in a circle with a diameter at most 1. So, assume we decrease the circle k until it touches 3 points of X, say A, B, C. If these points are vertices of an acute triangle we done, so suppose $\angle ACB \geq 90°$ see fig.2. We begin to further decrease the diameter of k keeping the points A and B on the circle. This can be done until the first of the following two events happens:

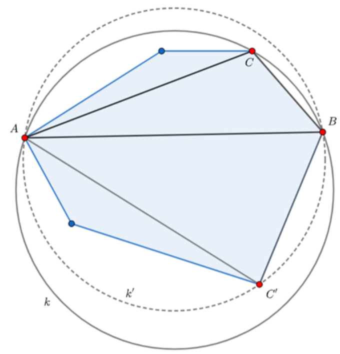
1. The circle hits a point $C' \in X$ (and C' and C lie on different sides of AB).
2. AB becomes a diameter of k.

If the second event happens first, we are done since X lies in a disk with diameter 1. In case of the first event, the triangle ABC' is acute. Next, since $\triangle ABC'$ is acute one of its angles, say $\angle AC'B$, is greater or equal to $60°$. It follows

the radius of $k' = \dfrac{AB}{2\sin\angle C} \leq \dfrac{1}{\sqrt{3}}$.

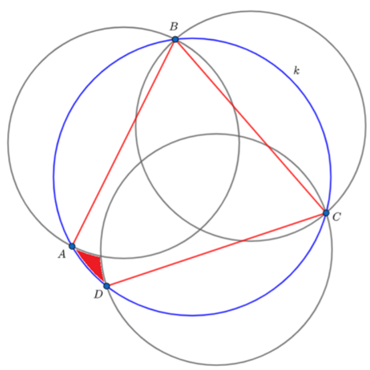
So, we proved that we can cover X with a circle k with radius $1/\sqrt{3}$. $\square$

Фигура 2



Now, let X be a good set. We cover it with a disk k of radius $1/\sqrt{3}$ - the blues circle on fig 3. We prove that if d is close enough to 1 (how close depends only on n but not on X) we can cover X with 3 disks with diameter d.

Фигура 3



Take a point $A \in k$ and construct the points $B, C, D \in k$ such that $AB = BC = CD = d$ - see fig 3. We construct three disks with diameters AB, BC, CD correspondingly. These disks cover all the points inside k except the red piece in fig 3. Note now that by taking d close to 1, the point D will be as close to A as we want. This means that we can make the red piece on fig 3 as small as we want.We take d so close to 1 so that the measure of the arc AD (on fig. 3) is less than $\frac{2\pi}{3n+1}$. If the red piece does not contain a point of X we are done. Otherwise we begin to rotate the point A and the entire configuration around k. It is not possible that every time the red piece hits a point of X since otherwise the number of points in X would be more than $3n$. Therefore we can cover X with 3 disks of diameter d, hence one of these disks contains at least n points. Thus, we proved that $m(n,d) \geq n$ and combined with $m(n,d) \leq n$ this yields

$m(n,d) = n$ providing d is close to 1. □

2 Bulgarian Winter Tournament

Problem 2.1. *(8.1) Given the equation*

$$2 + x\sqrt{9+6\sqrt{2}} = x\sqrt{5-2\sqrt{6}} + \sqrt{6} - 2\sqrt{3} + \sqrt{2}.$$

a) Write the root of the equation in the form $m - \sqrt{n}$, where m and n are natural numbers.

b) Factor the expression $a^3 - 3a^2 - 5a + 7$ into two non-constant factors with integer coefficients and calculate the value of this expression if a is the root found in a).

(Ivaylo Kortezov)

Solution. a) We calculate

$$\sqrt{9+6\sqrt{2}} = \sqrt{3}\sqrt{2+2\sqrt{2}+1} = \sqrt{3}(\sqrt{2}+1) = \sqrt{6}+\sqrt{3}$$

$$\sqrt{5-2\sqrt{6}} = \sqrt{3-2\sqrt{6}+2} = \left|\sqrt{3}-\sqrt{2}\right| = \sqrt{3}-\sqrt{2}.$$

The equation takes the form

$$x\left(\sqrt{6}+\sqrt{3}-\sqrt{3}+\sqrt{2}\right) = \sqrt{6}+\sqrt{2}-2\sqrt{3}-2$$

$$x\left(\sqrt{6}+\sqrt{2}\right) = \left(\sqrt{6}+\sqrt{2}\right)\left(1-\sqrt{2}\right),$$

whence (given $\sqrt{6}+\sqrt{2} > 0$) finally $x = 1-\sqrt{2}$.

b) We have $a^3-3a^2-5a+7 = a^3-a^2-2a^2+2a-7a+7 = (a-1)(a^2-2a-7)$. The product of $a-1 = -\sqrt{2}$ and $a^2-2a-7 = (a-1)^2-8 = 2-8 = -6$ is $6\sqrt{2}$. □

Problem 2.2. *(8.2) The bisectors of the sides AC and BC of the acute triangle ABC with $CB = 30^\circ$ intersect at point O. The points M and N on the sides AC and BC, respectively, are such that O is the midpoint of the segment MN. How many times is the product of the lengths of segments CM and CN greater than the product of the lengths of segments AM and BN?*

(Miroslav Marinov)

Solution. Let L be the midpoint of BC (respectively $OL \perp BC$) and K be the fifth of the perpendicular from M to BC. Then $OL \parallel MK$ and with $MO = ON$ it follows that OL is a midsegment in the triangle KMN – in particular $KL = LN$. On the other hand, we also have $BL = CL$, whence $BN = CK$.

From the right triangle CKM with $CM = 30^\circ$ we get $KM = \frac{1}{2}CM$ and $CK = \sqrt{CM^2 - KM^2} = \ frac\sqrt{3}2CM$ from the Pythagorean theorem. Therefore, $\frac{CM}{BN} = \frac{2}{\sqrt{3}}$.

Analogously, we have $\frac{CN}{AM} = \frac{2}{\sqrt{3}}$, from where finally $\frac{CM \cdot CN}{AM \cdot BN} = \frac{4}{3}$. □

Problem 2.3. *(8.3) Given a natural number n. We have $n+1$ balls numbered 1, 1, 2, 3, ... , n (only the first two are the same). We need to color these balls in n given colors so that every ball is a single color and every color is used at least once. We denote by a_n the number of possible colorings. Find the smallest n for which a_n is divisible by* 2024.

(*Ivaylo Kortezov*)

Solution. Exactly one of the colors will be used for two of the balls; let their numbers be a and b, such that $a \leq b$.

If $a > 1$, then we have $(n-1)(n-2)/2$ choices for a and b, and n choices for their color. The remaining $n-1$ balls (two of which are the same) must be colored in the remaining $n-1$ colors; for this we have $(n-1)!/2$ variants. Therefore, in this case the number of possible colorings is $\frac{1}{4}(n-1)!.n(n-1)(n-2) = \frac{1}{4}n!(n^2 - 3n + 2)$.

If $a = 1$, then there are $n!$ options for coloring all balls except a in n colors. Now there are n choices for the color of a. Thus, in this case there are $n!.n$ possible colorings.

Finally $a_n = \frac{1}{4}n!(n^2 - 3n + 2 + 4n) = \frac{1}{4}n!(n^2 + n + 2)$. If a_n is a multiple of $2024 = 2^3.11.23$ and $n < 23$, it must be $23|n^2 + n + 2$. Direct inspection shows that the smallest such n is 9, but $a_9 = 9!.23$ is not divisible by 11, and the next suitable n is 13, where $a_{13} = 13!.46$ is divisible by $2^3.11.23$. □

Problem 2.4. *(8.4) We will call a natural number Yambolian if it can be represented in the form $a^2+6ab+b^2$, where a and b are (not necessarily different) natural numbers. The number 36^{2024} is written as the sum of k number of (not necessarily distinct) Yambol numbers. What is the smallest possible value of k?*

(*Miroslav Marinov*)

Solution. We will first show that $k = 1$ is not possible, i.e. $a^2 + 6ab + b^2 = 2^{4048} \cdot 3^{4048}$ has no solution in natural numbers. If such a, b exist, then $a^2 + b^2$ is divisible by 3 and therefore a and b are divisible by 3. Writing $a = 3a_1$, $b = 3b_1$ and dividing by 3^2, we get $a_1^2 + 6a_1b_1 + b_1^2 = 2^{4048} \cdot 3^{4046}$; repeating this argument another 2023 times, we arrive at an equation of the form

$$u^2 + 6uv + v^2 = 2^{4048}$$

where u and v are natural numbers.

If v is even, then u is even; writing $u = 2u_1$, $v = 2v_1$ and dividing by 4, we get $u_1^2 + 6u_1v_1 + v_1^2 = 2^{4046}$; repeating this several times, we arrive at an equation of the form

$$s^2 + 6st + t^2 = 2^{2A},$$

where s, t, A are natural numbers, t is odd and $A\ geq2$ (since the left side is greater than or equal to 8). The latter is equivalent to $(s+3t)^2 - 8t^2 = 2^{2A}$. Now from module 8 we see that $(s+3t)^2$ is divisible by 8, so $s+3t$ is divisible by 4, i.e. $(s+3t)^2$ is divisible by 16. But then $8t^2$ must be divisible by 16, which is impossible for an odd t, a contradiction. Therefore, $k=1$ is not possible.

For an example with $k=2$ let us first notice that

$$36 = [1^2 + 6 \cdot 1 \cdot 1 + 1^2] + [3^2 + 6 \cdot 3 \cdot 1 + 1^2]$$

and now multiplying by $(6^{2023})^2$ leads to

$$36^{2024} = [(6^{2023})^2 + 6 \cdot 6^{2023} \cdot 6^{2023} + (6^{2023})^2] + [(3\ cdot 6^{2023})^2 + 6 \cdot (3 \cdot 6^{2023}) \cdot 6^{2023} + (6^{2023})^2]$$

i.e. 36^{2024} is the sum of the numbers $a_1^2 + 6a_1b_1 + b_1^2$ and $a_2^2 + 6a_2b_2 + b_2^2$, where $a_1 = b_1 = b_2 = 6^{2023}$ and $a_2 = 3 \cdot 6^{2023}$. □

Problem 2.5. *(9.1) Find all real x, y, satisfying*

$$(x+1)^2(y+1)^2 = 27xy$$

and

$$(x^2+1)(y^2+1) = 10xy.$$

(*Nedyalka Dimitrova*)

Solution. □

Problem 2.6. *(9.2) Let $p > q$ be primes, such that $240 \nmid p^4 - q^4$. Find the maximal value of $\frac{q}{p}$.*

(*Angel Gushev*)

Solution. □

Problem 2.7. *(9.3) Let ABC be a triangle, satisfying $2AC = AB + BC$. If O and I are its circumcenter and incenter, show that $\angle OIB = 90^\circ$.*

(*Konstantin Delchev*)

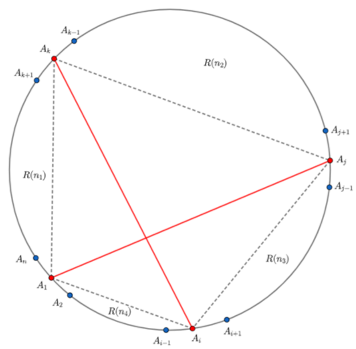
Solution. Let $D = BI \cap (ABC)$. We apply Ptolemy's theorem

$$AB.DC + BC.AD = AC.BD$$

which implies $BD = 2DA$. From $AD = DI = DC$ it yields I is midpoint of BD so $\angle OIB = 90^\circ$ □

Problem 2.8. *(9.4) There are 11 points equally spaced on a circle. Some of the segments having endpoints among these vertices are drawn and colored in two colors, so that each segment meets at an internal for it point at most one other segment from the same color. What is the greatest number of segments that could be drawn?*

Фигура 4



(*Mladen Vylkov*)

Solution. (Dragomir Grozev) Put n instead of 11 and let the points be $A_1, A_2, \ldots, A_n$. We may assume that the points are vertices of a regular n-gon. Let us first calculate the maximum number of diagonals that we can draw so that any diagonal meets no more than one other diagonal at an interior point. Note that we exclude the sides of the n-gon, and are interested only of its diagonals. We denote this number by $R(n)$. Obviously $R(2) = 0, R(3) = 0$. We will prove by induction that

$$R(n) = \left\lceil \frac{3(n-3)}{2} \right\rceil, n \geq 3$$

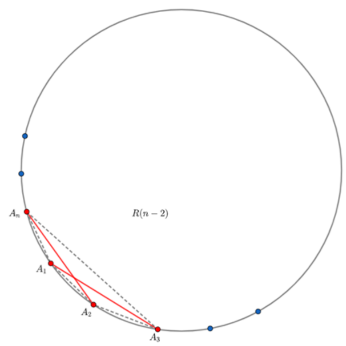
For small cases of n it is checked directly. For example $R(4) = 2, R(5) = 3, R(6) = 5$. Assume, the formula is already checked for all values less than n for some $n \in \mathbb{N}, n \geq 5$. Let A_1A_j and A_iA_k be diagonals of an extremal configuration of n points. Denote by G_1 the group of vertices that lie between A_1 and A_k (including A_1 and A_k) and let their number be n_1. In the same way let G_2, G_3, G_4 be the groups of vertices that lie respectively between A_k and A_j, A_j and A_i, A_i and A_1 and their corresponding numbers be n_2, n_3, n_4.

The key observation is that all the other diagonals connect points that are in a same group. The maximum number of diagonals that connect points in G_1 is $R(n_1)$. Note that if $n_1 \geq 2$ then A_1A_k connects non-adjacent vertices and this diagonal takes part in $R(n)$ but doesn't take art in $R(n_1)$ (the dotted diagonals on fig. 1). The same holds for the other groups. The red diagonals also should be added so we obtain,

$$R(n) = \sum_{i=1}^{4} R(n_i) + \sum_{i=1}^{4} 1_{n_i > 2} + 2.$$

Here, by $1_{n_i>2}$ we mean the indicator function that shows whether n_i is greater

Фигура 5



than 2, i.e. $1_{n_i>2} = 1$ if $n_i > 2$ and 0 otherwise. Hence,

$$R(n) = \sum_{i=1}^{4} \left\lceil \frac{3(n_i - 3)}{2} \right\rceil + \sum_{i=1}^{4} 1_{n_i>2} + 2 \qquad (1).$$

Let us prove that if A_i is connected to A_n instead of A_k, the corresponding right hand side in (1) will not decrease. If we make this change, it only affects the terms that depend on n_1 and n_2 and we have to check that

$$\left\lceil \frac{3(n_1 - 3)}{2} \right\rceil + 1 + \left\lceil \frac{3(n_2 - 3)}{2} \right\rceil \leq R(2) + \left\lceil \frac{3(n_2 + (n_1 - 2) - 3)}{2} \right\rceil .$$

This boils down to proving

$$\left\lceil \frac{3(n_1 - 3)}{2} \right\rceil + \left\lceil \frac{3(n_2 - 3)}{2} \right\rceil + 1 \leq \left\lceil \frac{3(n_1 + n_2 - 3)}{2} \right\rceil - 3$$

which can be directly verified, considering separately the parity of n_1 and n_2. Therefore, we can consecutively move the red segments on fig 1 and obtain a configuration as shown on fig 2 without decreasing the right hand side of (1)

That's why, we can write

$$R(n) = R(n-2) + 3, n \geq 5.$$

This immediately yields,

$$R(n) = \left\lceil \frac{3(n-3)}{2} \right\rceil$$

and completes the induction step. Back to the original problem. Remove the sides of the n-gon. Fix a color. The number of diagonals of that color is at most $\left\lceil \frac{3(n-3)}{2} \right\rceil$. The same holds for the other color. Adding the sides of the n-gon, which can be colored as we want, we get that the greatest number of segments is at most

$$2\left\lceil \frac{3(n-3)}{2} \right\rceil + n.$$

There is no problem to give an example of this number of segments that satisfy the condition. ☐

Problem 2.9. *(10.1) Find A_{2024}, where*

$$A_n = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 8 + \cdots + (2n-1) \cdot 2^n.$$

(*Nedyalka Dimitrova*)

Solution. Since

$$2A_n = 1 \cdot 4 + 3 \cdot 8 + \cdots + (2n-3) \cdot 2^n + (2n-1) \cdot 2^{n+1},$$

it follows

$$\begin{aligned} A_n = 2A_n - A_n &= (2n-1) \cdot 2^{n+1} - (1 \cdot 2 + 2 \cdot 4 + 2 \cdot 8 + \cdots + 2 \cdot 2^n) \\ &= (2n-1) \cdot 2^{n+1} - 2 \cdot (2 + 4 + 8 + \cdots + 2^n) + 1 \cdot 2 \\ &= (2n-1) \cdot 2^{n+1} - 2 \cdot 2 \cdot \frac{2^n - 1}{2-1} + 2 = (2n-3) \cdot 2^{n+1} + 6. \end{aligned}$$

Therefore $A_{2024} = 4045 \cdot 2^{2025} + 6$. ☐

Problem 2.10. *(10.2) Find all natural numbers k for which there exist natural numbers x, y such that the number $\frac{x^k y}{y^2 - x^2}$ is prime.*

(*Konstantin Delchev, Stanislav Harizanov*)

Solution. Let $d = (x, y)$ be the greatest common divisor of the numbers x and y. Then, $x = dx_1$, $y = dy_1$, where x_1, y_1 are coprime naturals. The expression in the condition can be rewritten as

$$\frac{d^{k-1}}{y_1^2 - x_1^2} x_1^k y_1 = p,$$

where we want p to be prime. But $(x_1^k, y_1^2 - x_1^2) = (x_1, y_1)^{\min(k,2)} = 1 = (y_1, y_1^2 - x_1^2)$, i.e. ., $x_1^k y_1$ is coprime to $y_1^2 - x_1^2$ and for p to be whole it takes $y_1^2 - x_1^2 \mid d^{k-1}$. Moreover, $y_1 > x_1$ and therefore $x_1^k y_1 > 1$. Therefore, to be p it is simply necessary that $x_1 = 1$, $y_1 = p$ and $y_1^2 - x_1^2 = d^{k-1}$ are fulfilled at the same time. If $k = 1$, then $1 = p^2 - 1 = (p-1)(p+1)$, which is impossible. At $k = 2$ we want $d = p^2 - 1$ and here the pair $(x, y) = (p^2 - 1, p(p^2 - 1))$ has the desired property for every prime p. Therefore, $k = 2$ is a solution. With $k > 2$ we want $d^{k-1} = p^2 - 1 = (p-1)(p+1)$. It is directly verified that $p = 2$ does not lead to a solution, since 3 is not the $k-1$th power of a natural number. Hence, p is an odd number and $8 \mid p^2 - 1$, whence $k - 1 \geq 3$. Therefore, $k = 3$ is not a solution. $k = 4$ is a solution at $p = 3$ and $d = 2$, i.e., $(x, y) = (2, 6)$. ☐

Problem 2.11. *(10.3) The inscribed circle in $\triangle ABC$ ($AC \neq BC$) is tangent to its sides AB , BC , and CA at points D , E , and F , respectively. Let P be the foot of the perpendicular from D to EF ($P \in EF$). If the circles circumscribed about $\triangle ABC$ and $\triangle EFC$ intersect for the second time at point Q, prove that $\angle PQC = 90^\circ$.*

(*Stoyan Boev*)

Solution. From the fact that C lies on the circle circumscribed about $\triangle FEQ$ and $CE = CF$ it follows that CQ is an exterior bisector of $\angle FQE$ and it remains to prove that QP is a bisector of $\angle FQE$, i.e. $QF : QE = FP : PE$. From $\angle QFC = \angle QEC$ and $\angle QAC = \angle QBC$ it follows that $\triangle QFA \sim \triangle QEB$, i.e.

$$QF : QE = AF : BE = AD : BD.$$

Let I be the center of the circle k inscribed in $\triangle ABC$ and the line DP intersects k for the second time at point R. Then

$$\angle REF = \angle RDF = 90^\circ - \angle DFE = 90^\circ - \angle DIB = \angle IBA$$

and analogously $\angle RFE = \angle IAB$, i.e. $\triangle FER \sim \triangle ABI$. But $RP \perp EF$, $ID \perp AB$, i.e. P and D are corresponding elements in similar triangles and $FP : PE = AD : BD$, which completes the proof. $\square$

Problem 2.12. *(10.4) Given a natural number $n \geq 3$. To find the smallest real number $k > 0$ with the following property: If G is a connected graph with n vertices and m edges, then it is always possible to delete no-more than $k \cdot \left(m - \left[\frac{n}{2}\right]\right)$ edges so that vertices can be colored in two colors and every undeleted edge has multi-colored vertices.*

(*Alexander Ivanov*)

Solution. Firstly, we will prove the following

Lemma: Let G be a connected graph with at least 3 vertices. Then either there exist two vertices connected by an edge whose removal (along with the outgoing edges) leaves G connected, or there exist two vertices of degree 1 (i.e., "leaves").

Consider an arbitrary "covering tree" of G and take as its "root" an arbitrary vertex that is not a "leaf". Let v be the farthest vertex from the "root" and u be its "ancestor". Let $v_1, v_2, \ldots, v_k$ be the "successors" of u. Clearly, they are all leaves on the tree.

Case 1. Among $v_1, v_2, \ldots, v_k$ there are two vertices connected by an edge in G. Then removing these two vertices leaves the tree (and therefore G) connected.

Case 2. Among $v_1, v_2, \ldots, v_k$ there are two vertices that are leaves in G. Then there are indeed at least two leaves in the output graph.

Case 3. Among $v_1, v_2, \ldots, v_k$ there is at most one vertex that is a leaf in G (b.o.o., let it be v_1). Then let us "connect" each of $v_2, \ldots, v_k$ to an arbitrary vertex in G other than u (such vertices exist, and none of these "connecting" edges are part of the covering tree, due to the extreme choice of u, i.e., each of them is part of a loop, all remaining edges of which are from the covering tree). Now we can remove u and v_1 and we will have a spanning tree again, so G remains connected.

This proves the lemma. With its help we can easily prove the following

Assertion: Let G be a connected graph with $n \geq 2$ vertices. Then we can color its vertices in two colors, so that if x and y are the number of "multicolored" and "single colored" edges, respectively, then $x-y\geq\left[\dfrac{n}{2}\right]$.

<u>Proof:</u> For $n=2,3$ the statement is immediately verified. Let $n\geq 4$ and G be a connected graph with n vertices. Let u and v be the two vertices from the Lemma. We "remove" u and v and color $G\setminus\{u,v\}$ according to the induction hypothesis. Now, it is not difficult to see that we can color $u\cup v$ such that the difference under consideration increases by at least 1. Indeed, this is clear if u and v are leaves, and otherwise case, considering the parity of the number of neighbors of $u\cup v$ in $G\setminus\{u,v\}$, we see that there is always such a way. The statement is proved by induction.

Let us now consider an arbitrary connected graph G with $n\geq 3$ vertices and m edges. We "color" it according to the Assertion: we have $x-y\geq\left[\dfrac{n}{2}\right]$; $x+y=m$, therefore

$$y\leq\frac{1}{2}\cdot\left(m-\left[\frac{n}{2}\right]\right)$$

and deleting y edges satisfies the condition. Thus, we got $k\leq\frac{1}{2}$.

To show that $k\geq\frac{1}{2}$ let us consider the complete graph with n vertices. A necessary and sufficient condition for having the coloring from the condition is that the graph obtained after deleting the edges is bipartite. Indeed, there must be no cycles of odd length in the graph, which is equivalent to the above.

Case 1. $n=2n_1$, $n_1\geq 2$. Then $m=\binom{n}{2}$. To reach a bipartite graph, we need to delete all the edges in each of the two groups of vertices, and the number of deleted edges is minimal when the two groups are of equal power and contain n_1 vertices. So, we need to delete at least

$$\binom{n_1}{2}+\binom{n_1}{2}=n_1^2-n_1$$

edges. From here

$$n_1^2-n_1\leq k\cdot\left(\binom{2n_1}{2}-\left[\frac{2n_1}{2}\right]\right)=k\cdot\left(\frac{2n_1(2n_1-1)}{2}-n_1\right)\implies k\geq\frac{1}{2}.$$

Case 2. $n=2n_1+1$, $n_1\geq 1$. Similarly, here we need to delete at least

$$\binom{n_1+1}{2}+\binom{n_1}{2}=n_1^2$$

edges and again

$$n_1^2\leq k\cdot\left(\binom{2n_1+1}{2}-\left[\frac{2n_1+1}{2}\right]\right)=k\cdot\left(\frac{2n_1(2n_1+1)}{2}-n_1\right)\implies k\geq\frac{1}{2}.$$

$\square$

Problem 2.13. *(11.1) The first, seventh, and seventeenth terms of an arithmetic progression are distinct and consecutive terms of a geometric progression. To* *find the difference of the arithmetic progression if its first term is a solution of the equation*

$$x^2 - 9x + x\sqrt{12-x} - 9\sqrt{12-x} = 0.$$

(*Adelina Chopanova*)

Solution. Let a_1 and d be the first term and the difference of the arithmetic progression, respectively. From the condition a_1, $a_1 + 6d$ and $a_1 + 16d$ are consecutive members of a geometric progression, i.e.

$$(a_1 + 6d)^2 = a_1.(a_1 + 16d) \iff d.(a_1 - 9d) = 0.$$

Since $d \neq 0$, we get that $a_1 = 9d$. Furthermore, we have $(x-9)(x+\sqrt{12-x}) = 0$ and $x \leq 12$. Then $x = 9$ or $\sqrt{12-x} = -x$, i.e. $x^2+x-12 = 0$ and $x \leq 0$, whence $x = -4$. Then $a_1 = 9$ and $a_1 = -4$, as $d = 1$ and $d = -\dfrac{4}{9}$, respectively. $\square$

Problem 2.14. *(11.2) Points D and E are on sides BC and AC of $\triangle ABC$. Lines AD and BE intersect at point S. Point F is on side AB and lines FE and FD intersect line l passing through C and parallel to AB at points P and Q. Prove that if $CP = CQ$, then the points C, S and F lie on the same line.*

(*Adelina Chopanova*)

Solution. From the similarities $\triangle PEC \sim \triangle FEA$ and $\triangle CDQ \sim \triangle BDF$ we obtain that

$$\frac{CP}{AF} = \frac{CE}{AE} \quad \text{and} \quad \frac{CQ}{BF} = \frac{CD}{BD}.$$

Therefore $\dfrac{CP}{CQ} = \dfrac{CE}{AE}.\dfrac{AF}{BF}.\dfrac{BD}{DC}$. By condition $CP = CQ$, it follows that

$$\frac{CE}{AE}.\frac{AF}{BF}.\frac{BD}{DC} = 1.$$

Applying Cheva's theorem for $\triangle ABC$ and the points F, D, and E, it follows that the lines AD, BE, and CF intersect in one point, i.e. point S lies on CF. $\square$

Problem 2.15. *(11.3) Given a rational number $q > 3$ such that $q^2 - 4$ is the square of a rational number. The row $\{a_i\}_{i=0}^{\infty}$ is defined as follows:*

$$a_0 = 2,\ a_1 = q,\ a_{i+1} = qa_i - a_{i-1},\ \textit{for each } i = 1, 2, \ldots$$

Do there exist a natural number n and nonzero integers $b_0, b_1, \ldots, b_n$ such that $\displaystyle\sum_{i=0}^{n} b_i = 0$ and if we write the number $b_0a_0 + b_1a_1 + \cdots + b_na_n$ in the form $\dfrac{A}{B}$, where A and B are co-prime integers, then is the number A free of squares?

(*Alexander Ivanov*)

Solution. We will prove that such numbers do not exist.

The quadratic equation $x^2 - qx + 1 = 0$ has two rational roots t and $\frac{1}{t}$ for which $t + \frac{1}{t} = q$. It easily follows by induction that $a_m = t^m + \frac{1}{t^m}$. Let's assume that goals exist numbers $b_0, b_1, \ldots, b_n$ satisfying the condition of the problem.

Lemma. Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with nonzero integer coefficients for which $c_{n-k} = c_k$ for each $k = 0, 1, \ldots, n$ and $\sum_{i=0}^{n} c_i = 0$. Then $f(x) = (x-1)^2 g(x)$, where $g(x)$ is a polynomial with integer coefficients.

Proof: From the condition we have that $f(1) = 0$. As

$$f'(x) = nc_n x^{n-1} + (n-1)c_{n-1}x^{n-2} + \cdots + c_1,$$

it

$$2f'(1) = (nc_n + (n-1)c_{n-1} + \cdots + c_1) + (nc_0 + (n-1)c_1 + \cdots + c_{n-1}) = n(c_n + c_{n-1} + \cdots + c_1 + c_0) = 0$$

and therefore $x = 1$ is a double root. The lemma is proved.

The polynomial $f(x) = b_n x^{2n} + b_{n-1}x^{2n-1} + \cdots + b_1 x^{n+1} + 2b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_{n-1}x + b_n$. satisfies the conditions of the lemma. It's not hard to see that

$$t^n \left(b_n \left(t^n + \frac{1}{t^n} \right) + b_{n-1} \left(t^{n-1} + \frac{1}{t^{n-1}} \right) + \cdots + 2b_0 \right) = f(t) = (t-1)^2 g(t).$$

Moreover, if $t = \frac{r}{s}$, $(r, s) = 1$, then from $\frac{r}{s} + \frac{s}{r} = q > 3$ it easily follows that $r \geq s + 2$, i.e. $r - s \geq 2$. Then $g\left(\frac{r}{s}\right)$ is a number of the form $\frac{l}{s^{2n-2}}$.

Finally $b_0 a_0 + b_1 a_1 + \cdots + b_n a_n$ is presented in the form $\frac{(r-s)^2.l}{r^n.s^n}$ and because the number $r - s \geq 2$ and $(r - s, r) = (r - s, s) = 1$, then the numerator will always square a prime number. $\square$

Problem 2.16. *(11.4) The sides and diagonals of a regular n-gon are colored in $k \geq 3$ colors. For each color i, between every two vertices of the polygon there exists a path consisting only of segments of color i. Prove that there exist three vertices of the polygon A, B, and C such that the line segments AB, BC, and AC are multicolor.*

(*Emil Kolev*)

Solution. We need to prove that in a complete graph with n vertices and k colors, where the induced graph on each color is connected, there exists a multicolor triangle. Denote the colors by $1, 2, 3, \ldots, k$ and recolor all edges that are in any of the colors $4, 5, \ldots, k$ into color 3. The new graph satisfies the condition of connectivity on each color and if there is a multicolored triangle for it, then the same triangle in the initial graph will also be multicolored. Therefore, we can consider that $k = 3$.

Suppose that the statement is not true for a graph G, as we can choose G to have a minimal number of vertices. It follows from the minimality of G that after removing any vertex, the new graph will not be connected by any of the colors, and let it be color 1. Denote by $G_1, G_2, \ldots, G_t$ the color connectivity components 1 after deleting vertex A. Since G is color 1 connected, there exist $A_i \in G_i$ for which AA_i is color 1. The segment A_1A_2 is not color 1 because G_1 and G_2 are different components of connectivity. Let it be of color 2. If A_1B is a color 1 segment of G_1, then the segment A_2B cannot be of color 1 because G_1 and G_2 are different connectivity components; cannot be of color 3, because then A_1A_2B is a multicolor triangle and is therefore of color 2. Analogously, it is proved that all segments between the points of G_1 and G_2 are of color 2. We obtained that all segments between any two connectivity components are either color 2 or color 3.

Now consider segments AX and AY in colors 2 and 3, respectively (such segments exist, since G is connected in each of the colors). Without limitation, we have the following two cases:

1. $X, Y \in G_1$, then one of the triangles AXA_2 and AYA_2 is multicolored.
2. $X \in G_1$ and $Y \in G_2$, then one of the triangles AXA_2 and AYA_1 is multicolored.

The resulting contradiction shows that for every graph with the given properties there exists a multicolored triangle. □

Problem 2.17. *(12.1) Maria and Bilyana play the following game. Maria has* 2024, *and Bilyana has* 2023 *of fair coins. Coins are tossed randomly - the probability of each individual coin being heads after the toss is* $\frac{1}{2}$. *Maria wins if there are strictly more heads among her coins than Bilyana's, otherwise Bilyana wins. What is the probability that Maria wins?*

(*Kristiyan Vasilev*)

Solution. Let p be the probability that Maria has more heads than Bilyana after tossing the first 2023 of Maria's coins. Then, for reasons of symmetry, the probability that Maria has fewer heads than Bilyana is also p, and therefore the probability that Maria and Bilyana have thrown an equal number of heads is $1 - 2p$. If Maria has thrown less heads than Bilyana to this moment, the probability of her winning is 0 (regardless of the last coin), if she flipped more heads, the probability of her winning is 1 (again, regardless of the last coin), and if they flipped an even number, the probability of her winning is $frac12$ (here the last coin must be heads).
Thus we get that the probability that Maria wins is $p + \frac{1-2p}{2} = \frac{1}{2}$. □

Problem 2.18. *(12.2) Given an equilateral and acute* $\triangle ABC$ *with* $AC > BC$. *Some point* P *in the interior of* $\triangle ABC$ *is such that* $\angle APB = 180° - \angle ACB$ *and some* AP *and* BP *intersect segments* BC *and* AC *at points* A_1 *and* B_1

, respectively. Let M be the midpoint of A_1B_1, and the circumscribed circles about $\triangle ABC$ and $\triangle A_1CB_1$ intersect for the second time at point Q. Show that $\angle PQM = \angle BQA_1$.

(*Kristiyan Vasilev*)

Solution. Let P' be symmetric to P with respect to the midpoint N of AB. Then we have that $AP'BP$ is a parallelogram, hence $AP'BC$ is inscribed. Hence we get that $\angle AQP' = \angle ABP' = \angle BAP$. On the other hand $\angle AQP = \angle AQC - \angle PQC = 180^\circ - \angle ABC - \angle AA_1B = \angle PAB$, where we used that P lies on the circle circumscribed about $\triangle A_1B_1C$. Thus we get that $\angle AQP' = \angle AQP$, from which it follows that P, Q, P' lie on one line, i.e. $N \in PQ$. We also have that $\triangle AQB \sim \triangle B_1QA_1$, which means that $\angle NQB = \angle MQA_1$ as corresponding elements. The latter is equivalent to $\angle PQB = \angle MQA_1$, from which it follows that $\angle PQM = \angle BQA_1$. □

Problem 2.19. *(12.3) Let $n \in \mathbb{N}$ and $\mathcal{A}$ be a nonempty family of nonempty subsets of $\{1, 2, \ldots, n\}$ with the following property – if $A \in \mathcal{A}$ and $A \subset B \subseteq \{1, 2, \ldots, n\}$, then $B \in \mathcal{A}$. Prove that the function*

$$f(x) := \sum_{A \in \mathcal{A}} x^{|A|}(1-x)^{n-|A|}$$

is strictly increasing in the interval $(0, 1)$.

(*Kristiyan Vasilev*)

Solution. Let $0 < p < q < 1$. Notice that $p^* = \frac{q-p}{1-p} \in (0, 1)$. We construct the sets X and Y as follows : For each element $i \in \{1, 2, \ldots, n\}$ we put i in X with probability p (independently of each other), and for each element $j \in \{1, 2, \ldots, n\} \setminus X$ we put j in Y with probability p^*. Then $\mathbb{P}(x \in X \cup Y) = p + (1-p)p^* = q$. It is easy to see that $f(p) = \mathbb{P}(X \in \mathcal{A})$, and $f(q) = \mathbb{P}(X \cup Y \in \mathcal{A})$, but since $\mathcal{A}$ has the property from the condition we have that $X \subseteq X \cup Y \Rightarrow f(p) < f(q)$. □

Problem 2.20. *(12.4) We will call a natural number m remarkable if there exist integers a, b, c, for which $m = a^3 + 2b^3 + 4c^3 - 6abc$. Prove that there exists a natural number $n < 2024$ such that for infinitely many prime numbers p, the number np e remarkable.*

(*Kristiyan Vasilev*)

Solution. **Lemma.** Let p be a prime number and $a, b, c \in \mathbb{Z}/p\mathbb{Z}$. Then there exist $x, y, z \in \mathbb{Z}$ such that $|x|, |y|, |z| < \sqrt[3]{p}$, $(x, y, z) \neq (0, 0, 0)$ and $ax + by + cz \equiv 0 \pmod{p}$.

Proof. Consider the set $M := \{(x, y, z) : x, y, z \in \{0, 1, \ldots, \lfloor \sqrt[3]{p}\ rfloor\}\}$. We have that $|M| > p$, i.e. in M there are two distinct elements (x_1, y_1, z_1)

and (x_2, y_2, z_2), for which $ax_1 + by_1 + cz_1 \equiv ax_2 + by_2 + cz_2 \pmod{p}$. Thus $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ satisfies the conditions of the lemma.

Now let $p \equiv 2 \pmod 3$. Then the comparison $x^3 \equiv 2 \pmod p$ has a solution a, since the function $x \mapsto x^3$ is injective in $\mathbb{Z}/p\mathbb{Z}$, hence it is surjective. One way to verify this is to see that $x^3 \equiv 1 \pmod p$ has only 1 for a solution, since $(3, p-1) = 1$ and hence the exponent of $x \pmod p$ is 1.

From the lemma, there exist x, y, z with $|x|, |y|, |z| < \sqrt[3]{p}$, for which $x + ay + a^2z \equiv 0 \pmod p$. From here we get that $x^3 + a^3y^3 + a^6z^3 - 3a^3xyz \equiv 0 \pmod p$. The latter is equivalent to $x^3 + 2y^3 + 4z^3 - 6xyz \equiv 0 \pmod p$. On the other hand $|x|, |y|, |z| < \sqrt[3]{p}$ gives $|x^3 + 2y^3 + 4z^3 - 6xyz| < 13p$. Also notice that if $x^3 + 2y^3 + 4z^3 - 6xyz < 0$, then the triple $(-x, -y, -z)$ will give a natural number divisible by p.

It remains to note that $x^3 + 2y^3 + 4z^3 - 6xyz \neq 0$. Indeed, if $x^3 + 2y^3 + 4z^3 - 6xyz = 0$, then we have $(x + \sqrt[3]{2}y + \sqrt[3]{4}z)((x - \sqrt[3]{2}y)^2 + (x - \sqrt[3]{4}z)^2 + (\sqrt[3]{2}y - \sqrt[3]{4}z)^2) = 0$, i.e. $x = \sqrt[3]{2}y = \sqrt[3]{4}z$ or $x + y\sqrt[3]{2} + z\sqrt[3]{4} = 0$, whose only integer solutions are $(x, y, z) = (0, 0, 0)$ (the first follows because $\sqrt[3]{2}$ is irrational, and the second follows from the fact that $x^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$ over the rational numbers). $\square$

3 Bulgarian Spring Tournament

Problem 3.1. *(8.1) Find all pairs (x, y) of real numbers for which*

$$4y^4 + x^4 + 12y^3 + 5x^2(y^2 + 1) + y^2 + 4 = 12y.$$

(Ivaylo Kortezov)

Solution. We have the inequalities $x^4 \geq 0$, $5x^2(y^2 + 1) \geq 0$ and $4y^4 + 12y^3 + y^2 - 12y + 4 = (2y - 1)^2(y + 2)^2 \geq 0$. The sum of the left sides is 0 if and only if each of them is equal to 0. The first two lead to $x = 0$, and the third to $y = -2$ or $y = \frac{1}{2}$. $\square$

Problem 3.2. *(8.2) Points A, B, Y and C lie in this order on circle k with center O, such that $BC = 2$ cm, $\angle BAY = 42^\circ$ and $\angle CAY = 78^\circ$. It is known that the circle ω through the points A, O and B is tangent to the line BY. The circle through the points A and C, tangent to the line CY, intersects ω for second time at the point N. To be found:*

a) the length of the segment BO; *b) the size of the angle $\angle YAN$.*

(Miroslav Marinov)

Solution. a) Clearly $\angle BAC = \angle BAY + \angle CAY = 120^\circ$, respectively $\angle BOC = 360^\circ - 2\angle BAC = 120^\circ$. Thus, if M is the midpoint of BC, then $OM \perp BC$ (because $BO = OC$), $\angle BOM = 60^\circ$ and $BM = \frac{BC}{2} = 1$. Let $BO = x$ and from triangle BOM we have $OM = \frac{x}{2}$ from $\angle OBM = 30^\circ$ and $x^2 = \left(\frac{x}{2}\right)^2 + 1^2$ from the Pythagorean theorem, respectively $x^2 = \frac{4}{3}$ and $x = \frac{2\sqrt{3}}{3}$.

b) We have $\angle ANB = \angle AOB = 180^\circ - 2\angle BAO = 180^\circ - 2\angle OBY = 180^\circ - 2(90^\circ - \angle YCB) = 2\angle YCB$. Hence $\angle ABY = \angle ABO + \angle OBY = 2\angle OAB = 180^\circ - \angle AOB = 180^\circ - 2\angle YCB$ and now from the other circle we calculate $angleANC = 180^\circ - \angle ACY = \angle ABY = 180^\circ - 2\angle YCB$. Therefore, $\angle ANB + \angle ANC = 180^\circ$, i.e. N lies on BC. It remains to consider that $\angle NAC = \angle BCY = \angle BAY$ from the touching and $\angle YAN = \angle CAY - \angle CAN = \angle CAY - \angle BAY = 36^\circ$. □

Problem 3.3. *(8.3) A three-digit natural number n is initially written on the board. Two players, A and B, take turns taking turns, with A going first. The turner reduces the number on the board by some divisor of his own (i.e, other than 1 and the number itself). For example, if at some point the number on the board is 6, it can be reduced by 2, then the number on the board will now be 4. Whoever cannot make a move loses, and the other wins. Player A is known to have a way to win as well as play B. What are all the possible n?*

(*Ivaylo Kortezov*)

Solution. If there is a prime number on the board, the player loses by definition. If there is an even number on the board that is not a power of 2, then the player can always reduce it by its odd divisor, leaving an odd number on the board. If the number on the board is odd and is reduced by its (odd) divisor a, i.e. a number of the form ab is replaced by $a(b-1)$, the resulting number is even and is not a power of the pair. Therefore, if the starting number is even and not a power of even, the player can always make a move guaranteeing that for his next move he will again receive a natural number of the same kind. Thus, an even number that is not a power of even is a winning position, and an odd number is a losing position.

It remains to analyze the cases when the board number is 2^m for a natural m. In this case, the reducer should be 2^k for naturally $k < m$. If $k < m-1$, then the resulting position $2^k(2^{m-k}-1)$ is even and not a power of even, so it would be a winning move for the adversary and therefore an unacceptable move. So, we need $k = m-1$. Playing this way, one player will get the even powers of 2 on the board and the other the odd ones. Given that 2 is a losing position, we conclude that even powers of 2 are winning positions and odd powers of 2 are losing positions.

The finally suitable n are the even numbers that are not odd powers of the pair. Among the given there are $900 : 2 - 2 = 448$ such (we excluded $128 = 2^7$ and $512 = 2^9$). □

Problem 3.4. *(8.4) The non negative real numbers x, y, z are such that $(x+y)(y+z)(z+x) = 1$. We denote by m and M respectively the smallest and largest possible values of the expression $A = (xy + yz + zx)(x + y + z)$.*

a) Find m and M.

b) Is there a triple of nonnegative rational numbers (x, y, z) satisfying the given equality for which $A = m$?

(*Miroslav Marinov*)

Solution. a) The inequality $(xy+yz+zx)(x+y+z) \geq 1 = (x+y)(y+z)(z+x)$ is equivalent to $xyz \geq 0$. Equality is reached only when one of the variables, say x, is 0 and the other two (in this case y and z) are whatever with $yz(y+z) = 1$; one possibility is $y = 1$ and $z = \frac{\sqrt{5}-1}{2}$. The inequality $(xy+yz+zx)(x+y+z) \leq \frac{9}{8}$ is equivalent to $9(x+y)(y+z)(z+x) - 8(xy+yz+zx)(x+y+z) \geq 0$, i.e. on $x^2y+xy^2+y^2z+yz^2+x^2z+xz^2 \geq 6xyz$. The latter is true from the inequality between the arithmetic mean and the geometric mean applied to the six terms on the left, with equality only at $x = y = z$ and $8x^3 = 1$, i.e. $x = y = z = \frac{1}{2}$.

b) Given the reasoning in a), it suffices to prove that the equation $yz(y+z) = 1$ has no solution in (positive) rational numbers. Assume the opposite and let $y = \frac{p}{r}$, $z = \frac{q}{r}$ is a solution (with natural p, q, r) in which we have reduced the fractions under a common denominator. Then $pq(p+q) = r^3$, as after truncating a common divisor of p, q, r, if necessary, we can consider that $(p, q, r) = 1$. In fact, here already $(p, q) = 1$, since otherwise their common prime divisor would also be that of r, contradiction with $(p, q, r) = 1$. Thus, the numbers p, q and $p+q$ are two by two mutually prime, and since their product is an exact cube, then necessarily $p = a^3$, $q = b^3$ and $p+q = c^3$ for some natural numbers a, b, c. Thus we obtained $a^3 + b^3 = c^3$ for some natural a, b, c, which is impossible (a special case of the so-called Fermat's Great Theorem). □

Problem 3.5. *(9.1) Solve the inequality:*

$$\frac{x^2 - |x-1| - 4}{x-4} \geq 2x - 1.$$

(*Nedyalka Dimitrova*)

Solution. Note that $x \neq 4$ and $|x-1| = x-1$ for $x > 1$, otherwise $|x-1| = 1-x$.

If $x > 1$ we get:

$$\frac{x^2 - 8x + 7}{x-4} \leq 0 \Leftrightarrow \frac{(x-7)(x-1)}{x-4} \leq 0 \Leftrightarrow x \in (4, 7]$$

If $x \leq 1$ we get:

$$\frac{x^2 - 10x + 9}{x-4} \leq 0 \Leftrightarrow \frac{(x-9)(x-1)}{x-4} \leq 0 \Leftrightarrow x \in (-\infty, 1]$$

Finally $x \in (-\infty, 1] \cup (4, 7]$. □

Problem 3.6. *(9.2) Given the triangle ABC and M - midpoint of AB. Given the angles $\angle ABC = 30^\circ$ and $\angle BCM = 105^\circ$. Prove that $CM.AC = BM.BC$.*

(*Konstantin Delchev*)

Solution. Le AA_1 be the height from point A in $\triangle ABC$. Note that it lies on the continuation of BC, because this triangle is obtuse. The triangle $\triangle AA_1B$ is right-angled with angle $30°$. Therefore, $AA_1 = AM = A_1M = BM = x$. Consequently, we directly find $\angle CMB = 45°$, $\angle A_1MA = 60°$, $\angle A_1MC = 75°$, $\angle MCA_1 = 75°$. So $\triangle MCA_1$ is isosceles. Then $CA_1 = x$ and $\triangle AA_1C$ is isosceles and right-angled, therefore $\angle ACA_1 = 45°$ and therefore $\angle ACM = 30°$.

We will calculate the area of $\triangle ACM$ in two different ways.

$S_{ACM} = \frac{AC.CM}{4}$ because $\angle ACM = 30°$ i.e. the height to AC is equal to $MC/2$. But also M is the middle of AB. So $S_{ACM} = \frac{S_{ABC}}{2} = \frac{BC.AA_1}{4} = \frac{BC.BM}{4}$ and we get what we are looking for. □

Problem 3.7. *(9.3) We call n lines in the plane three-way if they can be separated into three nonempty sets, X, Y, Z. Every two lines from the same set are parallel to each other, no two lines from different sets are parallel to each other, and no three lines intersect at a point.*

By S_n we denote the maximum number of regions into which n three-way lines can divide the plane. As a region we consider a connected part of the plane, not necessarily finite, whose boundaries are defined by three-way lines.

What is the largest n for which $S_n < 128$?

(*Konstantin Delchev*)

Solution. We will derive a general formula for the number of regions. Let the three sets have the number of elements $|X| = x$, $|Y| = y$ and $|Z| = z$, respectively. The first two divide the plane into a total of $(x+1)(y+1)$ regions. Each line of the third set intersects the others at $x + y$ points and is divided into $x+y+1$ parts. Each of them divides one of the already obtained areas into two. Thus we get a general formula:

$$S_{x,y,z} = (x+1)(y+1) + z(x+y+1) = x + y + z + xy + xz + yz + 1.$$

We note that $n = x + y + z$. Furthermore, we have that $3(xy + yz + zx) \le (x + y + z)^2 = n^2$ (equivalent to $(x - y)^2 + (y - z)^2 + (z - x)^2 \ge 0$). So $S_{x,y,z} \le n^2/3 + n + 1$ which is less than 128 for $n = 18$ (actually, $S_{6,6,6} = 127$, i.e. $S_{18} = 127$). Also, $S_{6,6,7} = 140$, so $S_n > 128$ for $n \ge 19$. Therefore, the answer is 18. □

Problem 3.8. *(9.4) For an odd natural number $n > 1$, we define the set of the different remainders of powers of the pair when dividing by n:*

$$S_n = \{a \mid a < n, \exists k \in \mathbb{N} : 2^k \equiv a \pmod{n}\}.$$

Are there different odd numbers m and r such that $S_m = S_r$?

(*Alexander Ivanov*)

Solution. No! There exists a natural number s such that $2^s \equiv 1 \pmod{n}$ (for example $s = \varphi(n)$ from Euler's theorem or because the power series of 2 modulo n is periodic). We have $2^{s-1} \equiv \frac{n+1}{2} \pmod{n}$ and so $x = \frac{n+1}{2} \in S_n$, but $2x = n+1 > n$ is not in S_n. Also, if $t \leq \frac{n-1}{2}$ is of S_n, then $2t < n$ is also. Therefore, the smallest natural number t such that $t \in S_n$ and $2t \notin S_n$, is $\frac{n+1}{2}$. Since this number is different for different n, we get what we asked for. □

Problem 3.9. *(10.1) The reals x, y satisfy $x(x-6) \leq y(4-y)+7$. Find the minimal and maximal values of the expression $x + 2y$.*

(*Nedyalka Dimitrova*)

Solution. Let $a = x + 2y$. Then $x = a - 2y$ and

$$\begin{array}{rcl}
(a-2y)(a-2y-6) & \leq & y(4-y)+7 \\
a^2 - 2ay - 6a - 2ay + 4y^2 + 12y & \leq & 4y - y^2 + 7 \\
5y^2 - 2(2a-4)y + (a^2 - 6a - 7) & \leq & 0 \\
D = (2a-4)^2 - 5(a^2 - 6a - 7) & \geq & 0 \\
4a^2 - 16a + 16 - 5a^2 + 30a + 35 & \geq & 0 \\
a^2 - 14a - 51 & \leq & 0 \\
(a-17)(a+3) & \leq & 0.
\end{array}$$

Finally, $a \in [-3; 17]$. □

Problem 3.10. *(10.2) Let ABC be a triangle and a circle ω through C and its incenter I meets CA, CB at P, Q. The circumcircles (CPQ) and (ABC) meet at L. The angle bisector of $\angle ALB$ meets AB at K. Show that, as ω varies, $\angle PKQ$ is constant.*

(*Alexander Ivanov, Miroslav Marinov*)

Solution. (Miroslav Marinov) Clearly $IP = IQ$ from the bisector CI in ω. Our aim is to show that $IK = IP$, as it would follow that I is the circumcenter of triangle PKQ, whence $\angle PKQ = \frac{1}{2}\angle PIQ = 90^\circ - \frac{1}{2}\angle ACB$.

Let the angle bisectors CI and LK intersect at the midpoint T of the arc $\widehat{AB}$ from k, not containing C. We have $\angle TAK = \angle TAB = \angle BCT = \angle ACT = \angle ALT$, thus $\triangle AKT \sim \triangle LAT$ and $TA^2 = TK \cdot TL$. Since $TA = TI$ from the trillium lemma, we deduce $TI^2 = TK \cdot TL$ and $\triangle IKT \sim \triangle LIT$. Hence

$$IK = IT \cdot \frac{LI}{LT} = AT \cdot \frac{LI}{LT}.$$

On the other hand, the circumcircles k and ω yield $\angle LPI = 180^\circ - \angle LCI = \angle LAT$ and $\angle PLI = \angle PCI = \angle ALT$. Hence $\triangle LAT \sim \triangle LPI$ and so

$$\frac{LI}{LT} = \frac{PI}{AT}$$

which completes the proof. □

Problem 3.11. *(10.3) See 9.4*

Problem 3.12. *(10.4) A graph G is called divisibility graph if the vertices can be assigned distinct positive integers such that between two vertices assigned u, v there is an edge iff $\frac{u}{v}$ or $\frac{v}{u}$ is a positive integer. Show that for any positive integer n and $0 \leq e \leq \frac{n(n-1)}{2}$, there is a divisibility graph with n vertices and e edges.*

(*Danila Cherkashin*)

Solution. We reason inductively on n, not writing the number 1 at any vertex. For $n = 1$ the requested is clear, for $n = 2$ an example with $e = 1$ is $(2, 4)$ and an example with $e = 0$ is $(2, 3)$. For $n = 3$ example with $e = 0$ is $3, 5, 7$, example with $e = 1$ is $2, 4, 7$, example with $e = 2$ is $2, 4, 10$, an example with $e = 3$ is $2, 4, 8$.

For $n \geq 4$ we have $n - 1 \leq \frac{(n-1)(n-2)}{2}$, so at least one of $e \geq n - 1$ and $e \leq \frac{(n-1)(n-2)}{2}$ is satisfied.

Let $e \geq n - 1$ first. In an example with $n - 1$ vertices and $e - (n - 1)$ edges, we add a vertex (of degree $n - 1$) by writing in it a prime number p greater than the numbers in the other vertices , then we multiply the numbers in the remaining vertices by p – this does not spawn new edges between the remaining vertices.

Now let $e \leq \frac{(n-1)(n-2)}{2}$. In an example with $n-1$ vertices and e edges, we add a vertex (of degree 0), writing in it a prime number p greater than the numbers in the other vertices, and do not change the numbers in the other vertices - - it does not spawn new edges. □

Problem 3.13. *(11.1) Let a be a real number.*

a) Find all values of a for which the inequality

$$x \log_{\frac{1}{2}} a^4 - x^2 > 3 + 2 \log_2 a^2$$

has a solution.

b) Calculate the limit

$$\lim_{a \to -\infty} \left(\sqrt{a^2 - a + 1} + a \right).$$

(*Adelina Chopanova*)

Solution. a) Since $\log_{\frac{1}{2}}(a^4) = -2. \log_2(a^2)$, then by putting $2 \log_2(a^2) = b$, we get the inequality $x^2 + b.x + 3 + b < 0$. For this inequality to have at least one solution, it is necessary and sufficient that $D = b^2 - 4b - 12 > 0$ whose solutions are $b < -2$ or $b > 6$, whence $\log_2(a^2) < -1$ or $\log_2(a^2) > 3$. From the properties of the logarithmic function, we get $a^2 < \frac{1}{2}$ or $a^2 > 8$ and $a \neq 0$. Final

$$a \in \left(-\infty; -2\sqrt{2}\right) \cup \left(-\frac{\sqrt{2}}{2}; 0\right) \cup \left(0; \frac{\sqrt{2}}{2}\right) \cup \left(2\sqrt{2}; \infty\right).$$

b) Since $a<0$, we get:
(1)

$$\sqrt{a^2-a+1}+a=\frac{(\sqrt{a^2-a+1}+a)(\sqrt{a^2-a+1}-a)}{\sqrt{a^2-a+1}-a}=\frac{-1+\frac{1}{a}}{-\sqrt{1-\frac{1}{a}+\frac{1}{a^2}}-1}.$$

Therefore

$$\lim_{a\to-\infty}\left(\sqrt{a^2-a+1}+a\right)=\frac{1}{2}.$$

□

Problem 3.14. *(11.2) Let $ABCD$ be a parallelogram and a circle k passes through A,C and meets rays AB,AD at E,F. If BD,EF and the tangent at C concur, show that AC is diameter of k.*

(Adelina Chopanova)

Solution. Let the tangent to k at point C intersect the rays $AB^{\rightarrow}$ and $AD^{\rightarrow}$ at the points M and N, respectively, and the lines BD, EF and the tangent intersect at point P. After applying Menelaus' theorem twice to $\triangle AMN$ and to the lines BD and EF, we get

$$\frac{AD}{ND}.\frac{NP}{MP}.\frac{MB}{AB}=1 \text{ and } \frac{AF}{NF}.\frac{NP}{MP}.\frac{ME}{AE}=1.$$

Hence

$$\frac{AD}{ND}.\frac{MB}{AB}=\frac{AF}{NF}.\frac{ME}{AE}. \tag{1}$$

Since $ABCD$ is a parallelogram, then $\frac{AD}{ND}=\frac{MC}{NC}=\frac{MB}{AB}$ (2). From the tangent and secant property $MC^2=ME.MA$ and $NC^2=NF.NA$ (3). From (1), (2) and (3) it follows that

$$\frac{MC^2}{NC^2}=\frac{AF}{NF}.\frac{ME}{AE}\Rightarrow\frac{ME.MA}{NF.NA}=\frac{AF}{NF}.\frac{ME}{AE}.$$

Therefore, $AM.AE=AN.AF$, i.e. the quadrilateral $EFNM$ is cyclic. Then $\angle AEF=\angle ANM$, whence $\widehat{AF}=\widehat{AEC}-\widehat{FC}$, i.e., $\widehat{AEC}=\widehat{AFC}$. It follows that AC is a diameter of k. □

Problem 3.15. *(11.3) Find all natural numbers n for which the number of positive divisors of $LCM(1,2,\dots,n)$ is a power of 2.*

(Emil Kolev)

Solution. For each prime p, the numbers of the interval $n \in [p^2, p^3)$ are not solutions, because the degree of p in the decomposition of the LCM is 2 and so contributes by a factor of 3 to the number of divisors. Therefore this number is not a power of 2. From Bertrand's postulate for a prime p there is a prime q with $p < q < 2p$, respectively $p^2 < q^2 < 4p^2 < p^3 < q^3$ for $p \geq 5$; also $3^2 < 5^2 < 3^3 < 5^3$. Therefore, each interval $[p^2, p^3)$ and $[q^2, q^3)$ for consecutive primes $3 \leq p < q$ intersects. We obtained that $n < 9$ and a direct check shows that only $n = 1, 2, 3$ and 8 are solutions. □

Problem 3.16. *(11.4) Given is a convex* 2024*-gon* $A_1A_2 \ldots A_{2024}$ *and* 1000 *points inside it, so that no three points are collinear. Some pairs of the points are connected with segments so that the interior of the polygon is divided into triangles. Every point is assigned one number among* $\{1, -1, 2, -2\}$*, so that the sum of the numbers written in* A_i *and* A_{i+1012} *is zero for all* $i = 1, 2, \ldots, 1012$*. Prove that there is a triangle, such that the sum of the numbers in some two of its vertices is zero.*

(*Emil Kolev*)

Solution. Clearly, if there are two adjacent points A_i and A_{i+1} with opposite numbers, the problem is solved. Without restriction, let $A_1 = 1$ and consider all segments A_iA_{i+1} for $i = 1, 2, \ldots, 1012$. Since $A_{1013} = -1$, among the considered segments there is an odd number whose ends are one positive and one negative number. These two numbers can be $\{-1, 2\}$ or $\{1, -2\}$ and let the number of segments with ends of the first kind be p and the number of segments with ends of the second species to be q. Due to symmetry, the number of segments A_iA_{i+1} for $i = 1013, \ldots, 2024$ ($A_{2025} \equiv A_1$) with ends $\{1, -2\}$ is equal to p. Therefore, all segments with endpoints $\{1, -2\}$ are $p + q$, which is an odd number.

Now consider any triangle that does not have two vertices with opposite numbers. We have the following possibilities for the three numbers:

$$(1, 1, -2), (1, 1, 2), (-1, -1, 2), (-1, -1, -2),$$
$$(2, 2, -1), (2, 2, 1), (-2, -2, 1), (-2, -2, -1).$$

Each of these triangles has an even number (2 or 0) of sides with ends 1 and -2. Therefore, the total number of segments with ends 1 and -2 (counted in multiples) is an even number. But every line segment inside the 2024-gon is counted twice (once from the two triangles in which it participates), and every line segment that is a side of the 2024-gon is counted once. The resulting contradiction shows that a triangle with the requested property exists. □

Problem 3.17. *(12.1) Given is a sequence* $a_1, a_2, \ldots,$ *such that* $a_1 = 1$ *and* $a_{n+1} = \frac{9a_n+4}{a_n+6}$ *for any* $n \in \mathbb{N}$*. Which terms of this sequence are positive integers?*

(*Borislav Kirilov*)

Solution. It can be shown by induction that

$$a_n = \frac{4.2^n - 3}{2^n + 3}, \forall n \geq 2$$

Thus, a_n is integer only when $n = 1$. □

Problem 3.18. *(12.2) Given is a triangle ABC and two points $D \in AC, E \in BD$ such that $\angle DAE = \angle AED = \angle ABC$. Show that $BE = 2CD$ iff $\angle ACB = 90^\circ$.*

(Nikolay Nikolov)

Solution. Let $\alpha = \angle A$, $\beta = \angle B$ and $\gamma = \angle C$. Then

$$\frac{BE}{\sin(\alpha - \beta)} \overset{(1)}{=} \frac{AB}{\sin(\pi - \beta)}, \quad \frac{CD}{\sin(\alpha - \beta)} \overset{(2)}{=} \frac{BC}{\sin 2\beta}, \quad \frac{AB}{\sin\gamma} \overset{(3)}{=} \frac{BC}{\sin\alpha}$$

and so

$$\frac{BE}{CD} = \frac{\sin 2\beta \sin\gamma}{\sin\beta \sin\alpha} \overset{(4)}{=} \frac{2\cos\beta\sin\gamma}{\sin\alpha} \overset{(5)}{=} \frac{2\cos\beta\sin\gamma}{\sin\beta\cos\gamma + \cos\beta\sin\gamma}.$$

Therefore (6) $BE = 2CD \Leftrightarrow \cos\gamma = 0 \Leftrightarrow \gamma = 90^\circ$. □

Problem 3.19. *(12.3) For a positive integer n, denote with $b(n)$ the smallest positive integer k, such that there exist integers $a_1, a_2, \ldots, a_k$, satisfying $n = a_1^{33} + a_2^{33} + \ldots + a_k^{33}$. Determine whether the set of positive integers n is finite or infinite, which satisfy:*

a) $b(n) = 12$; b) $b(n) = 12^{12^{12}}$.

(Borislav Kirilov)

Solution. a) From Fermat's theorem and $y^2 \equiv 1 \pmod{67} \Leftrightarrow y \equiv \pm 1 \pmod{67}$ it follows that any student number gives a remainder of 0, 1 or 66 when divided by 67 .Let us consider the numbers 12^{66k+1}, where $k \in \mathbb{N}$. They are presented as the sum of 12 student numbers. Furthermore, by Fermat's theorem $12^{66k+1} \equiv 12 \pmod{67}$. This shows that $b(12^{66k+1}) = 12$ for every $k \in \mathbb{N}$. Therefore, the set here is infinite.

b) For $P \in \mathbb{Z}[X]$ let us set $\Delta(P)(x) = P(x+1) - P(x)$. It is clear that if P is of degree d with leading coefficient a, then $\Delta(P) \in \mathbb{Z}[X]$ is a polynomial of degree $d - 1$ with leading coefficient ad. Consider the series of polynomials $P_1(x) = x^{33}$, $P_{k+1} = \Delta(P_k)$ for $k \in \mathbb{N}$. It is easy to see by induction on k that for each $x \in \mathbb{Z}, k \in \mathbb{N}$ the number $P_k(x)$ is a sum of 2^{k-1} student numbers. Furthermore, we have that $P_{33}(x) = 33!x + b$ for some $b \in \mathbb{Z}$. Since 1 and -1 are student numbers, each integer is the sum of at most $2^{32} + 33! < 12^{12^{12}}$ student numbers. Then the set here is empty, and therefore finite. □

Problem 3.20. *(12.4) Let $d \geq 3$ be a positive integer. The binary strings of length d are splitted into 2^{d-1} pairs, such that the strings in each pair differ in exactly one position. Show that there exists an alternating cycle of length at most $2d-2$, i.e. at most $2d-2$ binary strings that can be arranged on a circle so that any pair of adjacent strings differ in exactly one position and exactly half of the pairs of adjacent strings are pairs in the split.*

(*Lyuben Lichev*)

Solution. Consider a graph G with vertices the binary vectors of length d and edges connecting the pairs of vectors that differ in exactly one position. Let $\mathcal{M}$ be a complete d-dimensional pairing. We will prove that for each vertex $\mathbf{x} \in \{0,1\}^d$ of G we can find an alternating cycle of length no more than $2d-2$ among the distance vectors of the most many 2 of $\mathbf{x}$.

Let $\mathbf{x} \in \{0,1\}^d$ have as neighbors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ in G. Without loss of generality let $\{\mathbf{x}, \mathbf{x}_1\} \in \mathcal{M}$, and the vectors $\mathbf{x}_2, \ldots, \mathbf{x}_d$ form pairs in $\mathcal{M}$ respectively with $\mathbf{y}_2, \ldots, \mathbf{y}_d$ (which are at distance 2 from $\mathbf{x}$). Notice that each of the vectors $\mathbf{y}_2, \ldots, \mathbf{y}_d$ has exactly two neighbors among $\mathbf{x}_1, \mathbf{x}_2, \ldots,$ $mathbf{x}_d$ in G. We will consider two cases:

Case 1. Any of the vectors $\mathbf{y}_2, \ldots, \mathbf{y}_d$, say $\mathbf{y}_i$, is a neighbor of $\mathbf{x}_1$ in G. Then $\mathbf{x}, \mathbf{x}_1, \mathbf{y}_i, \mathbf{x}_i$ form an alternating cycle of length $4 \leq 2d-2$.

Case 2. None of the vectors $\mathbf{y}_2, \ldots, \mathbf{y}_d$ is a neighbor of $\mathbf{x}_1$ in G. Consider the following algorithm. At the beginning, let's place a token at vertex $\mathbf{x}_2$ and at each step:

- if the token is located at a vertex $\mathbf{x}_i$ for some $i \in \{2, \ldots, d\}$, we move it to a vertex $\mathbf{y}_i$;
- if the token is located at a vertex $\mathbf{y}_i$ for some $i \in \{2, \ldots, d\}$, we move it to the neighbor of $\mathbf{y}_i$ other than x_i.

Obviously, after no more than $2d-2$ runs of the algorithm, some vertex will be repeated. Since the algorithm describes a walk on the graph G where every odd edge is in $\mathcal{M}$ and every even edge is outside $\mathcal{M}$, it finds an alternating cycle after at most $2d-2$ moves, whence the statement also follows in this case. $\square$

4 Bulgarian National Olympiad - Regional Round

Problem 4.1. *(12.1) Let ABC be an acute triangle with midpoint M of AB. The point D lies on the segment MB and I_1, I_2 denote the incenters of $\triangle ADC$ and $\triangle BDC$. Given that $\angle I_1 M I_2 = 90°$, show that $CA = CB$.*

Problem 4.2. *(12.2) Let N be a positive integer. The sequence $x_1, x_2, \ldots$ of non-negative reals is defined by*

$$x_n^2 = \sum_{i=1}^{n-1} \sqrt{x_i x_{n-i}}$$

for all positive integers $n > N$. Show that there exists a constant $c > 0$, such that $x_n \leq \frac{n}{2} + c$ for all positive integers n.
(Kristian Vasilev)

Solution. (Dragomir Grozev) Applying AM-GM inequality to each term of the right-hand side, we get,

$$x_n^2 \leq \sum_{i=1}^{n-1} x_i \, , \; n > N.$$

Let us set $x_n = a \cdot n + \Delta(n)$, where a is a constant that will be determined later. Vaguely speaking, we want to pick a value for a such that $\Delta(n)$ has a smaller growth rate than n. For $n > N$ we obtain,

$$a^2n^2 + 2an\Delta(n) + \Delta(n)^2 \leq \frac{a(n-1)n}{2} + \sum_{i=1}^{n-1} \Delta(i)$$

$$2an\Delta(n) \leq \left(\frac{a(n-1)n}{2} - a^2n^2 \right) - \Delta(n)^2 + \sum_{i=1}^{n-1} \Delta(i) \qquad (1).$$

The plan is to divide both sides by n so let's determine a such that the term in the brackets of the right side of (1) be of degree n (not n^2 as it is now). It can be easily calculated that $a = 1/2$. Putting it in (1), we get

$$\Delta(n) \leq -\frac{1}{4} - \frac{\Delta(n)^2}{n} + \frac{1}{n} \sum_{i=1}^{n-1} \Delta(i).$$

Now, write

$$\Delta(n) \leq \frac{1}{n} \sum_{i=1}^{n-1} \Delta(i) \, , \; \forall n > N$$

which yields

$$\Delta(n) \leq \max\{\Delta(i) : i < n\},$$

hence $\Delta(n) \leq \max\{\Delta(i) : i \leq N\}$ for any $n > N$. This yields $x_n \leq n/2 + c, \forall n \in \mathbb{N}$ for some constant c. $\square$

Sharper estimate. We want to go a bit further, so let's write

$$\Delta(n) \leq -\frac{1}{4} + \frac{1}{n} \sum_{i=1}^{n-1} \Delta(i) \, , \; n > N$$

Define the sequence $\Delta'(n)$ as follows:

$$\Delta'(n) = -\frac{1}{4} + \frac{1}{n} \sum_{i=1}^{n-1} \Delta'(i) \, , \; \text{ for } n > N \qquad (2),$$

and $\Delta'(n) = \Delta'(n), \text{ for } n \leq N$. Clearly $\Delta(n) \leq \Delta'(n), \forall n \in \mathbb{N}$. We have

$$\Delta'(n+1) = -\frac{1}{4} + \frac{1}{n+1}\left(\sum_{i=1}^{n-1}\Delta(i) + -\frac{1}{4} + \frac{1}{n}\sum_{i=1}^{n-1}\Delta(i)\right), n > N \qquad (3).$$

Subtracting (2) from (3) yields

$$\Delta'(n+1) - \Delta'(n) = \frac{-1}{4(n+1)}\,,\ \forall n > N.$$

Summing up the above inequality gives

$$\Delta'(n) = \Delta'(N+1) - \frac{1}{4}\sum_{k=N+2}^{n}\frac{1}{k}\,,\ n \geq N+2.$$

Therefore, using an well known property of harmonic series,

$$\Delta'(n) \leq -\frac{\ln n}{4} + c$$

where c is a constant. Finally, we get

$$x_n \leq \frac{n}{2} - \frac{\ln n}{4} + c, n \in \mathbb{N},$$

where c is a constant.

Remark. Actually, the precise growth rate of the sequence is $x_n = \frac{\pi}{8}n + o(n)$ which can be seen here https://dgrozev.wordpress.com/2024/02/14/growth-rate-of-a-sequence-bulgarian-2024-mo-regional-round/

□

Problem 4.3. *(12.3) Let $A_0B_0C_0$ be a triangle. For a positive integer $n \geq 1$, we define A_n on the segment $B_{n-1}C_{n-1}$ such that $B_{n-1}A_n : C_{n-1}A_n = 2 : 1$ and B_n, C_n are defined cyclically in a similar manner. Show that there exists an unique point P that lies in the interior of all triangles $A_nB_nC_n$.*

Solution. (Dragomir Grozev) We have nested compact sets (closed triangles), so they have non empty intersection. We prove that they intersect in only one point. It's enough to prove that the three points A_n, B_n, C_n converge to a common point P. Assume it's false. Then there exists some subsequences of A_n, B_n, C_n (which for simplicity we denote again by A_n, B_n, C_n) that converge to points A, B, C respectively and $\{A, B, C\}$ consists of at least 2 elements. Assume, first A, B, C are distinct. Assume wlog that $\angle BAC \leq 60°$. Take n large enough such that A_n, B_n, C_n are close enough to A, B, C respectively. Consider the next triangle $A_{n+1}B_{n+1}C_{n+1}$. Its side $B_{n+1}C_{n+1}$ is far enough from A and so A is outside $\triangle A_{n+1}B_{n+1}C_{n+1}$. But A was a limit point of the sequence $A_n, n = 1, 2, \ldots$, contradiction. Suppose now, $B = C \neq A$. Then $\angle B_nA_nC_n < 60°$ (it tends to 0 actually) and we apply the same argument. We proved that there is a unique point P that's common for all the triangles.

Now it remains to prove a small trifle - namely P is in the interior of all the triangles. It was part of the Bulgarian text. Assume on the contrary P is on some side, say A_nB_n, for some n. But it easily follows that P is outside $\triangle A_{n+2}B_{n+2}C_{n+2}$ contradiction. $\square$

Remark. Observe that the proof remains the same if we only require that $A_{n+1}, B_{n+1}, C_{n+1}$ are on the sides B_nC_n, A_nC_n and A_nB_n respectively, but not too close to the vertices A_n, B_n, C_n, for example the distance from A_{n+1} to both points B_n, C_n is greater than $\varepsilon \cdot |B_nC_n|$ for some fixed $\varepsilon > 0$, and the same for B_{n+1} and C_{n+1} . $\square$

Problem 4.4. *(12.4) Find all pairs of positive integers (n, k) such that all sufficiently large odd positive integers m are representable as*

$$m = a_1^{n^2} + a_2^{(n+1)^2} + \ldots + a_k^{(n+k-1)^2} + a_{k+1}^{(n+k)^2}$$

for some non-negative integers $a_1, a_2, \ldots, a_{k+1}$.

5 Bulgarian National Olympiad - Final Round

Problem 5.1. *Is it true that for any positive integer $n > 1$, there exists an infinite arithmetic progression M_n of positive integers, such that for any $m \in M_n$, the number $n^m - 1$ is not a perfect power (a positive integer is a perfect power if it is of the form a^b for positive integers $a, b > 1$)?*

Solution. (Victor Kostadinov) The answer is yes. Fix a positive integer n and two large distinct primes $p, q > n$. Let $d_p = \mathrm{ord}_p(n), d_q = \mathrm{ord}_q(n)$, $c_p = \nu_p(n^{d_p} - 1), c_q = \nu_q(n^{d_q} - 1)$ and let $c = \nu_p(d_q), d = \nu_q(d_p)$. Pick two sufficiently large constants a, b, and let $M := d_pd_qp^{(a-1)c_p+bc_q-c}q^{ac_p+(b-1)c_q+1-d}$ and finally choose M_n to consist of $m = M(1 + iM)$ for $i = 1, 2, \ldots$. By LTE, we have

$$\nu_p(n^m - 1) = \nu_p(n^{d_p} - 1) + \nu_p(\frac{m}{d_p}) = c_p + c + (a-1)c_p + bc_q - c = ac_p + bc_q$$

(we used that $\gcd(1 + iM, p) = 1$) and similarly $\nu_q(n^m - 1) = ac_p + bc_q + 1$, which are consecutive, i.e. they can't have a common divisor greater than 1 and thus $n^m - 1$ can't be a perfect power. $\square$

Problem 5.2. *Given is a triangle ABC and the points M, P lie on the segments AB, BC, respectively, such that $AM = BC$ and $CP = BM$. If AP and CM meet at O and $2\angle AOM = \angle ABC$, find the measure of $\angle ABC$.*

Solution. (Marin Hristov) Let D be the reflection of P across C, so $AB = BD$ and O' be the circumcenter of $\triangle ABD$. As $\triangle AO'B \cong \triangle BO'D$ and $MB = CD$, we have $\angle O'CB = \angle O'MA$, so $MBCO'$ is cyclic. Now $\angle O'CM = \angle O'BM = \angle AOM$, hence $CO' \parallel AP$. Therefore O' must be the midpoint of AD, implying that $\angle ABC = 90°$. $\square$

Problem 5.3. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that*

$$f(af(b) + a)(f(bf(a)) + a) = 1$$

for any positive reals a, b.
(Lyuben Lichev)

Problem 5.4. *Do there exist 2024 non-zero reals $a_1, a_2, \ldots, a_{2024}$, such that*

$$\sum_{i=1}^{2024}(a_i^2 + \frac{1}{a_i^2}) + 2\sum_{i=1}^{2024}\frac{a_i}{a_{i+1}} + 2024 = 2\sum_{i=1}^{2024}(a_i + \frac{1}{a_i})?$$

Solution. (Victor Kostadinov) Suppose that such a_i exist. The condition rewrites as $\sum_{i=1}^{2024}(a_i + \frac{1}{a_{i+1}} - 1)^2 = 0$, which means that $a_i + \frac{1}{a_{i+1}} = 1$ for all $i = 1, 2, \ldots, 2024$. It's easy to obtain consecutively that for any $i = 1, 2, \ldots, 2024$, $a_{i+1} = \frac{1}{1-a_i}$, $a_{i+2} = 1 - \frac{1}{a_i}$, so $a_{i+3} = a_i$. Since $3 \nmid 2024$, all a_i are equal to some real k. However, $k + \frac{1}{k} = 1 \iff k^2 - k + 1 = 0$ doesn't have a real solution, contradiction. $\square$

Problem 5.5. *Let $\mathcal{F}$ be a family of 4-element subsets of a set of size 5^m, where m is a fixed positive integer. If the intersection of any two sets in $\mathcal{F}$ does not have size exactly 2, find the maximal value of $|\mathcal{F}|$.*

Problem 5.6. *Given is a triangle ABC and a circle ω with center I that touches AB, AC and meets BC at X, Y. The line through I perpendicular to BC meets the line through A parallel to BC at Z. Show that the circumcircles of $\triangle XYZ$ and $\triangle ABC$ are tangent to each other.*

Solution. (Victor Kostadinov) Let W be the midpoint of the major arc BAC, $A' \in (ABC)$ be such that $AA' \parallel BC$, T be the intersection of the circle with diameter AI and (ABC) and let ω touch AC, AB at E, F. We claim the two circles touch at T.

Firstly, observe that $Z \in (AEF)$, so

$$\begin{aligned}\angle ATZ = \angle AIZ &= 90^\circ - (\frac{\alpha}{2} + \gamma)\\ &= \frac{\beta - \gamma}{2} = \beta - (90^\circ - \frac{\alpha}{2})\\ &= \angle ABC - \angle WBC = \angle ABW = \angle ATW,\end{aligned}$$

hence T, Z, W are collinear. Let $TW \cap BC = P$. By spiral similarity and angle bisector theorem we have $\frac{PB}{PC} = \frac{TB}{TC} = \frac{BF}{CE}$, so by converse of Menelaus theorem for $\triangle ABC$ we obtain that EF, TZ, BC are concurrent at P. Thus, by power of point at P, we obtain that $PX \cdot PY = PE \cdot PF = PZ \cdot PT$, so $XYZT$ is cyclic.

Finally, observe that the center of (ZXY) lies on IZ, so (ZXY) touches AA' at Z. Hence, by shooting lemma, since W is the midpoint of the minor arc AA', we obtain that (ZXY) touches (ABC) at T and we are done. $\square$

6 TST for BMO 2024

Problem 6.1. *(Day 1, problem 1) Some cities of country Graphland are connected with roads provided that*

- *(i) from each city we can reach any other city.*

It turned out that every city A can choose its favourite number $1 \leq f(A) \leq 2024$ which is an integer, such that the following condition holds:

- *(ii) the favourite numbers of any two cities connected with a road are different.*

Let $1 \leq m \leq 2024$ be a given integer. A tourist arrives in the capital of Graphland. He can move from city A to city B (in this direction), if and only if there is a road connecting A and B and additionally

$$f(B) - f(A) \equiv m \pmod{2024}.$$

For which values of m it is guaranteed (no matter of the cities and roads, provided that (i) and (ii) hold) that the cities can choose their favourite numbers (complying with (ii)) in a way that the tourist can reach any city starting from the capital?
(Dragomir Grozev)

Solution. Answer: For each m coprime with 2024. Let us first assume that$(m, 2024) > 1$. Then the tourist, starting from the capital with a favourite number r, can reach only cities with numbers $km + r \pmod{2024}, k \in \mathbb{N}$. Since it is not a complete system of residues modulo 2024, it means that the set of all favourite numbers consists of less than 2024 numbers. Take now 2024 cities and connect every two of them with a road. This is an example for which (i) can hold if all colours (favourite numbers) are used. That is, in this particular case it's impossible to reach every city starting from the capital.

Let us assume now that $(m, 2024) = 1$. Denote by V the set of all cities and let $W \subset V$ be the maximal possible subset of vertices such that we can assign favourite numbers and starting from the capital v_0 we can reach every vertex in W. We prove $W = V$. Assume for the sake of contradiction $W \neq V$. Then, for any vertices $u \in W, v \in V \setminus W$ that are connected, we have

$$f(v) - f(u) \neq m \pmod{2024} \qquad (1)$$

Now, we construct a different mapping f_1 by changing the assigning of vertices in $V \setminus W$. We set,

$$f_1(v) \in [1..2024]\,,\, f_1(v) = f(v) - m \pmod{2024}, \forall v \in V \setminus W.$$

For any $v \in W$ we set $f_1(v) = f(v)$. Observe that under the new mapping f_1, condition (i) still holds. Indeed, if $f_1(u) = f_1(v)$ for some connected vertices $u \in W, v \in V \setminus W$, it implies $f(u) = f(v) + m$ and thus, we would reach v from v from u under the mapping f, which contradicts the maximality of W.

Therefore, the mapping f_1 satisfies (i), and moreover starting from v_0 we can reach any vertex in W. Since W is maximal, it's not possible to access any vertex outside W. It allows us to change the assignment of favourite numbers again using the rule

$$f_2(v) = f_1(v) - m \pmod{2024}, \forall v \in V \setminus W$$

and so on. Suppose now, $u \in W, v \in V \setminus W$ are connected. There exists k for which $f(u) + m = f(v) - km$, because $\{km : k = 1, 2, \ldots, 2024\}$ is a complete system of residues modulo 2024. But this means that under the mapping f_k, the vertex v is accessible from u, which contradicts tha maximality of W. $\square$

Problem 6.2. *(Day 1, problem 3) The segment BC is fixed in the plane. Let A be a point, such that $\triangle ABC$ is acute. Let BP ($P \in AC$) and CQ ($Q \in AB$) be altitudes in $\triangle ABC$ and O is the center of the circumscribed circle about ABC. The points K and L are symmetric to O with respect to the lines AB u AC respectively. The lines through P and Q, orthogonal to LP and KQ respectively, meet at T. Prove that when A varies, the point T lies on a fixed line.*
(Aleksander Ivanov)

Problem 6.3. *(Day 2, problem 1) Find all functions $f : \mathbb{R} \to \mathbb{R}$, that satisfy the following condition*

$$f\left(f(x) + xf(y)\right) = xf(y+1)\,, \forall x, y \in \mathbb{R}.$$

(Alexander Ivanov)

Solution. Let $f(0) = a, a \in \mathbb{R}$. We set $x = 0$ and obtain $f(a) = 0$. Next, $y = a$ yields

$$f(f(x)) = xf(a+1) \qquad (1).$$

Assume first that $f(a+1) \neq 0$. Then f is an injection. Indeed, assuming $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$, after puttig these values in (1) we get contradiction. Set $x = 1$ in the initial condition. Since f is injective, it follows

$$f(1) + f(y) = y + 1 \qquad (2)$$

Setting $y = 1$ yields $2f(1) = 2$, hence $f(1) = 1$. $f(y) = y + 1$. Now, (2) gives us $f(y) = y, \forall y \in \mathbb{R}$, which indeed is solution.

Let now consider the case $f(a+1) = 0$. Then

$$f(f(x)) = 0, \forall x \in \mathbb{R}.$$

This means that $f(y) = 0, \forall y \in \mathrm{Im}(f)$. Assume that there exists a point y_0, for which $f(y_0 + 1) \neq 0$. Setting $y = y_0$ in the initial condition yields

$$f\left(f(x) + xf(y_0)\right) = xf(y_0 + 1) \qquad (3).$$

For any $x_0 \in \mathbb{R}$ putting $x := x_0/(f(y_0+1))$ in (3), we get,

$$f\left(f(x) + xf(y_0)\right) = x_0.$$

Since x_0 is arbitrary, it means that f is surjective, which implies $\mathrm{Im}(f) = \mathbb{R}$. Therefore, $f(y) = 0, \forall y \in \mathbb{R}$, which contradicts the assumption $f(y_0 + 1) \neq 0$. So, in this case we get $f(x) = 0, \forall x \in \mathbb{R}$, which also is solution. $\square$

Problem 6.4. *(Day 2, problem 2) Given a scalene triangle ABC. On the rays $AC^{\rightarrow}$ and $BC^{\rightarrow}$, the points C_a and C_b are chosen, respectively, such that $AC_a = BC_b = AB$. We denote by O_c the center of the circumcircle about $\triangle CC_aC_b$. Analogously, we define the points O_a and O_b. Prove that the lines AO_a, BO_b and CO_c intersect at a point that lies on the circumcircle of the triangle ABC. (Alexander Ivanov)*

Solution. Let O and I be the centers of the circumscribed and inscribed circle of ABC, and A_1, B_1, C_1 be the centers of the arcs $\widehat{AB}, \widehat{AC}$, $widehatBC$ (not containing the third vertices) of the circumscribed circle. By symmetry with respect to AA_1 we have $A_1C_a = A_1B = A_1C$ and together with $O_cC = O_cC_a$ it follows that A_1O_c is the segment bisector of CC_a. Analogously, B_1O_c is the segment bisector of CC_b. Thus, the quadrilateral $OA_1O_cB_1$ is a parallelogram with $OA_1 = OB_1$, i.e. rhombus. In other words, O and O_c are symmetric about A_1B_1, and this is also true for C and I, i.e. CO_c is symmetric to OI with respect to A_1B_1. Analogously, AO_a and BO_b are symmetric on OI with respect to B_1C_1 and A_1C_1, respectively. Since I is the orthocenter of $A_1B_1C_1$, finally the lines AO_a, BO_b, and CO_c intersect at the Anti-Steiner point for the line OI and triangle $A_1B_1C_1$ (circumscribed about ABC). $\square$

Problem 6.5. *(Day 2, problem 3) Let $n \geq 4$ be an integer number and $S_n = \{1, 2, 3, \ldots, 2^n\}$. Two sets A, B are given, $A \subset S_n, B \subset S_n \setminus S_{n-1}$, such that $|A| = n+1, |B| = 2$. Is it possible $ab - 1$ be a perfect cube for any $a \in A, b \in B$? (Dragomir Grozev)*

Solution. Answer: NO. Let us argue by contradiction. Arrange the numbers in A and B in increasing order $1 \leq a_1 < a_2 < \ldots a_{n+1} \leq 2^n$ and $2^{n-1} < b_1 < b_2 \leq 2^n$. Apparently, there exists an index $i \leq n$ such that $a_i < a_{i+1} \leq 2a_i$. We denote:

$$a_ib_1 - 1 = q_1^3\,,\ a_{i+1}b_2 - 1 = q_2^3\,,\ a_ib_2 - 1 = s_1^3\,,\ a_{i+1}b_1 - 1 = s_2^3.$$

From $a_i < a_{i+1} \leq 2a_i$ и $b_1 < b_2 < 2b_1$ it folows

$$q_1 < s_1, s_2 < q_2 < 2q_1 \qquad (1).$$

Therefore

$$\left(q_1^3 + 1\right)\left(q_2^3 + 1\right) = \left(s_1^3 + 1\right)\left(s_2^3 + 1\right)$$

which yields

$$\left(q_1q_2\right)^3 + q_1^3 + q_2^3 + 1 = \left(s_1s_2\right)^3 + s_1^3 + s_2^3 + 1 \qquad (2).$$

Using (1) it can be seen that if $q_1q_2 > s_1s_2$ or $q_1q_2 < s_1s_2$ the equality (2) cannot hold. Thus, $q_1q_2 = s_1s_2$. Let us consider the function $f(x) = x^3 + \frac{C}{x^3}$ in he interval $x \in [q_1, q_2]$, where $C = (q_1q_2)^3$. I decreases in $\left[q_1, \sqrt{q_1q_2}\right]$ and increases in $\left[\sqrt{q_1 q_2}, q_2\right]$, hence it attains its maximum value in the ends of this interval, $f(q_1) = f(q_2) = q_1^3 + q_2^3$. Therefore, the equality (2), provided that (1) holds, is true only if $(q_1, q_2) = (s_1, s_2)$, which is the needed contradiction. $\square$

Problem 6.6. *(Day 2, problem 4) Let n be a natural number. King Arthur has invited $2^n - 1$ knights to an audience in Camelot. Merlin the Magician arranged the knights in a list numbered from 1 to $2^n - 1$. It turned out that any two knights with numbers $a, b, a < b$ are friends if and only if $0 \le b - 2a \le 1$. The king chose a natural number k and ordered Merlin to make a new list with the following requirement. For each $1 \le i \le 2^n - k - 1$, all friends of the knight with sequence number i (in the new list) must be in positions among $1, 2, \ldots, i + k$. Prove that the smallest k, for which Merlin can fulfil Arthur's wish, satisfies the condition*

$$\frac{1}{100} \cdot \frac{2^n}{n} \le k \le 100 \cdot \frac{2^n}{n}.$$

(Dragomir Grozev)

Solution. Let us construct a graph T with vertex-set which is the set of all knights, numbered as in the first list. Two vertices are adjacent if the corresponding knights are friends. It can be seen that T is a fully balanced binary tree - see fig. 1. We label each vertex with the knight's number in the first list. Let us assume the vertices can be arranged in a row $i_1, i_2, \ldots i_m$ as King Arthur requested, where $m = 2^n - 1$ and i_j refers to the label of the corresponding vertex. There is a unique path in T, $i_1 = v_1 v_2 \ldots v_\ell = i_m$ that connects the vertex labeled as i_1 and i_m. Clearly, $\ell \le 2(n-1) + 1$, because the longest path in T has length $2(n-1)$. Note that the distance between the positions of v_i and v_{i+1} in the new list, is at most k. This means that $(\ell - 1)k \ge m - 1$ which yields

$$k \ge \frac{2^n - 2}{2n - 2} > \frac{2^n}{4n},$$

which proves the lower bound for k.

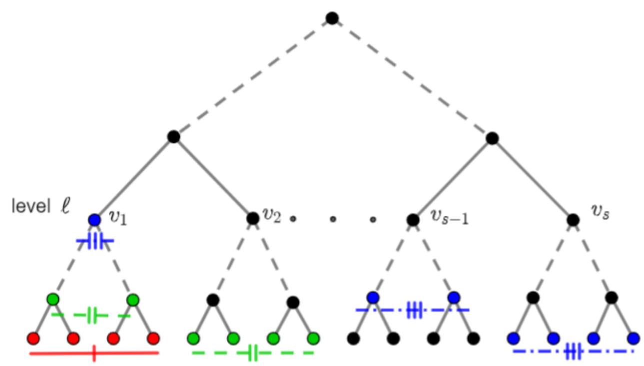
Now we will arrange the vertices of T in a list. Let ℓ be a natural number which will be determined later. Denote by $v_1, v_2, \ldots, v_s, s := 2^\ell$ the vertices of T on the ℓ-th level, and let $T(v_1), T(v_2), \ldots, T(v_s)$ be the subtrees with roots in these points - see fig. 1. Each of them has exactly $2^{n-1-\ell}$ leaves.

We successively put in a list the vertices of T as follows. First, we place the last level of vertices of $T(v_1)$, that is, its leaves. Then we put down the second to last level of $T(v_1)$ and the last level of $T(v_2)$. At the i-th step we place the i-th level of $T(v_1)$, counting from the bottom up (fig. 1), then the $i-1$-th level of $T(v_2)$ (from the bottom up) and so on, and finally - the last layer of $T(v_i)$ (i.e. its leaves). The number of vertices we place at the i-th step $i = 1, 2, \ldots, s - \ell - 1$ is equal to

$$\sum_{j=0}^{i-1} 2^{n-1-\ell-j} \le 2^{n-\ell}.$$

We follow these steps until one of the two events happens. 1) We reach the root v_1 of $T(v_1)$. 2) We place in the list the leaves of $T(v_s)$. The first event will

Фигура 6



happen after $n-\ell$ steps and the second one - after $s = 2^\ell$ steps. To ensure that the second event occurs first we choose ℓ to be the largest positive integer for which $s = 2^\ell \leq n - \ell$.

In this situation, at the s-th step we have put the leaves of $T(v_s)$ in the list. On the $s+1$-th step we put in the list the remaining vertices of T. The number of all vertices in $T(v_s)$ without its last level does not exceed $2^{n-\ell-1}$. The number of vertices in $T(v_{s-1})$ without its last two layers does not exceed $2^{n-\ell-2}$ and so on. Adding the vertices of T up to its ℓ-th level, we obtain that the number of the vertices ordered at the last step is at most

$$2 \cdot 2^\ell + 2^{n-\ell} \leq 2 \cdot 2^{n-\ell} \leq 8 \cdot \frac{2^n}{n}$$

since $2^{n+2} \geq n$. Clearly, for any vertex v, placed at position j in the first s steps, all of its neighbours in T, that are placed after it, are in positions with numbers not exceeding $j + 2 \cdot 2^{n-\ell} \leq j + 8 \cdot \frac{2^n}{n}$. Taking into account the number of vertices added in the last step, we get that in the constructed list the condition imposed by the king holds for

$$k := \left\lceil 16 \cdot \frac{2^n}{n} \right\rceil$$

This proves the upper bound. $\square$