Zdravka Paskaleva, Maya Alashka, Raina Alashka, Zvezdelina Stankova

BULGARIAN MATH 5 OR 9A TEXTBOOK

adapted for U.S. High Schools
used in many public schools abroad

Translated, Adapted, and Augmented
by
Zvezdelina Stankova
Founder and Director
of the Berkeley Math Circle

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PUBLISHING HOUSE

BERKELEY
MATH CIRCLE

Icons, Symbols, and Notation Used in the Textbook

D Definition

A Axiom

T Theorem

E Elementary construction

! Facts to be remembered!

Problem solving technique or clarification to the solution of the problem

Exercises are grouped by level:

1. For everyone
2. For excellent proficiency
3. Require thinking or skills beyond the current material

Write	Read
$\Rightarrow$; $\Leftrightarrow$	implies that; is equivalent to
LHS, RHS	left-hand side, right-hand side
$\alpha, \beta, \gamma, \pi, \varphi, \psi, \nu$	alpha, beta, gamma, pi, phi, psi, nu
$\Delta, \Sigma, \Omega\ (\omega)$	Delta, Sigma (= sum of), Omega
A', A_1	A-prime, A-sub-one
wrt	with respect to
$A\ \mathbf{z}\ a,\ A \in a,\ a\ \mathbf{z}\ A$	point A lies on line a
$M \equiv N,\ M = N$	M coincides with (is same as) N
$M \not\equiv N,\ M \neq N$	M does **not** coincide with N
$OA^{\rightarrow}, \angle AOB, \Delta ABC$	ray OA, angle AOB, triangle ABC
$a \times b = a \cap b = M$	lines a and b intersect in point M
$a \perp b;\ a \parallel b$	a and b are perpendicular/parallel
h_a, m_a, l_a (to side a)	altitude, median, angle bisector
s_{AB}	perpendicular bisector of AB
P_F, p_F, S_F	perimeter, half-perimeter, area of F
$\Delta ABC \cong \Delta A_1B_1C_1$	ΔABC is congruent to $\Delta A_1B_1C_1$
SAS, ASA, SSS, SSR	Δ congruence criteria:
	S = side, A = angle, R = right angle
$\Delta ABC \sim \Delta A_1B_1C_1$	ΔABC is similar to $\Delta A_1B_1C_1$
AA, RAR, RRR, LH	Δ similarity criteria:
	R = ratio, L = leg, H = hypotenuse
$k\,(O, r)$	circle k with center O and radius r
$\underrightarrow{t}$	translation along vector $\vec{t}$
$r\,(O, \alpha)$	rotation with center O and angle α
S_O, S_g	symmetry across point O/line g
AV (or D)	allowable values (or domain)
$\mathcal{D}$	discriminant $b^2 - 4ac$
$\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{I}, \mathbf{R}; \varnothing$	set of natural, integer, rational,
	irrational, real numbers; empty set
$a \in A,\ A \ni a$	a is in set A; set A contains a
$A \cup B$ (or $A + B$)	union (or sum) of sets
$A \cap B$ (or AB)	intersection (or product) of sets
$A \setminus B$ (or $A - B$), $\overline{A}$	difference, complement of sets
$A \times B$	Cartesian product of sets
$\left\lvert \begin{array}{l} f(x, y) = 0 \\ g(x, y) = 0 \end{array} \right.$	system of two equations with two unknowns x and y
$n!$	n factorial
$\nu(A)$	size of a set A (number of elements)
P_n, V_n^k, C_n^k	permutations, variations, combinations
	[of class k] from n elements, no rep's
Ω, ω	outcome space, outcome (omega)
$P(A)$	probability of event A

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Foreword

No two people will agree entirely on all features of the "best" math curriculum, nor how it should be taught. Consequently, you need to read this introduction with an open mind, accepting that there will be differences of opinion and in approach, knowing that the possible "error" in how much the outlined philosophy differs from your views is a proof of the *seriousness of the situation with the U.S. math education.*

We all want to improve the math curriculum **and** the math experience, **and** the math proficiency, **and** ultimately the mathematical success of our children. Do we want too much?

> From the education battlefield:
>
> *"You have done a masterful job of adapting this curriculum to the U.S. reality.*
>
> *The U.S. Education System needs this kind of prodding. The Common Core ideas are heading in the right direction but th`e instruments of change are too unwieldy and we cannot just stand by and wait to see what will happen.*
>
> *I hope that your efforts will be successful. Thanks for all of your hard work."*
>
> *Tom Rike*
> *Veteran math teacher*
> *Oakland High School*
>
> *Copyeditor of the Textbook Series*

We want a **lot**: that is clear. And a **lot** has to be done in order to achieve this goal: a lot of sacrifices, hard work, and compromises between the views of various worlds. For the U.S. is a land where cultures mix (or don't mix but co-exist), a land of opportunity, and a land of contradictions and extremes.

Our children come first: **before** our self-centered goals for self-improvement and well-being, **before** ourselves. Sacrificing personal comfort and personal self to the benefit of one's students (or one's children) is a sign of a great teacher (and a great parent).

When something powerful comes along and rings true, along with the listeners will come the frightened ones, who will be afraid to change, who will be scared to make the leap and try something new in the U.S.: something that has worked for a century elsewhere.

And such people will raise angry voices of distrust and will try to stamp out anything tender that has not yet taken root. But without such angry voices there will be no reality check, no feel for the seriousness of the situation.

This introduction addresses a **typical** situation in a U.S. school. Exceptions do exist. There is no point in starting with this textbook series:

- unless you are ready to be completely *honest, goal-oriented, and selfless* as a teacher, a parent, or an administrator;
- unless you are ready – instead of trying to squash something new because it differs from what you are used to – to *embrace a philosophical viewpoint and a practical attitude that have proven tremendously successful elsewhere* and that have been *adapted to the U.S. reality and tested in a pilot school in the San Francisco Bay Area.*

Thus, we hope that you take this new math curriculum seriously and enjoy it too.

Mathematics is a powerful universal language. Everyone needs to and can own it to the extent of his/her abilities and potential.

Let this new math curriculum be a big step for the students and community at your middle and high school towards reaching this goal.

Zvezdelina Stankova
Ph.D., Harvard University
Author, Translator, Adapter, and Co-Author
Teaching Professor of Mathematics, UC Berkeley
Berkeley Math Circle Founder & Director
Deborah and Franklin Tepper Haimo Award
for Distinguished College/University Teachers
Princeton Review's "300 Best Professors" in the U.S.

Reviews of the Program
by Educators and Parents

Dr. Norm Prokup joined the faculty at College Preparatory School in Oakland, CA (the preeminent high school in the SF East Bay Area) in 2003. He has taught at the Berkeley Math Circle and the BMC Summer Program, and very notably, he has been the Geometry instructor for the 8A-9A groups at the MTRW Program. He graduated from Cornell University and received his Ph.D. from the University of Rhode Island at Kingston, studying nonlinear difference equations that model two-stage "healthy-infected" populations, suggested by epidemiologists. He has taught math ever since, working with students from grade 6 through college and covering a wide range of subjects, from algebra to ordinary differential equations.

"How powerful is the Bulgarian approach to algebra and geometry?

It surpasses the traditional U.S. approach in two key ways:

1. *Bulgarian Math presents algebra and geometry simultaneously over three years, guaranteeing much better retention than students can get from the traditional sequence: Pre-algebra, Algebra 1, Geometry, and Algebra 2.*
2. *The Bulgarian approach is more thorough. Although the authors didn't have the American Common Core standards in mind, the texts work exceptionally well with Common Core. Students who learn algebra and geometry from these texts develop superior skills in problem-solving, logic and reasoning, algebraic manipulation, and computation.*

I grew up in northern Virginia. I had good teachers, many of whom cared enough about me to get angry when I wasn't doing my best. Growing up, math was my favorite subject, so I always started my math homework first. From engineering school, I learned the value of computation, precision, and fluency of execution. From mathematics, I learned that to really understand a subject, you must be able to take it apart and then put it back together. These values resound throughout the Bulgarian curriculum.

All my favorite math problems involve pictures, so I consider myself a geometer at heart.

- *Solid geometry, vector geometry, plane transformations, and other topics that are often marginalized in an American curriculum play a central role in the Bulgarian texts. Hence, the Bulgarian curriculum is ideal training for future physicists, engineers, and computer scientists.*
- *Students emerge from this program ready to study analytic geometry and functions. Thus, this curriculum would be a superior choice for students from all grades 6 to 12.*

I use the Bulgarian curriculum for my geometry classes at the "Math Taught the Right Way" Program at UCB. My students, who range from grade 7 to 10, have embraced this curriculum.

- *I love teaching from these texts, and I love sharing them with other teachers.*
- *I eagerly await each chapter as it emerges, newly translated into English, and*
- *I'll be first in line to buy each new volume as it becomes available.*

The miracle of the Bulgarian Math Program is that it works exceptionally well with a group of mature and industrious younger students. Indeed, the purpose of the Bulgarian texts is to provide the best possible training for tomorrow's mathematicians and scientists.

Dr. Norm Prokup

Elena Blanter received her Master's degree in mathematics from Saint Petersburg University in Russia. Elena's many years of experience in industry span mathematical risk modeling, statistical analysis, and development of financial analytical software. Elena is a co-founder of the Berkeley Math Circle Elementary Level (for grades 1-4) and has been teaching there since its inception in 2009. She is also a co-founder of the Stanford Math Circle Elementary Level at Stanford University, launched in 2011. Elena is thrilled to have the opportunity to bring her experience and passion to the Math Taught the Right Way Program (MTRW) at UC Berkeley.

When I taught a problem-solving class at MTRW in 2018, I was once again reminded of how ...

... fragmented and incomplete mathematics curriculum is in a typical American middle school. Even motivated and bright students do not have adequate mathematical language; they lose confidence when trying to solve a problem that is not given in a familiar format; they find themselves at a loss when presented with an atypical word problem, even when they have all the knowledge required to solve it.

I grew up in the former Soviet Union. As a child, I remember the thrill of getting a new math textbook at the beginning of each school year and looking through it in anticipation of learning new and exciting mathematics. Every Sunday morning during my school years I took a bus across town to attend a math circle class. *When my three children were growing up in the U.S., I wanted them to discover the beauty of mathematics.* In their elementary school years, I supplemented their standard school curriculum with the Singapore Math textbooks available in the U.S., with several editions of Russian math textbooks, and with the Math Circle Elementary material.

Like learning a new language, becoming proficient in math takes time and practice. The earlier one starts, the easier it is and the greater proficiency one achieves. In a typical Eastern European mathematics program, middle school children are taught different areas of mathematics, starting in 6^{th} grade and continuing through middle and high school. Such an approach supports giving children a solid mathematical foundation.

My children attended different public and private middle schools in the U.S. None of their math textbooks covered math subjects broadly and geometry was often absent from middle school math. *I wish the Bulgarian middle school textbooks had been available at the time. This program covers many areas of mathematics continuously over the course of 3 years, including algebra, geometry, logic, reasoning, statistics, and problem solving.* There is no one-year "crash" course in geometry or any other math discipline. *The Program supports math knowledge acquisition and retention at a steady pace throughout all middle school years.*

The exercises in textbooks are split into several levels to help with differentiation.

Math contest problems, interspersed throughout the book, can be assigned to advanced students.

Supplemental materials, offered as independent work, can add a "math circle" element and challenge even the most advanced students.

The Bulgarian program teaches children not just to look for the right formulas to plug in the numbers, or for the right template or a recipe to use; *it helps teach students how to approach and solve a problem as a mathematician: "Let's pause to understand the problem, let's take a piece of paper and a pencil, and let's think".*

I am excited to finally have an excellent middle school math program, and I love teaching it.

Elena Blanter

* * *

These textbooks emphasize the logical structures that seem to lack in the U.S. curriculum. Even starting in the 6A-B series, one can see the *build up in knowledge* – latter sections require results from previous ones and clearly state so – and *the cohesiveness of the algebra and geometry,* where the two subjects support each other. When used correctly, the books can explain most results in one lesson, using previous lessons. This structure helps students think mathematically. A true mathematician always asks "Why is this true?" when given a result. It is great to see this type of thinking exposed and instilled in students at such a young age.

In the U.S. math is commonly taught by memorizing algorithms without justifications: "It works because I told you so." And everyone knows that this does not sit well with a rebellious teenager. *The essence and beauty of math is whether or not an answer to a question can be found.* I was taught to question the results when I started 6^{th} grade axiomatic geometry, using translated French textbooks in Vietnam. And I am ecstatic to see that the present Bulgarian series *provides students the same support and opportunity to question and develop their understanding of mathematics.*

Harry Mai-Luu, Math Major at UCB
BMC/MTRW assistant and instructor

Here follow reviews of the Bulgarian Math curriculum by two more instructors, Elysée W.-Egolf and Ellen Kulinsky, of the Algebra, Geometry, and Problem Solving/Proofs materials at the "Math Taught the Right Way" (MTRW) Program at UC Berkeley.

I grew up in the U.S. with textbooks ...

... that were rote and unmotivated, with no applications for algebra students, no geometry or number theory for 6th graders, and no guidelines in writing mathematics. The curricula that are popular in the Bay Area at the time of writing have sought to resolve this issue by making the entirety of student education exploratory, without practice or summarization of what is expected - and in both paradigms, the textbook authors take dozens of pages to get to the point.

The current adapted Archimedes curriculum, by contrast, gives examples for students to explore and then guides them to correct conclusions. Algorithms are offered only after clarifying their logic and students are taught to work clearly and thoroughly through both numerical exercises and problems requiring more complex reasoning - and all this while calmly weaning reliance from calculators. It picks up where books like Singapore Math leave off, and offers students of all backgrounds and abilities entry into the deeper understanding they require to master future material.

Because the topics are covered so thoroughly and a workbook is included with extra problems, there is absolutely no need for teachers to create additional worksheets, which means that the teacher can focus on planning lessons and evaluating student work. The Bulgarian Math textbook is more relaxing for both student and teacher to work from.

Elysée Wilson-Egolf
BA in Mathematics, UC Berkeley
Urban School of San Francisco
Instructor at MTRW and BMC

These texts are such an improvement from ...

... the typical textbooks used in the American math education system. Rather than merely presenting formulas, these materials focus on their derivation and various applications. The result is remarkable: studying math becomes an engaging exploration for the students, rather than the tedious memorization and monotonous plug and chug students may be used to in the American math curriculum.

The books contain beautiful explanations and plenty of examples, easing students into challenging problems. And although the books cover a wide range of mathematical subfields (geometry, algebra, etc.) the students are able to see how concepts from different subfields are integrated, as they are in the real world, outside the classroom.

I attended a Math Circle while growing up, which exposed me to creative, thought-provoking problem-solving approaches, as well as the connections between mathematical subfields, so I knew that the math classes I was taking in middle and high school were not a fair representation of how enjoyable studying math could be. These are the first math books I have come across that bridge the gap and incorporate the depth, clarity, and excitement of a Math Circle, while being tailored to the American math standards.

These materials are the perfect tool for students and math teachers alike to truly experience and share the beauty and joy of studying mathematics.

Ellen Kulinsky
BA in Mathematics, UC Berkeley
Instructor at MTRW and BMC

Dr. Kelli Talaska, a long-time instructor at the Berkeley Math Circle and its Summer Programs, as well as the Algebra instructor for the 8th and 9th grade groups at the "Math Taught the Right Way," *shares* her view *on why* it is important in mathematics:

- to *share* the *"why"* (pun intended!),
- in addition to the *"what"* and the *"how,"*

and *how* this textbook series achieves that.

> *Again and again, I find that the Bulgarian curriculum has problems that are designed to force students to improve their critical thinking skills. Most often, this is a consequence of studying not only the most common sorts of examples and exercises, but also the strange boundary cases and modifications of standard problems. This gets students to take a step back and think about when and why our formulas and algorithms work, rather than blindly memorizing.*
>
> *For example, many of the 8B topics are addressed much later in the typical American math curriculum, often not until high school. Students with a competent background are absolutely prepared to study functions and their transformations, to examine important modeling concepts like direct and inverse variation, and to make judgment calls on the most efficient method for solving a particular system of linear equations. If they are trained to understand WHY, they can handle technical intricacies like domain restrictions and piecewise functions and inequalities involving absolute values.*
>
> *The Bulgarian textbooks hit these topics head on, and I have found that the students appreciate being given more of the full story from the beginning.*
>
> *Kelli Talaska, Ph.D.*
>
> *University of California at Berkeley*
> *Instructor at MTWR and BMC*

Here is a fresh and succinct quote from a 9th grade student at the "Math Taught the Right Way" on:

- *what* the 9A textbook has given him;
- *how* it differs from high school math.

> *The 9A textbook provides a wide range of problems that are more challenging and have helped me to think in a more nuanced way than most of the problems I have encountered in High School.*
>
> *Isaac Salemme*
> *MTWR Student*

Last but not least, many parents and teachers are concerned how this textbook series fits within the *Common Core Mathematics Standards*. To calm such an anxiety, here is a quote straight from the source:

> *This Middle and High School Math Program is deeply aligned with the Common Core Mathematics Standards.*
>
> *Dr. Phil Daro*
>
> *One of Three Principal Writers of the Common Core Mathematics Standards*

To understand this quote in its original context *"Within the Global US Picture,"* we strongly encourage you to carefully read the Introduction first, before continuing with the actual math material.

9A Contents

Let the Math Begin!

Chapter 1. Initial Review

Chapter 2. Functions and Linear Systems: Review with Extensions

Chapter 3. Systems of Equations of Degree 2 with Two Unknowns

Chapter 4. Review of Geometry

Chapter 5. Similar Triangles

Chapter 6. Rational Inequalities

Chapter 7. Classical Probability

Introduction to the Program

1. Mission Impossible

BERKELEY MATH CIRCLE

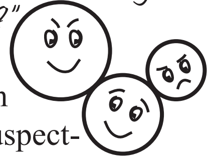
Every year, for the past 20 years, while working* with 500 students in grades 5-12 at the Berkeley Math Circle (BMC), I have been asked over and over again by parents, students, and teachers alike:

> *"Can you give us good textbooks for our middle and high school math curriculum? What we have is not working and it hasn't worked for years!"*

And for these 20 years I could think of only two sources that I would also give to my own children: my *old Bulgarian middle school textbooks*, and the *Art of Problem Solving books (AoPS)*. But **neither of these choices would have worked**, despite the fact that they are, in my opinion, among the best middle school textbooks in the world. Indeed, they are written for *advanced* audiences as measured by U.S. standards:

The very first sentence in my old *Bulgarian 6th Grade Algebra* textbook would stump many of my freshman college students: *"Since we have worked [in 5th grade] with the set of rational numbers* Q *and are familiar with its properties, we move on."* And without further ado, Lesson 9 went on to studying the subtle differences between functions and relations!

My old *Bulgarian 6th Grade Geometry* textbook did not lag behind. On page 42 it asked: *"Is the relation 'is tangent to' on the set of circles in the plane reflexive, symmetric, or transitive?"* Just to explain what the question means to an unsuspecting, typical 6th grader would take a whole lot of time!



Although the *AoPS books* are a wonderful resource, they are not designed as "textbooks" for a regular math class environment. I tend to recommend these books for my more advanced Math Circle students, who are guaranteed to have extra support outside of a typical school classroom.

And so, short of designing a new program myself, I was helpless to help anyone seeking a better middle or high school math curriculum....

2. The Find

....That is, until I asked my mom to send me some recent Bulgarian math textbooks, to teach my own kids in the U.S. She found the most popular textbooks in Bulgaria: the present series from "Archimedes." When I opened them, I was immediately taken by a main feature: *Each lesson is on exactly 2 pages!*

And each lesson consists of:

- *Several examples of the new concept(s) that illustrate how to understand the new material and how to solve related problems.*
- *Up to 10 exercises that differ from each other and that further strengthen the understanding and develop the skills of the reader.*
- *A clear summary of new theory, formulae, algorithms, and consequences to be learned.*

---

* I am a Teaching Professor of Mathematics at UC Berkeley. Originally from Bulgaria, I went through its rigorous mathematics curriculum. I founded BMC and has been running it since 1998 to help pass math enthusiasm on to the younger generation in the San Francisco Bay Area.

The textbooks are written in a *straightforward* and *simple manner*, capturing the *elegance* and *robustness of mathematics* itself. There is no "hide-and-seek":

- everything is *crystal clear;*
- *one look* at a lesson suffices to see what it will teach you;

- there is no need to look around on the Internet for other sources: the *textbook, workbook,* and *problem collection* are all one needs for a *successful course* in middle school mathematics.

A pilot 6th grade teacher shares:

"I feel that the Bulgarian textbooks are much more thorough than the American textbooks I am using for 7th grade. The truth is that had there also been a translated 7th grade Bulgarian textbook I could use this year, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, teacher

3. Origins

This middle and high school math program is based on one of today's leading programs in Eastern Europe, developed and implemented by the well-known Bulgarian mathematician and educator Professor Georgi Paskalev. His relatively new mathematical program specifically targeting middle and high schools in Bulgaria has conquered 60-70% of the Bulgarian education market during the decade and a half since its conception.

The new program:

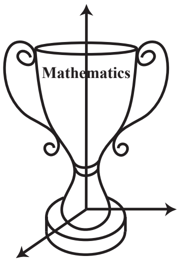
- has *partially departed* from the traditional "old school" Bulgarian program, and
- has *crossed boundaries* to meet the standards and demands of a global educational market, such as in the U.S.; yet,
- has *preserved the intrinsic pedagogical* and *mathematical values* of the old curriculum.

Quite appropriately, the company founded by Professor Paskalev is called "*Archimedes*": its publications live up to the name of the great Greek mathematician, scientist, and inventor from antiquity.*

4. Purpose

The present program extends the mathematics taught in elementary school and completes it to a comprehensive K-12 math program that prepares students:

- to enter high school and college with *confidence* in their math skills and knowledge;
- to be *successful* at any level of high school and first-year college math classes;



- to develop *maturity* and a *life-long relationship* with mathematics that:
  - ➢ will distinguish them among their peers;
  - ➢ will serve them well in any profession and in any endeavor in which they chose to engage.

* Archimedes' picture is from Wikipedia: https://la.wikipedia.org/wiki/Archimedes#/media/File:Archimedes1.jpg.

5. The Temple of Mathematics

The math education of every child is like a *building*: every year we hope that a new floor emerges on top of the previous floor. Unfortunately, more often than not, our children experience *damaging* (even though, perhaps, well-intended) *actions of math educators on any floor of this building*. For example:

> What happens when a teacher decides to spend 3 months in 4^{th} grade reviewing single-digit multiplication?

Well, the 4^{th} graders become really skilled with their multiplication tables, and, say,

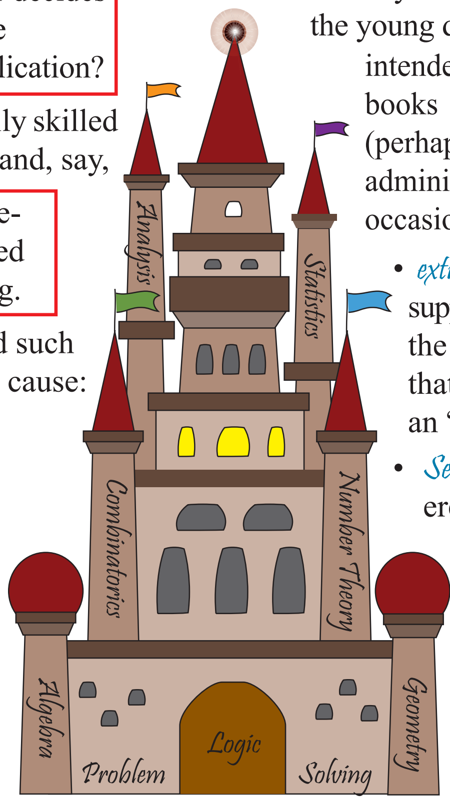
> A 6-foot thick wall of "single-digit multiplication" is erected on the 4^{th} floor of the building.

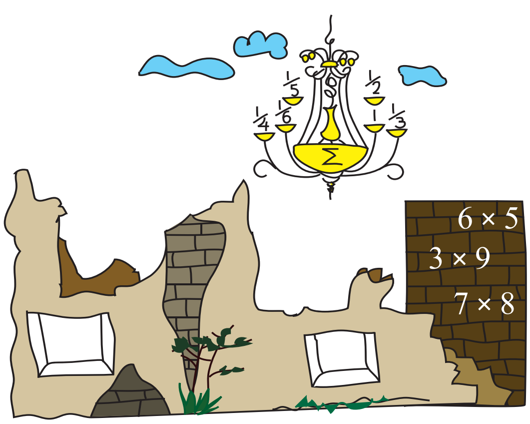
Alas, time is not stretchable, and such educational decisions inevitably cause:

> Another wall on that floor to be thin or non-existent: a "wall" of word problems, logical thinking, or a rich geometric vocabulary, etc.

As a result, when in 5^{th} grade the next teacher tries to build "walls" upon such previously expected but non-existent or weak walls, not surprisingly:

> The structure collapses! In the ensuing debris, time is wasted figuring out what and why it happened. Worse, a student may lose confidence in his/her math skills, in the teacher's abilities, in the integrity of the math program, and, sadly but not infrequently, in mathematics itself.





What may strike us as even *more dangerous* for the young developing mind (yet, still well-intended) is the desire of many textbooks and some teachers nowadays (perhaps, under pressure from school administration or for other reasons) to occasionally show off via:

- *extra-curricular activities* that are unsupported by and incompatible with the math building erected up to that moment. A typical example is an "enrichment" section on:
- *Series*! This topic, ordinarily covered in a college Calculus course, has made its debut in well-known, standard U.S. 6^{th} grade textbooks, alongside computer codes that intend to demonstrate the growth of the "partial sums of the series."

Let us reason logically here.

Question 1. How can one *truly understand* why the sum

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$$

adds up to infinity, but the sum

$$1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

does not, *before* having mastered, say, the *distributivity laws*?! Although U.S. students may be able to recite from memory these distributive laws (e.g., $a \cdot (b + c) = a \cdot b + a \cdot c$), by and large they will have no clue when, why, and how to apply these laws, let alone efficiently.

Question 2. Thus, we ask again: what effect will such a fancy and untimely introduction of high-level math ideas have on the budding young mind? To put it bluntly, the situation will be equivalent to hanging a fancy chandelier from a non-existent ceiling. Imagine the "aftermath" of such an action, the loss of time, energy, and confidence.

And we are **not** talking about the precocious youngster who can handle with ease and excitement any bombardment with advanced math material, whether on series, in combinatorics, abstract algebra, or other such topics! We have plenty of such young people at the Berkeley Math Circle and observe this phenomenon on a weekly basis all the time.

- We are talking about the *typical* U.S. student, whose primary or only source of learning math is through math classes in school. → The math structure for this student will be *jeopardized* by such disorderly and out-of-context teaching.

Just like when learning:

To play the piano, one might first want to learn to read music before attempting *Rachmaninoff's Piano Concerto No. 2;*

To skate, one ought to practice a figure-8 before performing an *Axel jump;*

To write well, one should first concentrate on sentences and paragraphs before producing a literary analysis on, say, "Realism vs. Local Color in *Mark Twain's novels*"...

... So there should be *order, balance,* and *harmony* in everything we do, teach, and learn. We **cannot** lower our standards when teaching mathematics to young minds. On the contrary, we must be especially careful because of the *hierarchical structure of mathematics.*

And thus, the present textbook series* has found the *golden ratios*:*

between
- mathematically sound and
- pedagogically sound;

between
- the practical and
- the inspiring.

The textbook series promotes at all times

Simplicity and Elegance: In anything we do, especially in mathematics.

Courage: Not being afraid to make mistakes and to learn from them.

Diversity of Ideas: Via multiple solutions to the same problem.

First and foremost: *Learning to Think Deeply:* Make connections with previously learned material.

These are not just catchy, "politically correct" phrases intended to sell more copies of this textbook. They truthfully describe this math program and help build a solid math structure, with adjoining "walls, windows, and doors," through which to move from topic to topic and make connections between various concepts, formulas, and ideas.

For someone who has not worked deeply with mathematics before, the words *love, creativity, and excitement* are usually light years away from having anything to do with math. Yet, here they all are, in a testimony of a pilot 6th grade parent, leading us to the next topic – *how a student interacts with and feels about mathematics.*

A pilot 6th grade parent shares:

"Our daughter has learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems. This curriculum keeps her interested and excited."

Eve Maidenberg, parent

* The pun was not intended in the former, but it was intended in the latter instance! ☺

6. Relationship with Mathematics

With some notable exceptions, especially in Geometry, the actual math content of these textbooks does not "look" much different from just about any standard U.S. high school textbook. After all,

A *linear equation* $5x - 7 = 11$ looks just like a linear equation, no matter if it is displayed in a U.S., Uruguayan, or Bulgarian textbook.

The *Pythagorean Theorem* will still say $a^2 + b^2 = c^2$ for the two legs and the hypotenuse of a right triangle.

A shortcut to finding out if a number can be *divided by 3* (without remainder) will still be to check if the sum of its digits is itself divisible by 3.

... because *mathematical truth is universal*.

Beyond the content, two *fundamental questions* that a good teacher must ask about any textbook are:

Question 1: How are the math facts *presented?* In what order and what style? Is there a balance between computational (algorithmic) parts and theoretical (abstract) parts?

Question 2: What *relationship* between the student and the material is developed? The deeper the relationship, the more lasting and more useful the math knowledge.

This textbook series creates the environment for a *strong bond between the student and the material,*

by a *gradual climb:*

in the *difficulty and breath* of math content;

in the expectations of *quality, depth, and rigor* of student work;

by building a robust *Temple of Mathematics:*

always *depending on previous knowledge* and reinforcing it via new applications and without needless repetition;

always looking ahead and *linking current material to new things* to discover.

At any time throughout the school year the place and importance of the concepts, ideas, and techniques studied are *clear within the "bigger picture" of Mathematics*. Unfortunately, just as described on the previous pages, meddling with textbooks and curricula is a wide-spread phenomenon in the U.S. middle and high school math education! Here are four sure ways to *ruin the structure and purpose* of any math textbook, whether good or bad:

1. Omitting sections.
2. Changing the order of the material in the textbook.
3. Cluttering lessons with handouts from other curricula.
4. Sacrificing depth and teaching to the answers!...

Remember that no one wants to climb up a flimsy structure! As a result, over time the top floors of the Math Temple will become uninhabitable if the actions in the red box run amok in math lessons. Instead, the deep thought and math wisdom put into creating these textbooks must be respected, or there is no point in starting with this program. Thus, the present textbooks must be *followed just as systematically* as one studies a new language, violin, or chess.

7. A Cultural Shift

Several features of the textbooks, which are typically ignored or de-emphasized in standard U.S. high school curricula, represent a *cultural shift* in how mathematics is viewed and studied in the U.S., but which are very well-known around the world and have been adopted by many countries for decades, if not centuries.

7.1. Reading Mathematics. Every math problem has "words" – these are concepts expressed in different ways, whether as numbers, standard everyday words, or by visual aids such as diagrams and figures. Emphasizing the different forms of "words" as a regular part of math language and math communication and *learning to read, interpret, and write in "words"* is a major part of the new curriculum. The pilot parent's comments below addresses this and more:

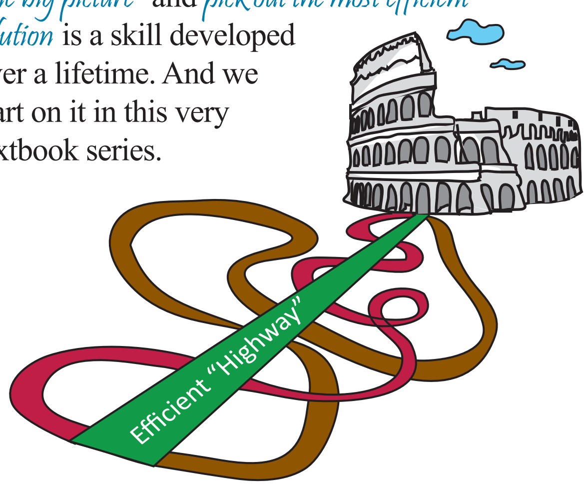
> A pilot 6th grade parent shares, 7 months after the new curriculum was launched:
>
> *"As a teacher and teacher educator, I am familiar with several math programs that have been used throughout the past twenty years in California. The Bulgarian math program is exceptional in the way it supports students' deep understanding of mathematical concepts, develops students' ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.*
>
> *I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics.*
>
> *Jacqueline Regev, parent and educator*

7.2. Logic and Communication. Learning to know *correct from flawed reasoning* and being *able to explain* one's mathematical ideas smoothly and convincingly to others will represent a gradual move towards *a more mature relationship with mathematics*.

7.3. Writing Mathematics. "Showing your work" is only the beginning to understanding how math really works. All problems have answers in the back of the textbook. Bringing back just an answer on a homework problem from the textbook will be worth no credit. *The explanation that leads to the answer* is what counts in our study of high school mathematics. Even harder than learning to correctly interpret problems is learning to consistently *write solutions in a correct, complete, and clear way.* (See Lesson 5 in 9A textbook.)

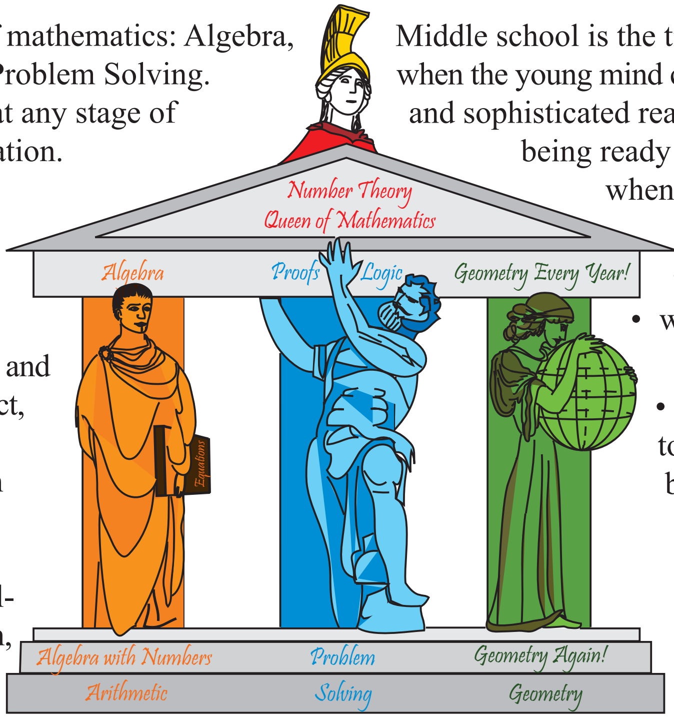
7.4. Multiple Solutions. Although there is usually (but not always!) only *one* correct answer to a math problem, the beauty of mathematics is that there may be *different solutions leading to that answer.* These are not "subjective opinions"; rather, they are objective mathematical creations that obey the laws of Logic. Students will learn that each solution (and even each incorrect attempt) has its value and usefulness in the long run and that one needs to *open up to others' ideas and ways of thinking* as a path to enriching oneself, to becoming more proficient and, ultimately, wiser.

7.5. Efficient Solutions. Yet, among "all the roads [that] lead to Rome," there might be a shortest or an easiest to follow. The ability to *see "the big picture"* and *pick out the most efficient solution* is a skill developed over a lifetime. And we start on it in this very textbook series.



8. The Pillars of Mathematics

There are *three pillars* of mathematics: Algebra, Geometry, and Logic/Problem Solving. They must be present at any stage of pre-college math education.



8.1. Algebra*

There is so much fuss to distinguish between Pre-Algebra, Algebra 1, and Algebra 2... when, in fact, there is no such thing as "Pre-Algebra" as a math subject, nor are there "Algebra 1 or 2"! In a well-designed and well-executed K-12 program,

Arithmetic

↓ gradually turns into

Algebra $\xrightarrow[\text{leads to}]{\text{naturally}}$ *Real Analysis.***

Alas, the typical U.S. student:

- hops along uneven and vaguely defined passages between Pre-Algebra, Algebra 1, Algebra 2, Pre-Calculus, and Calculus;
- experiences a choppy up approach of poorly matched Algebra curricula from grade to grade;
- is taught to be more concerned about memorizing algorithms*** and passing test benchmarks,

than what is truly important:

- *making connections* and fitting the many pieces together in a giant "jigsaw puzzle" to see the *"big picture;"*

and little by little,

- building a *solid Pillar of Algebra* in his/her Temple of Mathematics.

Middle school is the time when math clicks: when the young mind can handle logical leaps and sophisticated reasoning. Yet, instead of being ready for the "big moment," when middle school knocks on the door, teachers and parents alike start:

- worrying about students *"missing algebra skills"*;
- pulling the class back to arithmetic operations because all of a sudden *"pre-algebra" is too hard;*
- going in circles;
- losing time, energy, and confidence; and
- ultimately, *weakening the math structure*.

8.2. The Missing Algebraic Link

Why is this happening? The major Algebra mishaps in U.S. high schools are due to events that have already occurred in middle and elementary school, and are NOT necessarily due to poor arithmetic skills. Expecting that a middle or high school student will leap from *arithmetic operations with numbers,* e.g.,

$$97 \cdot 2016 = 195{,}552,$$

to *algebraic operations with symbols,* e.g.,

$$(a + b)\,(a - b) = a^2 - b^2,$$

is asking for trouble. There is a whole array of missing intermediate steps, to which we refer as *"Algebra with Numbers"* and which can and should be started very early in the learning process.

How early? On the next page we list some examples from *early grades in Bulgaria.* They also represent well what happens in the algebraic progression in other countries around the world.

---

* Algebra was present in ancient Greek mathematics mostly as *geometrical* algebra. The word "algebra" is Latin for the Arabic "al-jabr," or "the reunion of broken parts." In our "divine" interpretation of mathematics, the Greek mathematician *Diophantus, the Father of Algebra,* is supporting the math structure on the left and is the only non-deity in the drawing; yet, as a mortal, he had tremendous world influence in both Algebra and Number Theory.

** a.k.a. *Calculus* in first 2 years of U.S. college education.

*** Automated "recipes" for some types of problems.

Sample "Algebra with Numbers" Problems from Elementary School in Bulgaria

1st grade. Calculate $7 + 9 + 3$ in two ways.

- **Expected written solution:**

$$(7 + 9) + 3 = 16 + 3 = 19;$$
$$7 + 9 + 3 = 10 + 9 = 19.$$

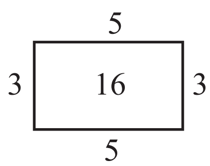
2nd grade. Fill in the blanks. Where have we seen such calculations?

$$\boxed{5 + 5} + \boxed{3 + 3} = 2 \cdot \lozenge + 2 \cdot \lozenge \;=\; \lozenge$$

- **Expected written solution:**

$$\boxed{5 + 5} + \boxed{3 + 3} = 2 \cdot 5 + 2 \cdot 3 \;=\; 10 + 6 = 16.$$

I remember! The sides of a rectangle that are opposite are equal. The perimeter of the rectangle is twice one of the sides plus twice the other side.



3rd grade. Calculate the expression:

$$1000 - (132 \div 3) \cdot 4.$$

- **Expected written solution:**

$$1000 - (132 \div 3) \cdot 4$$
$$= 1000 - 44 \cdot 4 \;= 1000 - 176 = 824.$$

4th grade. Check whether the equality

$$(a + b) - c = a + (b - c)$$

is true for $a = 347$, $b = 217$, and $c = 105$.

- **Expected written solution:**

$$(a + b) - c = (347 + 217) - 105$$
$$= 564 - 105 = 459;$$
$$a + (b - c) = 347 + (217 - 105)$$
$$= 347 + \;112 = 459.$$

Since we got the same answer 459, the equality is true in this case.

5th grade. Find the unknown number x if

$$8\frac{4}{5} \div x + 5.3 = 25 \cdot 0.3.$$

- **Expected written solution:**

$8.8 \div x + 5.3 = 7.5$	$8.8 \div x = 2.2$
$8.8 \div x = 7.5 - 5.3$	$x = 8.8 \div 2.2 = 4.$

This sequence of five random exercises in "Algebra with Numbers" starts in 1st grade in Bulgaria and continues all the way to 6th grade. And then *the "true" Algebra commences,* without hiccups, without drama, and with huge success, because:

- students have been *preparing for Algebra for a long time,* even if no one called it "Algebra";
- a *continuum in studying Algebra* has provided smooth transitions at every step of the way.

And just as playing the piano for the 6th year will likely encompass some harder, lovelier pieces, so will studying Algebra for the 6th year conquer tougher, nicer problems.

This short list of problems, culminating in the 5th-grade equation with fractions and decimals, also explains why we could NOT start our 6th graders in the U.S. on the 6th grade Bulgarian curriculum: they were NOT ready for it! We roughly estimated that they were lagging behind their Bulgarian peers by half a year in terms of algorithmic manipulations; and they were at least a year behind in algebraic skills and math maturity.

Of course, the easiest thing was to give up. But anyone could do that. We could do better. We did MUCH better! In fall 2015 we started the new curriculum in the pilot middle school:

- by first covering all of the Bulgarian 5th grade curriculum in 6th grade here; and
- only then moving to the Bulgarian 6th grade curriculum in the last third of the 6th grade here and covering half of it.

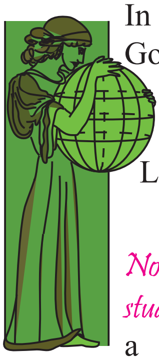
"Were you successful with the Algebra part?" you ask. Let a 6th grade pilot teacher respond:

A pilot 6th grade teacher:

"The Bulgarian curriculum is more rigorous than others I have used as it introduces Algebra in early grades. Variables are integrated into daily lessons so that students become as familiar with manipulating equations with x and y as they do with real numbers. This will pave the way for an easy transition to Algebra and set them up for success in higher grades."

Laurie Gold, teacher

8.3. The Geometry Neverland



In our drawing of the pillars of mathematics, Geometry is represented by Gaia, the Greek Goddess of Earth ("geo" = Earth in Greek). The truth is that Geometry is the *"Cinderella,"* with U.S. math education being its "step-mom." In the K-8 grades Geometry appears sporadically here and there, with few (if any) connections to the rest of the math curriculum. Later, in U.S. high school curriculum, there is *one intense year of Geometry,* during which:

No other math courses are studied. This in itself is a recipe for trouble: the few connections between Geometry and the rest of mathematics that were hinted before in the math curriculum are mostly forgotten during that year.

Geometry is studied in isolation!

4 years worth of Geometry are compressed in one year and they are presented to, by and large, students unprepared for such a huge single "dose" of Geometry.

Like making bread, Geometry knowledge and intuition need time to "rise" and develop!

Proofs are introduced through this Geometry course, often in the notorious "2-column format." Truly powerful Geometry theorems are not reached. As a result, many students not only do **not** learn how to do proofs, but also...

They end up, by association, disliking Geometry itself!

Question: Why care so much about a high school **Geometry** course, when *Calculus AP credits* seem to be the "admissions ticket" to top U.S. colleges?

Because everything starts at an early age and builds up over the years, regardless of whether it is Algebra, Geometry, Logic, playing an instrument, or learning to write excellent essays. And if we do not have the *vision of what awaits the young learner* in a few years from now, we will be doing a big disservice to the young learner, and we will be cutting off his or her chances to succeed in the future.

Knowledge and skills in Geometry cannot be amassed miraculously in a year. *Geometric intuition develops over time,* not instantaneously. To package and present several years worth of Geometry in one course to the untrained mind, and to top it off with introducing formal proofs (a hard topic even in a college curriculum) inevitably leads to a poor outcome.

Worst of all, by the time the high school Geometry course comes, *"the train is already gone":* it is too late to repair the damage that was systematically done over the K-8 years to the Geometry Pillar in the Temple of Mathematics. Consequently, the young budding math mind has *missed the creative and formative time* in middle school for a fundamental geometric development.

Insufficient Geometry knowledge and skills *will impede* performance in Algebra and *will reverberate* throughout a person's whole life, including college and post-college career.

The truth again:

Geometry has been sacrificed in favor of

- Arithmetic in elementary school;
- Algebra in middle school, and
- Pre-Calculus/Calculus in high school.

The *Geometry Pillar will never be built* with the strength and height it should have in order to hold up a robust, elegant, and long-lasting Temple of Mathematics.

The lack of serious or consistent Geometry in early grades makes it impossible to overturn the "algebraic dominance" in the K-12 curriculum. Can this be changed? Can Geometry "Neverland" be replaced by *a healthy symbiosis between Geometry and the rest of mathematics?* Let's look at what could have been.

8.4. A Personal Lamentation
Or ... the Golden Geometry Days of My Childhood

From my 30 years of experience teaching just about every undergraduate math course in college, the most mystifying formulas to my students, the least well understood, and (it goes without saying) the least frequently proven in pre-college years are the *trigonometric* ones. During the socialist times *in Bulgaria,* trigonometry was part of the standard 8th grade Geometry curriculum; the theory behind such formulas was systematically studied, and the formulas themselves were rigorously proven via geometric or, as needed, algebraic methods and applied to various problems for years in a row.

In fact,

Starting in 6th grade

We had *two separate textbooks:*

- one in Algebra;
- one in Geometry.

We had *separate homework, tests, and two grades:*

- one in Algebra;
- one in Geometry.

During the week, we *alternated studying:*

- Algebra on MWF;
- Geometry on TTh.

Most importantly, *we had a day between lessons in the same subject,* which allowed us:

to think deeply about what we studied in class;

to try various approaches on the homework problems;

and ultimately, to make the Algebra and the Geometry material *"our own."*

Although in two formally separate subjects, our studies of Algebra and Geometry were synchronized and mutually enriching. *A constant deep relationship between Geometry and Algebra* was reinforced at all stages of pre-college education: by using geometric facts to supply algebraic reasoning, and by using algebraic formulas to solve Geometry problems.

The Geometry Pillar of Mathematics was gradually built upwards throughout every single grade. There was no week without Geometry, especially from middle school onward. To this day, I distinctly remember the yearly Geometry themes that we studied, starting in 6th grade:

6th grade: *Plane Geometry* start of proofs

7th grade: *Circle Geometry* with proofs

8th grade: *Trigonometry* with proofs

9th grade: *Vector Geometry* with proofs

10th grade: *Solid Geometry* with proofs

11th grade: *Elective Geometry* advanced topics

Former Bulgarian Geometry Curriculum in Middle and High School

Every theme automatically included applications of the Geometry theory and techniques to *problem solving* and contributed to a *balanced and rich math curriculum.*

This is the *ideal symbiotic model of Geo-Algebra* that I would like for my children and grandchildren.

Unfortunately, high school math curriculum in the U.S. is driven, by and large, by college requirements and, let's face it, earning "bonus entrance points" to the more competitive universities. Hence, it is ultimately a *Calculus-driven curriculum,* regardless of whether students complete Calculus courses in high school, or not.

In turn, middle school math curriculum aligns itself with high school priorities. Backtracking, the pre-requisite to Calculus is Pre-Calculus, and the pre-requisite to Pre-Calculus is Algebra. Thus, Algebra "wins" by default, due to an established *Algebra-driven* and *Algebra-dominated environment.*

8.5. A Reality Check: Geo-Algebra in the U.S.?!

It is unrealistic (although not entirely impossible) to imagine that the pre-college curriculum in the U.S. can quickly change so much that it will embrace Geometry as an equal counterpart of Algebra, will split textbooks and classes into Algebra and Geometry ones, and will employ the *Geo-Algebra symbiotic model.*

The present textbook series capitalizes on a compromise between Eastern European and U.S. math educational systems and realities.

On the other hand, times **have** changed. *The Bulgarian curriculum has moved part-way towards the U.S. one;* e.g., newer Bulgarian textbooks feature both Algebra and Geometry, while preserving the mathematical integrity and pedagogical insights of a tremendously successful curriculum from the near past.

At a Glance: The New Middle and High School Curriculum Adapted to the U.S. Reality by Topic and Semester

6A Textbook

- *Decimals* operations, algebra w/ numbers; efficient calculations; motion & word problems
- *Divisibility* criteria for divisibility; GCD and LCM
- *Plane Geometry* altitudes; parallel lines; quadrilateral: perimeter, area; apply algebra
- *Solid Geometry* rectangular parallelepiped: surface area, volume; applications
- *Write:* correct, complete, and clear solutions.

6B Textbook

- *Fractions, %'s* operations; part of a whole; applications
- *Exponents* rules; properties
- *Rational #'s* operations with pos./neg. numbers; absolute value; equations
- *Graphing, Data* coordinates, diagrams/tables, arithmetic mean
- *Plane Geometry* circles, discs, polygons: perimeter, area; regular n-gons; constructions
- *Recognize:* expressions and equations. *Split:* solutions into "given," "to find," and calculations. *Draw:* pictures and diagrams.

Math 1 — 7A Textbook

- *Proportions* properties, applications
- *Polynomials* variables, numerical values, degrees, operations, algebraic identities; factoring in degree 2
- *Solid Geometry* types of prisms and pyramids; surface area, volume; constructions, applications
- *Include:* relevant formulas with variables; then substitute given numerical data. *Apply:* algebra tools to solve geometry problems.

Math 2 — 7B Textbook

- *Equations* linear, factored quadratic, parametric; w/ absolute value; modeling, practical applications
- *Plane Geometry* basic notions: pairs of angles, perpendicular/parallel lines; criteria, axioms, theorems
- *Congruent Δ's* criteria, isosceles Δ, perpendicular and angle bisectors, altitudes, medians
- *Understand:* the structure of mathematics. *Start proofs:* using definitions, primary concepts, axioms, and theorems.

Math 3 — 8A Textbook

- *Alg. Inequalities* properties, one variable, linear; inequalities vs. equations
- *Square Roots Quadratic Eq'ns* irrational numbers; algebraic expressions; quad. formula
- *Parallelograms Trapezoids* properties, criteria, proofs, constructions, applications
- *Intro to Vectors* operations; applications
- *Isometries:* translations, rotations, central symmetries, reflections
- *Devise:* algorithms for geometric constructions. *Link:* facts w/ given info.

Math 4 — 8B Textbook

- *Functions* graphs; linear, modular, quadratic functions; applications
- *Systems of Eq'ns and Inequalities* linear w/ one or two unknowns, parameter, abs. value, modeling
- *Rational Expressions* operations, equations, applications
- *Combinatorics* permutations, variations, combinations
- *Circles and Angles* tangents & lines; two circles; arcs, chords & angles; applications
- *Determine:* the hypothesis and what to prove. *Construct:* a proof as a logical sequence with words and math symbols.

Optional Topics

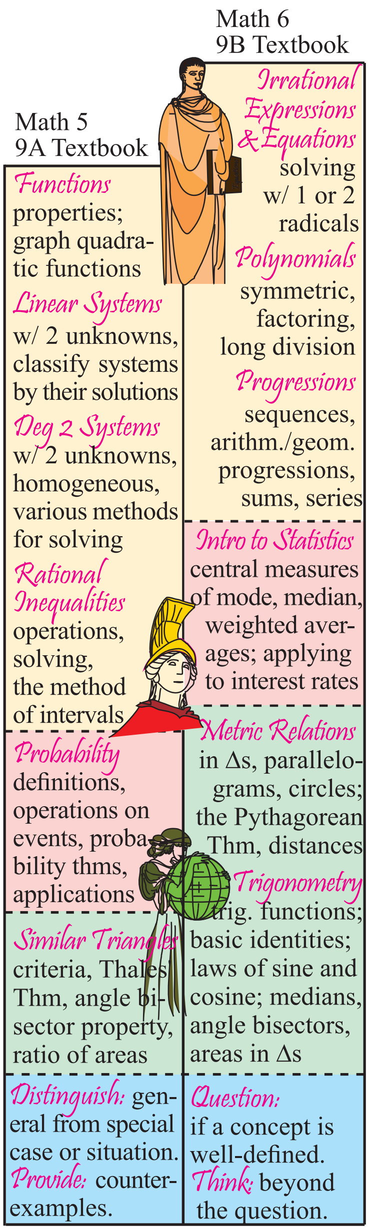
- 7A: *Algebra of Polynomials* alg. identities; degree 3 factoring
- 8B: *Equations and Roots* Vieta's formulas; Bi-quadratic eqn's
- 8A: *Geometric Inequalities* between sides; Δ Inequality
- 7B: *Euclidean Constructions* ruler & compass
- 8A: *Vectors* midsegments & centroids
- 8B: *Circles and Polygons* geometric loci; circumcircles and incircles; famous pts in Δ
- *Find:* multiple ways to solve. *Distinguish:* between faulty and valid reasoning. *Learn:* from own experience and others' solutions.

Legend

Elective Algebra	*Algebra*
Elective Geometry	*Geometry*
Number Theory, Combinatorics, Probability, Statistics	
Logic and Proofs Techniques	

8.6. Math Bridge from U.S. High School to College
Or ... Cravings for Calculus

If it were up to me, *up through 11th grade* high school students would study *a good balance of Algebra and Geometry topics, mixed with some Number Theory, Combinatorics, Probability, and Statistics.* And only *in 12th grade,* with a solid preparation and math maturity under their belts, would they venture into advanced concepts such as *limits, continuity, and derivatives,* which comprise about half of Calculus BC. Thus, during their senior year, they would study thoroughly and deeply:



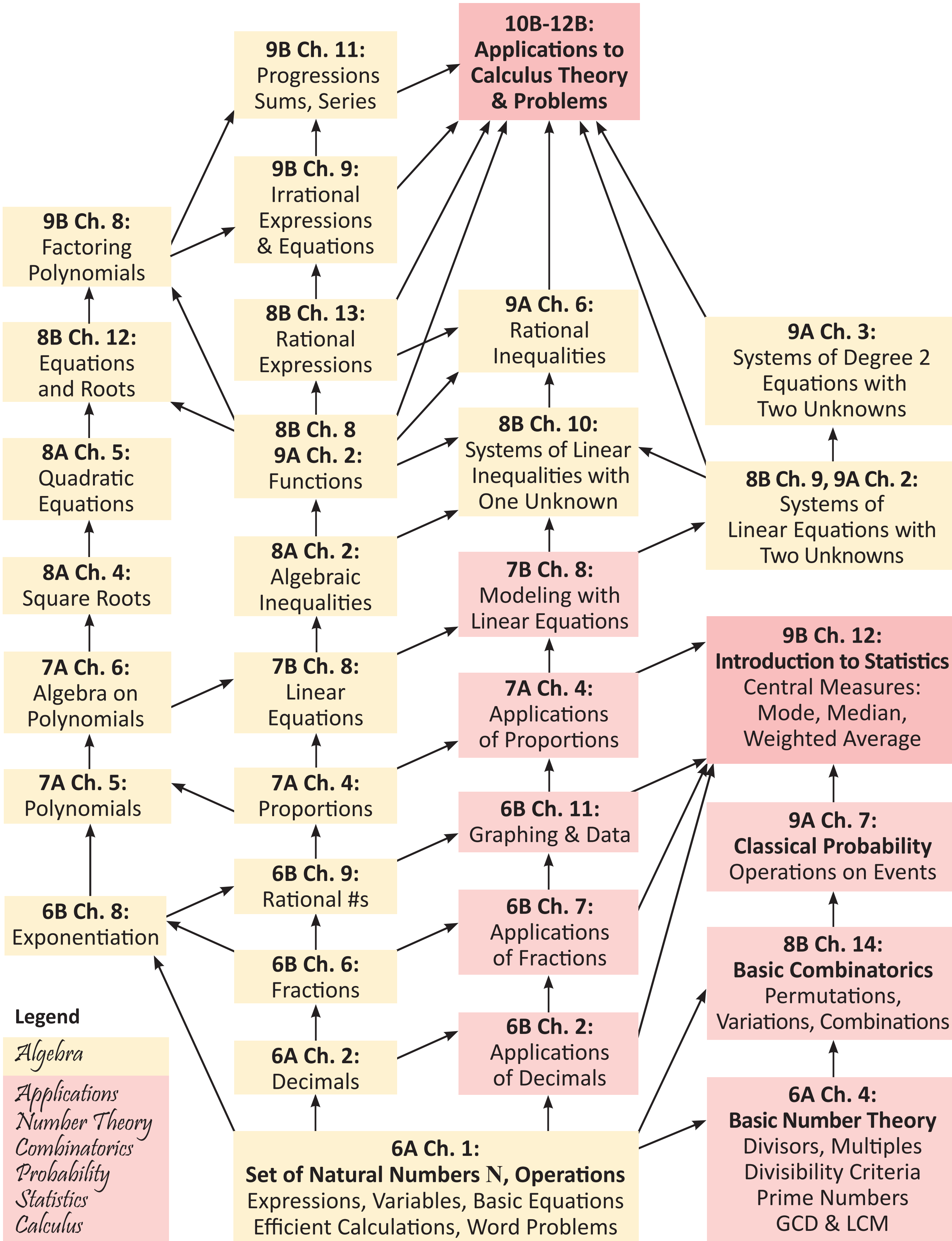
- sophisticated definitions like the *ε-δ definition of limit,*
- and powerful statements like the *Intermediate* and *Mean Value Theorems,* and the occasionally "dangerous" *L'Hôpital's Rule,*
- along with their *proofs* and relevant *problem-solving techniques.*

A rigorous treatment of this material would take time and sustained effort from both teachers and students, and a whole year may be barely enough. But this would be *"the right prequel to college math courses"* and *"the right way to learn Calculus."* Our students would thus march into college with fortified math knowledge and skills, and a clearer understanding of how mathematics works, better prepared for success in their college freshman math courses. Hopefully, many would then continue into STEM disciplines, which require more mathematics.

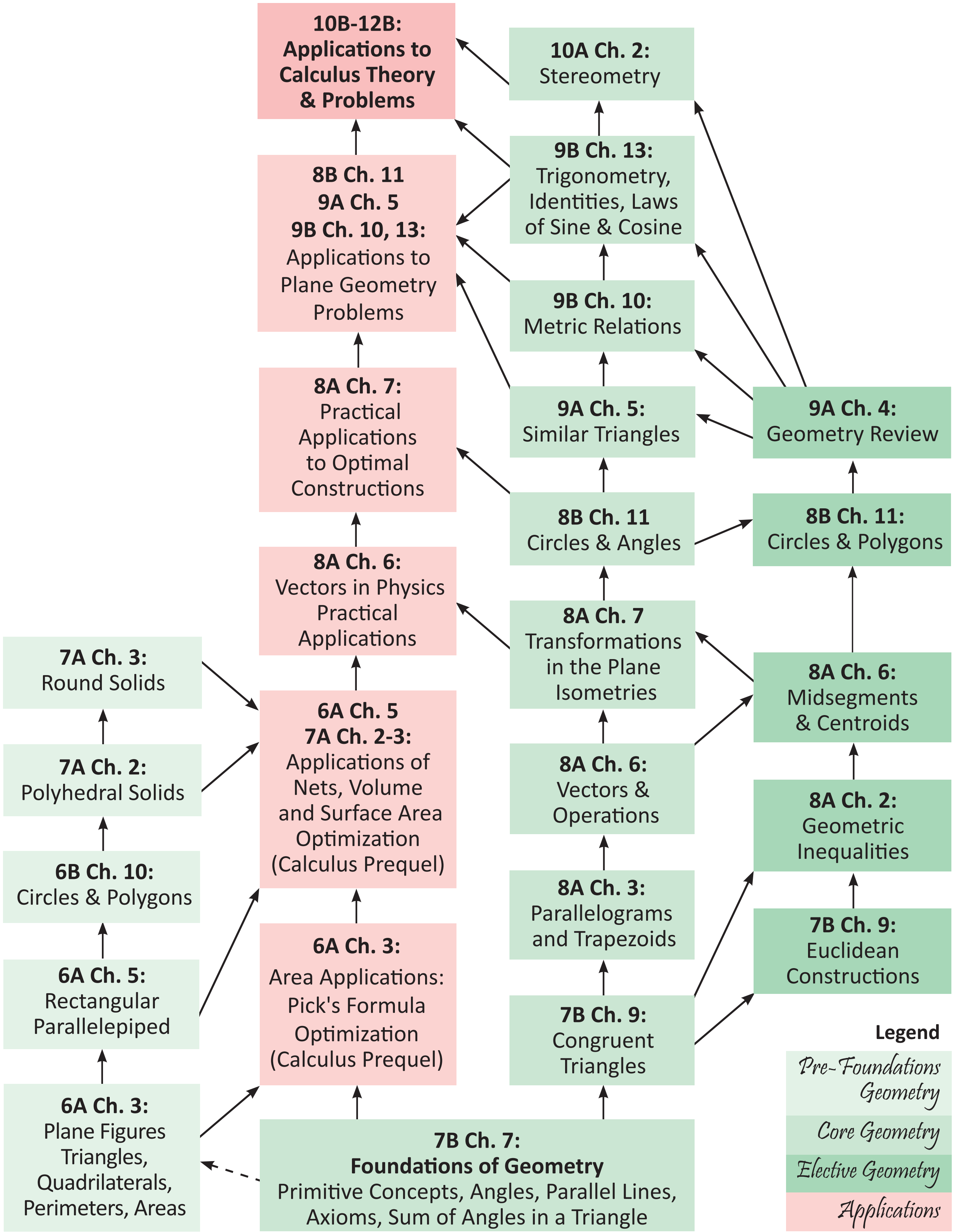
Instead, U.S. middle and high school curricula sprint up the pyramid of Mathematics, finding shortcuts and avoiding any "scenic math vistas," only to enable students to get through as many AP classes as possible before applications to U.S. colleges are due. But these high school sophomores and juniors are *unprepared in many ways* for *Differential* and *Integral Calculus* (not to mention the latest trend of *Multivariable Calculus* and *Linear Algebra* on college applications) and later the vast majority are *shocked and indeed humbled by* the true required depth in typical U.S. college math courses. Unfortunately, this is no concern of high school or other programs that offer "accelerated" paths to and through Calculus, with promises of "excellent prospects for acceptance to top universities." In fact, as US college math departments know very well, only a small minority of students demonstrate a real mastery of Calculus on AP exams: most students come out of an AP Calculus course without the skills expected by the exam, and, needless to say, without a rigorous understanding of the concepts.

The present textbook series *will get students through every topic needed in Calculus,* without rushing and without compromises. *Calculus will start permeating just about every chapter in the 11A Textbook,* taught with the same level of detail and rigor as any other topics in this series. And like these other topics, Calculus is developed over multiple years precisely so that it will be learned well. With 5 years of Algebra and Geometry from 6A to 10B, students of Bulgarian math will encounter their first Calculus topics with strong fundamentals and the appropriate math maturity to appreciate the rigorous development of the subject. Still, Calculus is just the beginning of one branch of higher mathematics: Real Analysis. As such, Calculus is **not** the goal of these textbooks. The goal is to *become aware* of all the fundamental pieces of an ever expanding and beautiful mathematical world, to *learn how* to put them together, and to *understand why* they work this way.

8.7. Structure of Algebra Topics and Applications in 6th-9th Grades Curriculum



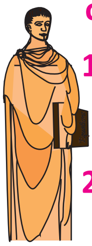
8.8. Structure of Geometry Topics and Applications in 6th-9th Grades Curriculum



8.9. The Main Principles

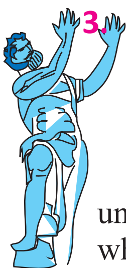
The Main Principles in designing this middle and high school math curriculum are

Continuity + Balance + Integrity
of the 4 basic building math blocks:



1. **Algebra** is developed from an early stage, using numbers at first and then variables more and more frequently.
2. **Geometry** is developed alongside Algebra as an equal partner.

Algebra & Geometry support/reinforce each other by applying each subject to the other.



3. **Logic, Proofs, and Problem Solving*** comprise the skeleton of the Math Temple, providing the "glue" that keeps together all parts of mathematics.

The Logic/Proof-Writing Topics appear uniformly and constantly throughout the whole middle and high school curriculum.

4A. Number Theory, or the Queen of Mathematics,** makes a grand appearance through the topics of *Divisibility and Square Roots* in the 6A, and 8A textbooks, to aid both logical development and practical algebra skills; e.g., finding *GCD*, *LCM*, number of divisors, etc., efficiently. If you wonder what Number Theory is all about, try to decide if 9,876,543,210 is divisible by 15 *without actually dividing by* 15! The world of *whole* numbers is how Number Theory can be approximately described in early school age.

4B. Combinatorics, Statistics, Probability and Their Applications are ubiquitous in the curriculum. Among other topics, they include counting techniques, data graphing, optimization, approximations, and calculating chances, and are often aided by physical and mental experimentations.

9. Differentiate and Scaffold

Talking about differentiation in math classes is a "politically correct" and enticing buzzword to mention when advertising a curriculum. Yet, by and large, this talk is not matched by actions to truly support <u>all</u> students... because it <u>is</u> hard to scaffold math lessons and because *it takes an enthusiastic math specialist to successfully cope with diversity in the math classroom.*

When mathematics is taught:

- in an organized and sound way,
- building a solid Math Temple for everyone,

the number of students who "fall through the cracks" or lag behind is minimized. Some students, though, need something *beyond* the regular curriculum that will still keep them topic-wise in the same math vicinity as the rest of their classmates. To this end, the exercises in the **9A Textbook** are split into:

1. *Basic* (mandatory) problems for everyone;
2. *High Proficiency* (recommended) problems;
3. *Harder/Math Contest* (elective) problems.

Part I of the **9A Workbook** is a **Problems Collection** of over 2,250 problems, matching the material in every chapter of the textbook. Some problems are grouped into multiple-choice tests, and some are open-ended and organized by topic. The material is gradated in three levels:

- **Level A:** basic problems developing fundamentals and providing practice for everyone.
- **Level B:** recommended problems for excellent proficiency that will extend the knowledge of those who want to move forward.
- **Level C:** preparation for math contests for the more adventurous students.

Part II of the **9A Workbook** provide worksheets with all **Lesson Exercises.** While everyone should complete a substantial portion of them in class, these are an absolute must for students who need more time with the material.

---

* The role of Logic, Proofs, and Problem Solving is played by the Greek Titan *Atlas,* often pictured "holding the Earth on his shoulders." Naturally, Atlas represents the *central Pillar of Mathematics,* which keeps the whole structure together.

** Number Theory is widely regarded as the Queen of Mathematics. Some of the most famous math problems have been in Number Theory; e.g., Fermat's Last Theorem. In our Math Temple, Number Theory is played by *Athena,* the Greek Goddess of Mathematics (also of wisdom, courage, inspiration, war strategy, arts, literature, and others). In this textbook series, Athena also supports the areas of Combinatorics, Probability, Statistics, and Applied Mathematics.

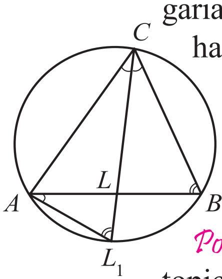
9.1. Examples of Scaffolding

The 9A Textbook features the exercises below on *Degree 2 Systems, Similar* Δ*s,* and *Probability,* reaching all parts of the difficulty spectrum:

1. *(Basic).* Solve $\begin{cases} x^2+y^2=29 \\ y^2-x=23. \end{cases}$

2. *(High proficiency).*

CL is an angle bisector in ΔABC. Prove that $CL^2 = CA \cdot CB - AL \cdot BL$.



3. *(More advanced material and ideas).*

In the **Monty Hall show**, there are 3 closed doors and a prize hidden behind one of them. As a show contestant, you choose one of the 3 doors. The host then randomly opens one of the other two doors that does not have the prize. You are given the choice to stick to your original door and open it, or to switch to the other unopened door. What should you do?

Test 2 in Lesson 91 further challenges the reader:

Question 11 (True or False). When the outcome space Ω is *infinite*, it is possible that an event with probability 0 actually occurs, while for a *finite* Ω this *cannot* happen.

Answer (cf. the **Appendix**)**:** True. For a *finite* outcome space Ω, by the (ratio) definition of classical probability, $P(A)=0$ iff $\nu(A)=0$; i.e., $A=\varnothing$, and $\varnothing$ *never* occurs. If Ω is *infinite*, we cannot use this definition of probability: $\nu(\Omega)=\infty$ will mess up the denominator of the ratio in $P(A)$! From this viewpoint, this question is not "fair": we have not yet introduced probabilities for an *infinite* Ω!

Still, think what will happen if the set of our outcomes were the *heights of people*? The individual probability of being, say, exactly 2 meters tall is 0 (why?) but it is certainly possible that some people are exactly 2 meters tall!

When **using definitions**, we must consider whether our situation allows for the conditions for these definitions or if new definitions are needed. One way or another, before deciding if a statement is true or false, we need all concepts in it to be **well-defined**.

9.2. What are Optional Topics?

This U.S.-adapted program starts in 6th grade with the 5th grade Bulgarian material; that is, in the beginning, it lags a full year behind its Bulgarian counterpart. By the end of 8th grade, it has has caught up with its Bulgarian counterpart.

The harder parts of the 7th-8th grade material on *Degree 3 Polynomials, Vieta's Formulas, Geometric Inequalities, Euclidean Constructions, Midsegments and Centroids, and Circles and Polygons,* are optional and intended as *elective* topics. The optional geometry topics are considerably deeper than the standard U.S. high school material and they provide a superb preparation for anyone who would like to be challenged in Geometry.

There are no optional 9A topics: the *Systems of Degree 2 Equations, Similar triangles, Rational Inequalities, and Classical Probability* are a must for any 9th grader. The review topics though on *Functions, Systems of Linear Equations,* and *Geometry* are optional and can be studied as much as needed, depending on the depth of previous understanding and preparation.

9.3. Does the Scaffolding Work?

Does the program really benefit all levels of students? Here is some anecdotal evidence.

Answer 1: A month after the start of the pilot 6th grade program, the teacher received a letter from a parent of a student who was struggling tremendously in the beginning, not participating at all in class, and submitting blank tests:

"Thank you so much for this feedback! She is struggling to catch up. I feel like she is learning more math now in one month than she did all last year. Thank you for your help!"

6th grade pilot parent

The student gradually progressed to a very good performance both on tests and in class, showing enthusiasm and understanding of material to which she had not even been exposed before but was now fired up to learn.

Answer 2: Several months after the start of the pilot program, a parent of a student who had always excelled in math wrote to the 6th grade pilot teacher:

> *"I just wanted to take a moment and thank you for all of your hard work this year with the 6th grade math students. Teaching is sort of like farming. You never know how your crops are going to turn out, but I have noticed with my son that his work has improved, and I really like the way he is talking about math."*
>
> *6th grade pilot parent*

The new program has worked exceptionally well for a number of students who had earlier been identified as having severe problems with learning mathematics. These students have gradually improved and excelled with the new curriculum, and questions about their future success with mathematics have become obsolete. Incidentally, a few of the girls improved so rapidly that 2 months into the program they had to be moved to the more advanced 6th-grade pilot class.

Answer 3: Here is the truth as seen through the eyes of the very first pilot teacher in the U.S. who worked with this program in 6th grade and achieved what was impossible before, and to whom the above letters were addressed:

A pilot 6th grade teacher shares:

> *"This book has been a tremendous resource for my students, and I wish I had it when I was in school. The lessons are laid out in a very clear manner. The math is approachable or challenging to meet the needs of a child at any level.*
>
> *There are many math books that claim to reach every person. This book successfully does so, and it is one of the best."*
>
> *Natalya St. Clair, teacher*

co-author, "The Art of Mental Calculation"

10. The Value of a Teacher

As with any textbooks, the present textbooks are just that: *a basis for learning.* They will come alive in the classroom *only through the teacher.*

Let's not deceive ourselves: there is no universal panacea for the middle or high school math problems in the U.S. And although the present textbooks have been hailed by pilot parents as:

- the best textbooks they have ever seen;
- the most suitable for their children;
- a resource to easily follow and learn from at home if a child is absent from school or did not understand something in class...

...still, *a mediocre teacher will do a mediocre job even with the best textbook in hand.*

The fact that many middle (and some high!) school math positions are filled by non-specialists in math is alarming and is one of the major reasons for the current general downfall of U.S. middle school math, dragging down the U.S. high school math too. A non-specialist does not have a "bird's eye view" of the full middle and high school math curriculum. Although locally, he/she might be able to teach some of the material well, a lot of damage is done by a teacher's limited knowledge and *lack of understanding of what is important in the long run, even if hard to teach.*

The situation is similar to parachuting the whole math class into the middle of a jungle, giving each student a flashlight, and waiting outside the jungle to see who will come out alive. A better chance of "survival" will be ensured if the students are *given a map of the area, taught how to read and use it*, and *equipped with the necessary tools, knowledge, and skills* to efficiently cope with the difficulties that will inevitably occur on the way out of the jungle. And *the teacher has to be there, with the students, every step of the way,...* not just waiting for them outside of the "danger zone." I personally never thought of school math in this "jungle vision" until my own children experienced several different U.S. math programs and the image of the "jungle" more and more powerfully resembled the reality of pre-college math education.

More often than not, non-specialist math teachers in middle and elementary school, for a variety of reasons, decide to change the order, depth, and emphasis of the math program they are teaching from. Not knowing the "big picture" of how mathematics will develop over the high school and then college years (and perhaps, not caring about this big picture, since it is not part of their day-to-day job duties), their actions unintentionally destroy the logical structure and goals of the program, whether it is a good program or not. (See earlier discussion of this situation in Section 5: The Temple of Mathematics.) Fortunately,...

The new math program:

Is *straightforward to follow:* every lesson is on only 2 pages!

When each lesson in the textbook is on 2 pages only, with 6 to 10 exercises, and when the concepts are practiced in the next lessons and incorporated in new situations for better understanding, it is *very hard to justify staying on one particular lesson for a week or so* (and killing everyone's enthusiasm in the process).

(Most importantly!) **Ensures accountability from the teachers.**

The program is so crystal clear that *anyone can track where the students are and what they have covered* at any time during the school year. By "anyone" we mean parents, administrators, and random visitors alike. It is essential, though, that a copy of *the textbook resides at every student's home:* there is nothing to hide in the program, while there is so much to learn from reading the textbook at home!

Does not require search for extra materials elsewhere: the textbook, workbook, and problem collection *contain everything for a successful completion of the program.*

Due to the *natural transparency of the material,* it is impossible to hide behind phrases like "too much," "too hard," or "not well organized" "to be covered all in one year." And the *program does depend on the steady and full completion of the material* in every middle school year, in order for the next year to be successful.

A Good Choice for a Teacher for this curriculum would take into account:

The Mathematical Level

A teacher with a *solid math background* who can *see several steps beyond middle school,* e.g., the high school curriculum, and can recognize:

- the math concepts,
- types of solutions, and
- features of analytical thinking

that are indispensable and must be taught and reinforced in middle school, despite how hard it might be to teach them to a young audience unaware of them.

The Pedagogical Level

A teacher who can *quickly adapt to new situations* and turn them into an educational advantage for the students.

A teacher who is constantly attuned to the weaknesses and strengths in students' background and skills, and who accordingly emphasizes features of the curriculum in a balanced way to match and improve the classroom situation.

A teacher who *can communicate mathematics to young students.*

The Level of Dedication

A teacher who is *dedicated to the well-being of the students* and who will sacrifice his/her own desire to "shine on the career ladder" to provide a robust math education to the young charges.

A teacher *who believes in and is ready to undertake and follow the new curriculum closely;* who is ready to face the mathematical and cultural difficulties that will inevitably arise in introducing such a program.

11. Can a U.S. Middle or High School be Successful with this New Math Curriculum?

What kind of question is this?! Let us analyze the situation as a mathematician would, putting aside our own ambitions and de-politicizing math education.

11.1. The U.S. Gene Pool

Of course, on the average, U.S. students can do just as well as (if not better than) their Bulgarian peers, and, for that matter, any peers around the world. As far as the "gene pool" of the U.S. is concerned, it is probably the best mixture in the world.

The cream of the crop among U.S. students have performed at the top level and have occasionally made the number 1 team in the world (!) at the International Mathematical Olympiads (IMO)*, the preeminent world math competition for pre-college students that tests **not** speed but *depth and originality of mathematical thinking.*

The new math program emulates:

the IMO spirit of deep and rich math culture	a mature attitude towards mathematics	dedication to long-term goals

It is true that very few students will end up being math Olympians, and we will not give false hopes that the current program is training students for the IMOs. However, the expectation is that:

Everyone who is touched by this program will build a solid Temple of Mathematics, where later, if desired, we may hang "fancy chandeliers" and "brilliant art pieces." More importantly, the personal connection with mathematics will remain all through life to be enjoyed and cherished.

11.2. Can Girls Succeed in Math?

Upon coming to the U.S., I was shocked by this question. I was raised in a provincial town in Bulgaria and made it as a high school student to two IMOs, winning silver medals. There was another girl on the Bulgarian team for the 2 years I was there.

It took the U.S. 25 years of participating at the IMOs before the first U.S. girl, Melanie Wood, qualified for the IMOs. I was privileged to train the U.S. national team when Melanie competed at the IMOs, also earning two silver medals. Later, I participated in the training of the other two U.S. girls who went on to win gold medals at the IMOs.

To cut the long story short: anything is possible and *gender does not matter as far as mathematical talent and success are concerned.*

Gender apparently does matter in the social and cultural aspects of education. This in turn severely handicaps many girls in the U.S. who might otherwise become excellent mathematicians.

Decades after competing at the IMOs, I learned that Bulgaria is the top country in the world in sending more than 21 girls so far** to the IMOs. Germany and Russia are next, with 19 and 15 girls, respectively. The USA has only 3 girls so far.

Perhaps *it is time for the U.S. educational system to follow the example of other programs from around the world* that have been hugely successful in raising gen-

* The IMO takes place every summer in a different country. Each country's team has 6 students. As opposed to standardized tests, there are only 6 problems over 2 days, 4.5 hours each day. The solutions are written as rigorous mathematical proofs.
** See *"Cross-Cultural Analysis of Students with Exceptional Talent in Mathematical Problem Solving"* by Titu Andreescu, Joseph A. Gallian, and Jonathan M. Kane, http://www.ams.org/notices/200810/fea-gallian.pdf.

erations of women who are highly educated in math and other STEM (Science, Technology, Engineering and Mathematics) disciplines, and who have been brought up to think of men and women as intellectually equal.

11.3. The Needed Cultural Shift

The current middle school math program originating in Bulgaria necessitates a cultural shift in the U.S. school community so that the program can take root and yield results. For starters, the school must make

math a top priority in its academic curriculum

and continue unwaveringly in its determination to overturn the middle school math situation to the huge benefit of its students.

11.4. Continuity of the Curriculum

It is most important and advantageous for everyone that:

The **same** teacher continues with the **same** students throughout middle school, and then another teacher in high school.

The alternative has struck me as a very bad way to organize middle (and high) school math education. Indeed, when a person teaches the same grade year after year he/she:

- gains no understanding of the "big picture" of pre-college math education;
- shares no responsibility for the continuation of the program in the next grade (and a lot of time will be wasted in the beginning of each school year for the new teacher to test and get attuned to the new students);
- has no incentive (or responsibility) for reaching the final goals of the program.

Yes, I am acutely aware of the counterargument:

"I do not want a bad math teacher to teach my kid for all of middle school. I would rather bite the bullet of a bad math teacher one year, so as to get the good math teacher the other year."

In summary, many parents are (unwittingly) willing for their kids to endure on the average a mediocre middle school education. Can you imagine what kind of damage the "bad" teacher would do to the Temple of Mathematics during that, hopefully, single year of "bad" math lessons? Would the next, "good" teacher have the opportunity, or even the incentive, to rebuild the Temple AND cover the new material on top of that? Probably not.

This kind of "averaging math teaching" is another major reason for the tremendous difficulties in U.S. middle school math education. *A teacher must be an asset for math education:* equally important (if not more important) than the actual math curriculum! And so, the truth must be spoken:

- a bad math teacher must be let go, to prevent more irreversible damage to our kids;
- a good math teacher must be treasured and given opportunities to further develop.

11.5. Local Support

There is no easy remedy and no miracle solution to math education problems anywhere around the world. The new math Program will need all the support it can get from parents and students, in addition to teachers and administrators. The Program will depend on the school community's belief in it and commitment to it.

Approaching the end of the first pilot year, here is what the Head of School shared, after extensive observations and conversations with pilot teachers, parents, and students:

"I have only heard GREAT comments about math during and after parent-teacher conferences in March. People are very pleased with the direction things are going with the Bulgarian curriculum. We are very excited to continue to grow the program next year!"

Debra Massey
Former Head of Pilot School
Tehiyah Day School, El Cerrito, CA

12. Within the Global U.S. Picture

At a sectional meeting of the Mathematical Association of America (MAA) in 2014, I heard a captivating presentation by *Dr. Phil Daro,* one of the three principal writers of *the Common Core Mathematics Standards* (CCSS-M). The talk addressed the biggest education problem in school mathematics, on which the CCSS-M writing team focused:

"*The mile wide, inch deep*" configuration of the American school curriculum.

Consequently, Dr. Daro would ask any teacher:

"Are you teaching

Mathematics or *Answer-Getting Strategies?*"

This question fully resonated with the educational philosophy with which I was brought up back in Bulgaria. Dr. Daro explained that the core change incorporated in the CCSS-M is:

A shift in the perspective toward mathematics taught in schools:

Less emphasis on a fragmented multitude of special solution methods, especially tricks that avoid underlying mathematical principles.

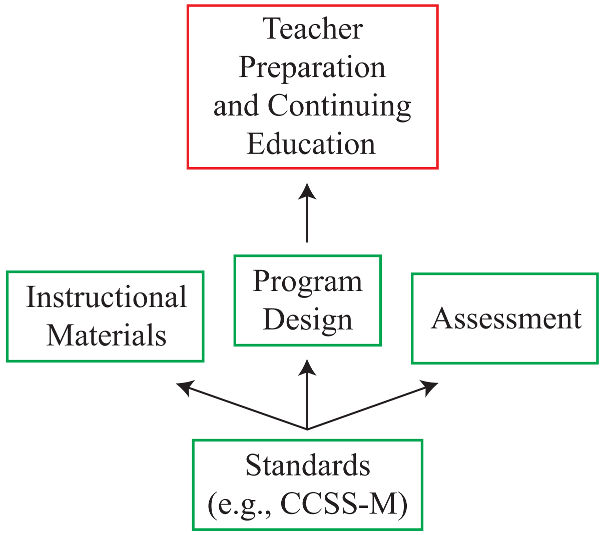
More emphasis on core principles and practices that unify and focus mathematical thinking.

Dr. Daro assembled the major pieces of the gigantic math education picture in the U.S.:

"Standards can help solve this kind of curriculum architecture problem, but we all face more detailed and difficult challenges of instructional materials, assessment, and program design, as well as the even more [difficult] area of teacher preparation and continuing education."

Dr. Phil Daro
A Principal Writer
Common Core Mathematics Standards

Visually, the scheme below represents the levels of change in the U.S. math education that are envisioned by the CCSS-M writing team:



At the time, I had no idea that I would soon launch the translation, adaptation, and implementation effort of the present curriculum. Inspired by Dr. Daro's talk, I invited him to meet the then-leaders of Tehiyah Day School in person. Perhaps, not entirely by serendipity, this meeting marked the path for the new math curriculum: as it turned out, the new Bulgarian curriculum matched the CCSS-M very closely.

During the 2016-2017 academic school year, I spent a day every week working with the pilot classes, training the pilot teachers, and learning myself how to adapt the curriculum to the U.S. reality. The instructional materials, program design, and assessment of the adapted math curriculum for 6th-7th grades have been published, and the adaptation of the 8th grade curriculum is ready.

Coming full circle, two years after his memorable MAA presentation, Dr. Daro reviewed the U.S.-adapted version of our new math program and assessed it as a

"Middle and High School Math Program that is deeply aligned with the Common Core Mathematics Standards".

13. How to Work with the 9A Materials

The 9A Textbook is for use in class as the main source of material to be covered:

- The *Textbook Examples* are the core of the class discussion and work.
- The *Textbook Exercises* can be done in class or assigned for homework.
- *Answers to all Exercises* are in the back of the textbook. Thus, writing only answers brings no credit, while good reasoning and organized calculations may earn a lot of credit.

9A Workbook Part II consists of *Lesson Worksheets,* identical to the textbook examples, with spaces for writing solutions. These worksheets are intended for classwork or for review at home.

9A Workbook Part I, or the *9A Problems Collection (PC),* contains over 2,250 problems. The PC:

- will save the teacher time in searching for more problems of various difficulty.
- can be used for independent work by the student, to assign homework, to conduct tests, and for practical work with applications.

The problems in the 9A PC correspond to each chapter in the 9A Textbook, grouped in 3 levels:

100 **Level A: For Everyone**.

- Level A problems are for all students. They will strengthen and extend the basic knowledge and skills needed to understand the 9A material.

200 **Level B: For Excellent Preparation**

- Level B problems should be addressed by any student who wishes to master the 9A material and obtain excellent preparation for future math courses.

300 **Level C: For Math Contests**

- Level C problems are intended for students who wish to go beyond the 9A textbook material and/or to prepare for math contests. These problems will prove to be significantly more challenging.

14. Biographical Sketches

The idea for the *Archimedes Publishing House* was conceived in 1982 when Professor Georgi Paskalev collected all entrance exams given thus far to Mathematics Departments in universities across Bulgaria and wrote a book with their solutions. The occasion was the upcoming exam of his own son, who actively participated in the preparation of the book. Until 2004, Professor Georgi Paskalev continued to study the competitive university entrance exams and published a variety of preparation materials. These books have enjoyed continual interest, large circulations, and many editions.

In 1989 Professor Georgi Paskalev, together with his wife Zdravka Paskaleva and their grown-up children Plamen and Nadezhda, founded publishing house *"Georgi Paskalev,"* specializing entirely in mathematics materials. Ten years later Plamen Paskalev assembled a team of authors and published a collection of tests to prepare for admissions to high schools. This book became the first publication of *"Archimedes,"* the successor to *"Georgi Paskalev."* The new publishing house is directed by Plamen Paskalev, and it still bases its activities on the creative duo of Georgi Paskalev and Zdravka Paskaleva. During 1997-2004, these two authors created a complete series of math textbooks for 5th-12th grades, which set new standards in the methodology of teaching mathematics and has ever since enjoyed popularity in teaching circles around Bulgaria.

Over the years, Archimedes Publishing House grew to include books in informatics, information technology, and chemistry, as well as translated academic and journalistic literature.

“Archimedes” is an active member of *Association "Bulgarian Book"* and an active participant in the publishing conferences of the *Union of Mathematicians in Bulgaria.* In 2010 the *European Council for Scientific and Cultural Community* awarded the *"Golden Book"* to “Archimedes” for its contributions to the development of Bulgarian culture.

Professor Georgi Paskalev (1932-2004) taught for many years at Sofia University. He co-founded the Mathematics Department at Plovdiv University (PU) and was the Vice Provost of PU (1970-1972) and also a longtime Chair of the Departments of Mathematics and Mechanics at PU and the University of Structural Engineering & Architecture (USEA) in Sofia.

Professor Paskalev was known for his ability to teach difficult math disciplines in a clear and accessible manner. He published numerous research papers in various areas of mathematics and mechanics. He mentored many mathematicians and authored more than 100 books, problems collections, and textbooks in mathematics. In 2004 the Academic Council of PU named its auditorium after him. Another auditorium at USEA in Sofia is also named in his honor.

Zdravka Paskaleva (1938-2017) was a well-known and experienced specialist in mathematics and the methodology of mathematics, with 20 years experience as a teacher in leading high schools and 10 years as a leader in teachers' education and development, having observed and reviewed over 1,000 lessons by teachers at various schools. From 1985 until 2010 she was the head of the *Education Center "Archimedes"*.

Maya Alashka received a degree in mathematics from Sofia University (SU), with a specialty in elementary school education. She has taught math for 40 years at all pre-college levels. As an affiliate of SU, she has trained many future teachers. She has participated in various education projects and co-authored a number of popular textbooks, problem collections, and other education materials for all pre-college levels.

Plamen Paskalev is an Assistant Professor of Mathematics, author of dozens of scientific publications in mathematics and mechanics, collections and guides for students, and articles on the challenges in math education. He has taught at USEA, the University of National and World Economy, and the German Language High School in Sofia. He has directed "Archimedes" since its creation in 1999. In 2004 he founded and has since chaired the *Union of Publishers of Textbooks & Curriculum Materials in Bulgaria*.

Professor Zvezdelina Stankova (Zvezda) was drawn into the world of mathematics when, as a 5th grader, she joined the math circle at her school in Bulgaria and, three months later, won the Regional Math Olympiad. She represented her home country at two *International Mathematical Olympiads* (IMOs), earning silver medals.

As a freshwoman at Sofia University, Zvezda won a competition to study in the U.S. and completed her undergraduate degree at Bryn Mawr College in 1993. She did her first math research in enumerative combinatorics at two summer programs in Duluth, Minnesota. The resulting papers contributed to her *Alice T. Schafer Prize* for Excellence in Mathematics by an Undergraduate Woman, awarded by the *Association for Women in Mathematics*. In 1997, Zvezda received a Ph.D. from Harvard University, with a thesis on moduli spaces of curves (in algebraic geometry). She also earned a high school teaching certification in the state of Massachusetts and later in California.

As a postdoctoral fellow at the Mathematical Sciences Research Institute (MSRI) and UC Berkeley in 1997-1999, Zvezda co-founded the *Bay Area Mathematical Olympiad* and created the *Berkeley Math Circle* (BMC). She trained the USA national team for the IMOs for six years, including the memorable year of 2001 when three of the six team members were BMCers and the USA tied with Russia for second place in the world overall. She worked at Mills College 1999-2016, and she joined the faculty at UC Berkeley in 2016.

Zvezda's inspiring style and passion to teach were recognized by the *Mathematical Association of America* (MAA): in 2004 she received the first *Henry L. Alder Award* for Distinguished Teaching by a Beginning College or University Mathematics Faculty Member. In 2011 MAA honored her with the highest math teaching award in the U.S., the *Deborah and Franklin Tepper Haimo Award* for Distinguished College or University Teachers who are widely recognized as extraordinarily successful and whose teaching effectiveness has shown to have influence beyond their own institutions. Zvezda was featured in the *Salutes Program of the ABC 7 News* in spring 2011. In 2012, she was listed in *Princeton Review's "300 Best Professors."*

In fall 2015 she introduced a new middle school math program at Tehiyah Day School in El Cerrito, CA, and in fall 2016 she founded the "Math Taught the Right Way" program for middle and high school students at UC Berkeley. Both programs are based on this textbook series.

Zvezda's most enduring passion remains working at the BMC with young students motivated to discover new mathematical wonders. She spends a lot of time with her girl and boy, studying foreign languages and playing the piano with them, and teaching them mathematics in the "Bulgarian" way.

Tom Rike graduated from San Francisco State College in 1968. He taught in the Oakland Unified School District from 1970 until he retired in 2003. He now volunteers at Oakland High School three days a week. He served as the high school liaison for BMC and BAMO until 2009.

Tom has been fascinated by giants of mathematical thought such as *Archimedes*, *Euler*, and *Gauss*, and has shared their works at the BMC sessions on a number of occasions: using the *arbelos* from the *Book of Lemmas* by Archimedes; showing Euler's solution to the *Basel Problem*; and demonstrating Gauss's proof from *Disquisitiones Arithmeticae* that a 17-gon is *constructible*. It is Archimedes' *lever*, on which he "balanced" his *Mass Point* article in Volume I of *"A Decade of the Berkeley Math Circle - the American Experience."* For a number of years, Tom also ran a math circle at Oakland High.

Tom has been involved as a coach and director of the *East Bay Mathletes*, a monthly competition among local high schools since 1978. In 1983, he helped found in Oakland a middle school math competition, *All-Star Mathletes*. Among his other interests are the game of go, opera, chamber music, languages, and the writings of *James Joyce*.

This series of textbooks has tremendously benefited from Tom's unsurpassed expertise in U.S. pre-college mathematics and from his generous and unbounded enthusiasm for new and sound ways to teach mathematics to youngsters.

Stankova's and Rike's bio sketches are partially taken from "A Decade of The Berkeley Math Circle," AMS/ MSRI's Math Circles Library.

Dr. Kelli Talaska has been a student and teacher of mathematics nearly all her life, starting with frequently volunteering to write the problem of the day in fifth grade. Her mathematics education really took off in her years as a student and later a counselor at the *Honors Summer Math Camp* (for high school students) at Texas State University. She spent two summers working on original mathematics research at the *Duluth Summer Program* for undergraduates. After completing her degree at Texas A&M University in 2002, Kelli spent two years teaching at Lowell High School in San Francisco.

She later returned to graduate school at the University of Michigan and received her Ph.D. in 2010, with a thesis in algebraic combinatorics. Kelli was an NSF Postdoctoral Fellow at UC Berkeley from 2010-2013, and returned to UC Berkeley as a lecturer in 2015, after starting her tutoring and enrichment business, *Octopus Math*.

Kelli relishes the chance to work with students of all ages and to expose students to the idea that mathematics involves not just algorithms and memorization, but also incredibly rich opportunities to experiment and to look for patterns and connections. She has been involved with the *Berkeley Math Circle* since 2015 and has also had the opportunity to teach at the *Santa Cruz Math Circle* and at *Epsilon Camp*.

Kelli really enjoys hands-on math explorations, including logic puzzles, geometric constructions, and paper cutting and folding activities. Outside of math, Kelli is an avid scuba diver and loves to talk about fascinating ocean critters.

Dr. Norm Prokup joined the faculty at College Preparatory School in Oakland, CA in 2003. He has been the MTRW Geometry instructor for the 8A-9A groups. He graduated from Cornell University and received his Ph.D. from the University of Rhode Island at Kingston, studying nonlinear difference equations that model two-stage "healthy-infected" populations, suggested by epidemiologists. He has taught math ever since, working with students from grade 6 through college and covering a wide range of subjects, from algebra to ordinary differential equations.

Elysée Wilson-Egolf, although always fond of and interested in mathematics, spent most of her early schooling years studying literature and world languages. Strong encouragement by her grandfathers and her parents' provision of practical education in woodworking, repair, and agriculture led her to apply for study in college as a mechanical engineer. As an undergraduate at *UC Berkeley* she was exposed to real mathematics for the first time. She immediately applied for a change in degree to mathematics major and was soon spending full days in the math/stats library in Evans Hall in UCB campus.

After graduating with her BA in 2013, she remembered the patience and skill of her Russian and Polish instructors with gratitude and looked for opportunities to share their mathematical taste with others. She built on her years of after-school tutoring as a teenager and started working with the Beginners' group at the *Berkeley Math Circle's Summer Program* in 2016. Subsequently she has taught 6th grade algebra and both 6th grade and 7th grade problem-solving with MTRW; at BMC, she has taught Beginners I and II and Intermediate I groups. Although she likes to share any kind of elegant proof or surprising result, she is especially fond of classical geometry, topology, and analysis: her favorite lessons with BMC involve repeating decimal expansions and limits, the combinatorics of 4th-dimensional objects, and graph theory. A quote by Elysée regarding the Bulgarian Math Program appears earlier in this chapter.

For several years since the fall of 2016, she also worked with Bentley Upper School, a private high school in Lafayette, where she used the Bulgarian textbooks to teach Honors Geometry and enjoyed tying school topics to their applications in practical living. She is now headed for a new exciting job at the Urban School of San Francisco, where she will be teaching an *inquiry-based curriculum* to high school students as a way to stimulate their creativity and problem-solving abilities.

When not learning more mathematics or teaching her students, she still translates poems and walks in the mountains.

15. Math Taught the Right Way

In addition to running the Tehiyah Day School math program on this textbook series for 3 years, we continued the weekly program *"Math Taught the Right Way"* (MTRW) for a 3rd year, extending the Berkeley Math Circle (BMC) at UC Berkeley.

In spring 2019, MTRW had an enrollment of about 130 students from 72 schools and 27 towns in the extended San Francisco Bay Area. The four program groups were selected by age and proficiency in the previous groups. As in the spirit of the BMC, younger, more advanced students were promoted to higher groups. The spring groups were named 6B, 7B, 8B, and 9B.

6B Group:	**Instructor:**
• *6B Algebra*	Roy Zhao
• *6B Geometry*	Harry Main-Luu
• *6B Problem Solving*	Elena Blanter
7B Group:	
• *7B Algebra*	Elena Blanter
• *7B Geometry*	Harry Main-Luu
• *7B Problem Solving*	Elysée W.-Egolf
8B Group:	
• *8B Algebra*	Kelli Talaska
• *8B Geometry*	Norm Prokup
• *8B Proofs*	Elysée W.-Egolf
9B Group:	
• *9B Algebra*	Kelli Talaska
• *9B Geometry*	Norm Prokup
• *9B Proofs*	Zvezdelina Stankova

In the 15 weekly 3-hour sessions in spring 2019:

The *6B group* covered all of the 6B curriculum, plus some 7B *Geometry of Polyhedra*. *6B Problem Solving* concentrated on the *Common Fractions* problems and math contests.

7B Algebra covered *Integer Expressions* and *Algebra on Polynomials*. *7B Geometry* studied *Foundations of Geometry*, *Congruent Triangles,* and *Euclidean Constructions*. *7B Problem Solving* worked its way through *Bulgarian Math Contests* for 6th-7th grades, learning about modular arithmetic, the pigeonhole principle, set theory, angle chasing and others.

8B Algebra covered *Square Roots, Quadratic Equations,* and *Vieta's Formulas*. *8B Geometry* covered *Vectors, Operations, Midsegments,and Centroids*. *8B Problem-Solving* was based on *Rational Expressions* from the 8B curriculum and Chapters 5-6 on *Proofs* and *Induction* from "*A Decade of The Berkeley Math Circle - the American Experience,*" vol. I, edited by Stankova and Rike, as well math contests for 8th-9th grades on *Parity, Tiling, Divisibility,* and *Irrationality.*

9B Algebra reviewed *Linear Systems* and moved onto *Systems of Degree 2 Equations*. *9B Geometry* covered *Similar Triangles* and *Metric Relations*. *9B Proofs* finished *Circles and Polygons* from the 9A curriculum and and studied *Inversion in the Plane* from *"A Decade of the Berkeley Math Circle - the American Experience,"* vol. I-II, edited by Stankova and Rike.

16. BMC Summer Program

The Berkeley Math Circle Summer Program 2019 can be viewed as a third component of the MTRW Program, which completes the 6th-9th grade curriculum, as well as some math circle topics. Here is the list of faculty who taught at the BMC Summer Program in 2019:

- Chris Overton, Ph.D., Ayasdi
- Elena Blanter, M.A., MTRW
- Elizabeth Wang, UCB undergraduate
- Elysée Wilson-Egolf, B.A. UCB, Urban School of San Francisco
- Espen Slettnes, BMC
- Hai Main-Luu, BA in Math, UCB
- Kelli Talaska, Ph.D., UC Berkeley
- Laura Pierson, Harvard, BMC alumna
- Matyas Sustik, Ph.D., WalmartLabs
- Nikhil Sahoo, UC Berkeley, undergraduate
- Norm Prokup, Ph.D., College Prep, Oakland
- Paul Zeitz, Ph.D., USF, Proof School
- Pratima Karpe, M.A., BMC
- Quan Lam, Ph.D., Office UC President
- Roy Zhao, UCB grad. Student, MTRW
- Zvezdelina Stankova, Ph.D., UC Berkeley

In the 10 daily 3-hour sessions in summer 2019, the four program groups covered the following:

The *Beginners group* (with Hai Main-Luu, Elena Blanter, Elizabeth Yang, and Elysée Wilson-Egolf) applied a problems-based approach to survey the 6B Chapters 6-10, using almost entirely the 6B Workbooks.

Intermediate I Algebra class covered 7B *Modeling and Applications of Equations* with Kelli Talaska. Norm Prokup led the group in *Geometry* through 7B *Euclidean Constructions* and 8A *Geometric Inequalities*. Elysée Wilson-Egolf taught the students the structure of mathematics and proofs, targeting *Problem Solving* techniques through 6th-7th grade math contests.

Intermediate II Algebra class covered 8B *Combinatorics* and 9A *Classical Probability* with Kelli Talaska. Norm Prokup led the group in *Geometry* through 9B *Trigonometric Functions of an Acute Angle*. In *Math Circles sessions,* the students learned about

- *Gaussian Integers* w/ Laura Pierson;
- *Number Theory* w/ Matyas Sustik;
- *Generating Functions* w/ Paul Zeitz;
- *Compass Constructions* w/ Hai Main-Luu;
- *Geometry Applications* w/ Zvezda Stankova.

The *Advanced group* worked hard and was challenged on the following topics:

- *Group Theory* w/ Quan Lam;
- *Graph Theory* w/ Espen Slettnes;
- Number Theory w/ Matyas Sustik;
- Modular Arithmetic w/ Roy Zhao;
- Polya Enumeration w/ Paul Zeitz;
- Classification of Surfaces w/ Nikhil Sahoo;
- Compactness: Behavior Modification for Unruly Spaces w/ Chris Overton;
- Inversion in the Plane w/ Zvezda Stankova;
- School-Choice Mechanism w/ Pratima Karpe;
- *Compass Constructions* w/ Hai Main-Luu.

Attending the Summer Program were 133 students from 84 schools in 40 towns, coming from 6 states/countries around the world.

17. Acknowledgments

The adaptation of language and math notation for the U.S. reality in this middle and high school math program, as well as the technical and mathematical correctness of the series, is due to the dedication and selflessness of Tom Rike, my long-time collaborator and co-editor, and Kelli Talaska, Norm Prokup, and Elysée Wilson-Egolf, instructors at BMC and MTRW. You are all *amazing*!

Natalya St. Clair was the very first pilot teacher, who moved across the country to bravely undertake the new math Program in fall 2015 with the two 6^{th} grade classes at Tehiyah Day School. Her enthusiasm for mathematics was evident each and every day in her lively lessons, sharing with her students her joy in teaching math. She also had numerous suggestions for adaptation based on her daily experience with the Program on U.S. turf, and offered constructive criticism of the revised version of the series. George Khasin and Laurie Gold continued the pilot Program in spring 2016 with a steady rhythm and care for the students, and Andrea Jedwab piloted the 7^{th} and 8^{th} grade and continued with the 6^{th} and 7^{th} grade classes in 2016-2018.

Going further back, Elise Prowse, the former Interim Head of the pilot school and a math educator herself, recognized the power and depth of the new Program (while it was still in the original Bulgarian language!) and laid down the path of its implementation in practice. Debra Massey, the former Head of School and a parent herself, strongly backed the Program and supported it all the way through the crucial first two pilot years.

The middle school math Program at Tehiyah Day School would not have had a stable basis to land on had it not been for the Singapore Math Program in grades K-5, introduced in 2003 by Professor Alexander Givental of UC Berkeley and his wife, Laura Givental, the leader of the Berkeley Math Circle-Elementary Division for grades 1-4. The mere fact that Singapore Math preceded the pilot Math Program already immensely increased the chances of success of this pioneering Project.

It goes without saying that the vision of Professor Georgi Paskalev and his collaborators, Zdravka Paskaleva and Maya Alashka, has provided the generating power for this translation and adaptation effort. The success of the original Bulgarian textbook series is no surprise. The novel ideas of the overall design, the simplicity and elegance in its organization, and the balance between depth and breath of the material, are unsurpassed.

Any textbook series needs a Publisher. And what more suitable Publisher of the new textbook series is there than the original "Archimedes" Publishing House! Its Director, Professor Plamen Paskalev, undertook the Project without reservations, investing in a future based on the common ground of two very different math educational systems: the U.S. and the Bulgarian systems.

Angelina Avramova, the graphic designer of the original series and the technical editor of the new series, prepared the materials for publication with efficiency and professionalism that would put any anxious collaborator at ease and that would make any publisher proud.

Looking forward, Dr. Phil Daro, one of the three Principal Writers of the Common Core Standards, reviewed and endorsed the translated and adapted Program. Dr. Doug Ensley, the Deputy Executive Director of the American Math Competitions made available about a dozen AMC8 problems for inclusion in the current 7B Textbook.

The daily work of the pilot 6^{th}-8^{th} grade students and the constant support of their parents at Tehiyah moved this program from paper to real life and gave it meaning and purpose. The translation and adaptation effort would have been inconceivable without the understanding and love of my family: my husband, Dima, and our children, Lily and Oleg.

With gratitude,
Zvezdelina Stankova
Translator, Adapter, and Co-Author

10. PROPERTIES OF QUADRATIC FUNCTIONS: INCREASE AND DECREASE, MINIMA AND MAXIMA

EXAMPLE 1 Draw the graphs of the functions:

a) $y=\frac{1}{2}x^2+4x+6$;

b) $y=-\frac{1}{2}x^2+4x-1$.

Solution:

a)

1. We find the vertex of the parabola:

$$\left| \begin{array}{l} x_v=-\frac{b}{2a}=-\frac{4}{2\cdot\frac{1}{2}}=-4 \\ y_v=\frac{1}{2}(-4)^2+4\cdot(-4)+6=-2 \end{array} \right. ,\ V(-4,-2).$$

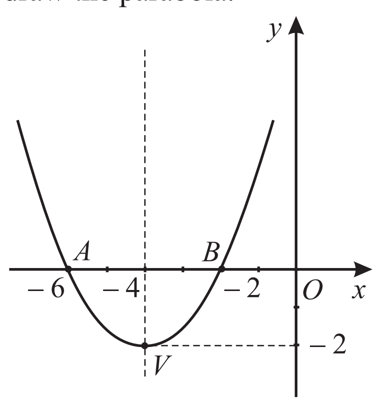
2. We find two more points A and B on the parabola.

$x_A=x_V-2=-6$,

$y_A=\frac{1}{2}(-6)^2+4\cdot(-6)+6=0$, $A(-6,0)$;

$x_B=x_V+2=-2$, $y_B=y_A=0$, $B(-2,0)$.

3. We draw the parabola.



b)

1. We find the vertex of the parabola:

$$\left| \begin{array}{l} x_v=-\frac{b}{2a}=-\frac{4}{2\cdot\left(-\frac{1}{2}\right)}=4 \\ y_v=-\frac{1}{2}\cdot 4^2+4\cdot 4-1=7 \end{array} \right. ,\ V(4,7).$$

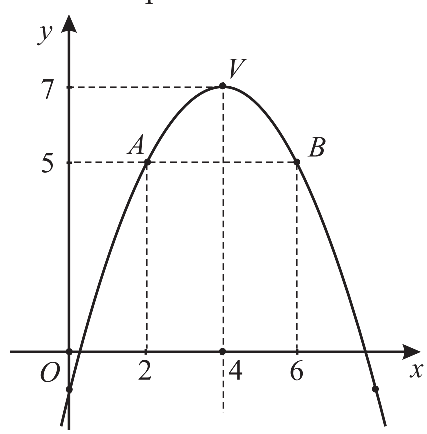
2. We find two more points A and B on the parabola.

$x_A=x_V-2=2$,

$y_A=-\frac{1}{2}\cdot 2^2+4\cdot 2-1=5$, $A(2,5)$;

$x_B=x_V+2=6$, $y_B=y_A=5$, $B(6,5)$.

3. We draw the parabola



Increase and decrease of a function

On the interval $(-\infty, 0)$, the graph of the function $y=x^2$ "slopes downward" since the value of the function decreases as the input increases.

On the interval $(0, +\infty)$, the graph of the function $y=x^2$ "slopes upward" since the value of the function increases as the input increases.

D10 The function $y=f(x)$ is called **increasing on an interval** if for any two values of x_1 and x_2 on this interval such that $x_1<x_2$, it is true that $f(x_1)<f(x_2)$.

The function $y=f(x)$ is called **decreasing on an interval** if for any two values of x_1 and x_2 on this interval such that $x_1<x_2$, it is true that $f(x_1)>f(x_2)$.

Thus, the function $y=x^2$, $D:\ x\in(-\infty,+\infty)$
| is **decreasing** on the interval $(-\infty, 0)$,
| is **increasing** on the interval $(0, +\infty)$.

Properties of a quadratic function

We will consider some properties of the function $y = ax^2 + bx + c$, $a \neq 0$.

Let $\boldsymbol{a} > \mathbf{0}$.

Consider the function in *Example 1a*,

$$y = f(x) = \frac{1}{2}x^2 + 4x + 6.$$

From its graph, we notice that the function decreases on the interval $(-\infty, -4)$ and increases on the interval $(-4, +\infty)$.

Let $\boldsymbol{a} < \mathbf{0}$.

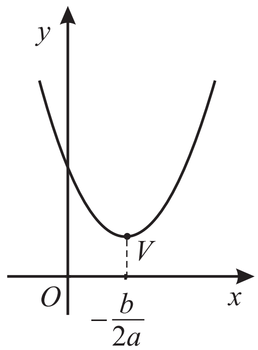
Consider the function in *Example 1b*,

$$y = f(x) = -\frac{1}{2}x^2 + 4x - 1.$$

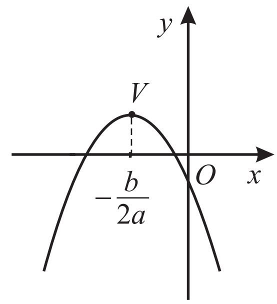
From its graph, we notice that the function increases on the interval $(-\infty, -4)$ and decreases on the interval $(-4, +\infty)$.

The properties of the functions and their graphs in the examples above are true for any quadratic function $y = ax^2 + bx + c$, $a \neq 0$.

$a > 0$



$a < 0$



T5

When $a > 0$:

Property 1. The graph of the function is a parabola.

Property 2. The function is:

- *decreasing* on the interval $\left(-\infty, -\frac{b}{2a}\right)$;
- *increasing* on the interval $\left(-\frac{b}{2a}, +\infty\right)$.

Property 3. The function has a smallest value, which is attained at $x = -\frac{b}{2a}$, and it has no largest value.

When $a < 0$:

Property 1. The graph of the function is a parabola.

Property 2. The function is:

- increasing on the interval $\left(-\infty, -\frac{b}{2a}\right)$;
- decreasing on the interval $\left(-\frac{b}{2a}, +\infty\right)$.

Property 3. The function has a largest value, which is attained at $x = -\frac{b}{2a}$, and it has no smallest value.

EXERCISES

Find the largest value of the function y and the value of x for which it is obtained:

1. $y = -x^2 + 4$; **2.** $y = -x^2 - 4x + 5$; **3.** $y = -x^2 + 6x - 6$;
4. $y = -x^2 + 8x$; **5.** $y = -4x^2 - 4x + 9$; **6.** $y = -9x^2 + 30x - 11$.

Find the smallest value of the function y and the value of x for which it is obtained:

7. $y = x^2 - 2x + 9$; **8.** $y = x^2 - 8x$; **9.** $y = 9x^2 + 12x - 5$;
10. $y = 4x^2 + 20x + 23$; **11.** $y = 36x^2 - 180x + 209$; **12.** $y = 4x^2 - 12x - 1$.

Find the intervals over which function y increases or decreases:

13. $y = -x^2 + 9$; **14.** $y = 2x^2 - 4x$; **15.** $y = -x^2 + 2x + 6$;
16. $y = x^2 + 6x$; **17.** $y = 2x^2 - 8x + 7$; **18.** $y = -3x^2 + 30x - 7$.

25. SOLVING SYSTEMS OF EQUATIONS OF DEGREE 2 WITH TWO UNKNOWNS. EXERCISES

EXAMPLE 1 Solve the system $\begin{vmatrix} x^2 + 2xy - 3y^2 = 0 \\ x^2 + y^2 - 10y = 0. \end{vmatrix}$

Solution:

We factor the expression $x^2 + 2xy - 3y^2$:

$x^2 + 2xy - 3y^2 = x^2 - xy + 3xy - 3y^2 = x(x-y) + 3y(x-y) = (x-y)(x+3y).$

We arrive at the system $\begin{vmatrix} (x-y)(x+3y) = 0 \\ x^2 + y^2 - 10y = 0, \end{vmatrix}$

which is equivalent to the union of the following two systems:

$\begin{vmatrix} x - y = 0 \to y = x \\ x^2 + y^2 - 10y = 0 \end{vmatrix}$ $\qquad$ $\begin{vmatrix} x + 3y = 0 \to x = -3y \\ x^2 + y^2 - 10y = 0 \end{vmatrix}$

$\begin{vmatrix} y = x \\ x^2 - 5x = 0 \to x_1 = 0, x_2 = 5 \end{vmatrix}$ $\qquad$ $\begin{vmatrix} x = -3y \\ y^2 - y = 0 \to y_3 = 0,\ y_4 = 1 \end{vmatrix}$

$\begin{vmatrix} x_1 = 0 \\ y_1 = 0 \end{vmatrix} \cup \begin{vmatrix} x_2 = 5 \\ y_2 = 5 \end{vmatrix}$ $\qquad$ $\begin{vmatrix} x_3 = 0 \\ y_3 = 0 \end{vmatrix} \cup \begin{vmatrix} x_4 = -3 \\ y_4 = 1 \end{vmatrix}$

Answer: The solutions to the given system are (0, 0), (5, 5), and (–3, 1).

D19 An equation of the type $ax^2 + bxy + cy^2 = 0$, where at least one of the numbers a, b, and c is different from zero, is called a **homogeneous equation** of degree 2 with two unknowns.

A homogeneous equation is satisfied by the pair $x = 0$ and $y = 0$. This solution is called **the zero solution** of the homogeneous equation. Any other solution in which at least one of x and y is not zero is called a **non-zero** solution of the homogeneous equation.

T11 After dividing a homogeneous equation by y^2 (or by x^2), we obtain a quadratic equation with respect to $\frac{x}{y}$ $\left(\text{or } \frac{y}{x}\right)$.

D20 A system with at least one homogeneous equation is called **homogeneous.**

EXAMPLE 2 Solve the homogeneous system $\begin{vmatrix} x^2 + xy - 2y^2 = 0 \\ xy + x - y = 1. \end{vmatrix}$

Solution:

We check that the pair (0, 0) is not a solution to the system.

We divide the two sides of the first equation by y^2 $(y \neq 0)$ and we obtain the equation

$\left(\frac{x}{y}\right)^2 + \frac{x}{y} - 2 = 0$, which is quadratic with respect to $\frac{x}{y}$.

We set $\frac{x}{y} = u$ and solve the equation $u^2 + u - 2 = 0$.

We obtain $u_1 = 1$ and $u_2 = -2$; i.e., $\frac{x}{y} = 1$ and $\frac{x}{y} = -2$.

The given system is equivalent to the union of the following two systems.

$$\begin{cases} x = y \\ xy + x - y = 1 \end{cases} \qquad \begin{cases} x = -2y \\ xy + x - y = 1 \end{cases}$$

$$\begin{cases} x = y \\ y^2 = 1 \to y_1 = 1, y_2 = -1 \end{cases} \qquad \begin{cases} x = -2y \\ 2y^2 + 3y + 1 = 0 \to y_3 = -1,\ y_4 = -\dfrac{1}{2} \end{cases}$$

$$\begin{cases} x_1 = 1 \\ y_1 = 1 \end{cases} \cup \begin{cases} x_2 = -1 \\ y_2 = -1 \end{cases} \qquad \begin{cases} x_3 = 2 \\ y_3 = -1 \end{cases} \cup \begin{cases} x_4 = 1 \\ y_4 = -\dfrac{1}{2} \end{cases}$$

Answer: The solutions to the given system are (1, 1), (–1, –1), (2, –1), and $\left(1, -\dfrac{1}{2}\right)$.

EXAMPLE 3 Solve the system $\begin{cases} x^2 + xy + y^2 = 7 \\ 4x^2 - xy - 4y^2 = -14. \end{cases}$

Solution:

The system does not contain a homogeneous equation, but the expressions containing the variables are homogeneous. To obtain a homogeneous equation, we multiply the first equation by 2 and then add the two equations term by term:

$$\begin{cases} x^2 + xy + y^2 = 7 \quad | \cdot 2 \\ 4x^2 - xy - 4y^2 = -14 \end{cases} \Leftrightarrow \begin{cases} 2x^2 + 2xy + 2y^2 = 14 \\ 4x^2 - xy - 4y^2 = -14 \end{cases} |+ \Leftrightarrow \begin{cases} 6x^2 + xy - 2y^2 = 0 \\ x^2 + xy + y^2 = 7 \end{cases}$$

We check that the pair (0, 0) is not a solution to the system, and we divide the first equation by y^2 $(y \neq 0)$.

We arrive at the system: $\begin{cases} 6\left(\dfrac{x}{y}\right)^2 + \dfrac{x}{y} - 2 = 0 \\ x^2 + xy + y^2 = 7. \end{cases}$

We set $\dfrac{x}{y} = u$, to arrive at the equation $6u^2 + u - 2 = 0$, from which we obtain $u_1 = \dfrac{1}{2}$ and $u_2 = -\dfrac{2}{3}$; i.e., $\dfrac{x}{y} = \dfrac{1}{2}$ $(y = 2x)$ and $\dfrac{x}{y} = -\dfrac{2}{3}$ $(3x = -2y)$.

The given system is equivalent to the union of the following two systems:

$$\begin{cases} y = 2x \\ x^2 + xy + y^2 = 7 \end{cases} \qquad \begin{cases} -2y = 3x \to y = -\dfrac{3}{2}x \\ x^2 + xy + y^2 = 7 \end{cases}$$

$$\begin{cases} y = 2x \\ x^2 = 1 \to x_1 = 1, x_2 = -1 \end{cases} \qquad \begin{cases} y = -\dfrac{3}{2}x \\ x^2 = 4 \to x_3 = 2,\ x_4 = -2 \end{cases}$$

$$\begin{cases} x_1 = 1 \\ y_1 = 2 \end{cases} \cup \begin{cases} x_2 = -1 \\ y_2 = -2 \end{cases} \qquad \begin{cases} x_3 = 2 \\ y_3 = -3 \end{cases} \cup \begin{cases} x_4 = -2 \\ y_4 = 3 \end{cases}$$

Answer: The solutions to the given system are (1, 2), (–1, –2), (2, –3), and (–2, 3).

EXERCISES Solve the systems:

1. $\begin{cases} x^2 - 3xy + 2y^2 = 0 \\ x^2 + y^2 - 4y = 0; \end{cases}$

3. $\begin{cases} x^2 - 4y^2 = 9 \\ xy + 2y^2 = 18; \end{cases}$

5. $\begin{cases} x^2 + xy = 15 \\ xy + y^2 = 10; \end{cases}$

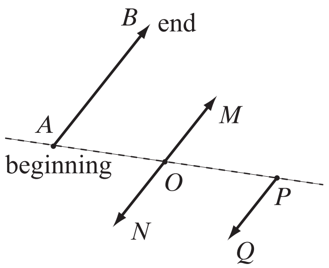
2. $\begin{cases} 2x^2 + 5xy - 18y^2 = 0 \\ xy + y^2 = 12; \end{cases}$

4. $\begin{cases} 5x^2 + xy - y^2 = -1 \\ 4x^2 + xy - y^2 = -2; \end{cases}$

6. $\begin{cases} x^2 + xy = 6 \\ y^2 + xy = 3. \end{cases}$

39. VECTORS AND OPERATIONS

D7 A **vector** is a segment whose endpoints are ordered; e.g., $\vec{AB}$.



Elements: beginning A; direction $A \to B$; length $|AB|$

Zero vector: $\vec{AA} = \vec{0}$; $|\vec{AA}| = 0$

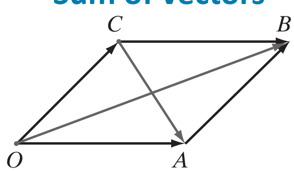
Collinear Pairs:

- *uni-directional*: $\vec{AB} \uparrow\uparrow \vec{OM}$
- *counter-directional*: $\vec{AB} \uparrow\downarrow \vec{ON}$
- *equal*: $\vec{ON} = \vec{PQ}$ $(\vec{ON} \uparrow\uparrow \vec{PQ},\ |\vec{ON}| = |\vec{PQ}|)$
- *opposite*: $\vec{ON} = -\vec{OM}$ $(\vec{ON} \uparrow\downarrow \vec{OM},\ |\vec{ON}| = |\vec{OM}|)$

$$\vec{PQ} = -\vec{OM}$$

Operations with vectors

- **Sum of vectors**



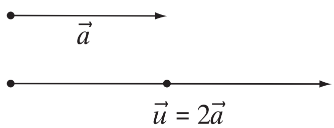
- **Difference of vectors**

T24 Two vectors with a common beginning are added by **the parallelogram rule**: $\vec{OA} + \vec{OC} = \vec{OB}$.

Adding more vectors: $\vec{OA} + \underbrace{\vec{AB} + \vec{BC}}_{} = \vec{OC}$.

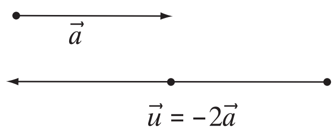
Two vectors with a common beginning are subtracted by **the triangle rule**: $\vec{OA} - \vec{OC} = \vec{CA}$; i.e., $\vec{CA} = \vec{OA} - \vec{OC}$.

- **Product of a vector with a number**



$\vec{a}$ and $2\vec{a}$ are collinear,

$k > 0,\ \vec{a} \uparrow\uparrow \vec{u},\ |\vec{u}| = |2||\vec{a}|$.



$\vec{a}$ and $-2\vec{a}$ are collinear

$k < 0,\ \vec{a} \uparrow\downarrow \vec{u},\ |\vec{u}| = |-2||\vec{a}|$.

$$\vec{AB} = -\vec{BA}$$
$$\vec{AB} + \vec{BA} = \vec{AA} = \vec{0}$$

T25

$$(\alpha+\beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$$
$$\alpha(\vec{u}+\vec{v}) = \alpha\vec{u} + \alpha\vec{v}$$
$$\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u},$$

for any numbers α and β and any vectors $\vec{u}$ and $\vec{v}$.

$\vec{AB}$ (as a sum) $= \vec{AO} + \vec{OB}$
$\vec{AB}$ (as a difference) $= \vec{OB} - \vec{OA}$,
where O is any point in the plane.

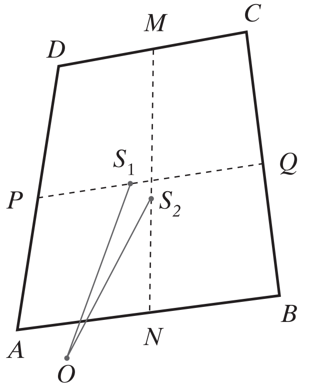
$$\vec{a} + \vec{0} = \vec{a} \qquad \vec{a} + (-\vec{a}) = \vec{0} \qquad \vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \vec{a} - \vec{a} = \vec{0}$$

T26 **Collinear vectors and points:** $\vec{OA} = k\,\vec{OB}$ *if and only if* points O, A, and B lie on one line.

T27 **Vectors and midpoints:** For any segment AB and points O and M in the plane,

$$\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$$ *if and only if* point M is the midpoint of AB.

EXAMPLE 1 Prove that the midpoints of the segments whose endpoints are midpoints of the opposite sides in an arbitrary quadrilateral coincide.



Proof:

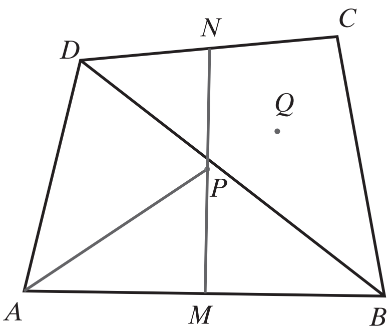
We denote the midpoints of sides AB, BC, CD, and DA by N, Q, M, and P, respectively. Let S_1 and S_2 be the midpoints of PQ and MN, respectively. We choose an arbitrary point O in the plane. Then

$$\vec{OS_1} = \frac{1}{2}(\vec{OP} + \vec{OQ}) = \frac{1}{2}\left[\frac{1}{2}(\vec{OA} + \vec{OD}) + \frac{1}{2}(\vec{OB} + \vec{OC})\right] =$$

$$= \frac{1}{4}(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}).$$

Analogously, $\vec{OS_2} = \frac{1}{4}(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}) \Rightarrow \vec{OS_1} = \vec{OS_2} \Rightarrow S_1 \equiv S_2$.

EXAMPLE 2 In quadrilateral $ABCD$, M and N are the midpoints of sides AB and CD, respectively, and P is the midpoint of MN. Point Q, inside ΔBCD, satisfies the following vector equality: $\vec{AQ} = \frac{1}{3}(\vec{AB} + \vec{AD} + \vec{AC})$. Prove that points A, P, and Q lie on one line.



Proof: Points A, P, and Q will lie on one line if $\vec{AP} = k\,\vec{AQ}$.

$$\vec{AP} = \frac{1}{2}(\vec{AM} + \vec{AN}) = \frac{1}{2}\left[\frac{1}{2}\vec{AB} + \frac{1}{2}(\vec{AD} + \vec{AC})\right]$$

$$\vec{AP} = \frac{1}{4}(\vec{AB} + \vec{AD} + \vec{AC}), \qquad \vec{AB} + \vec{AD} + \vec{AC} = 4\vec{AP} \qquad (1)$$

$$\vec{AQ} = \frac{1}{3}(\vec{AB} + \vec{AD} + \vec{AC}), \qquad \vec{AB} + \vec{AD} + \vec{AC} = 3\vec{AQ} \qquad (2)$$

From (1) and (2) we obtain $4\vec{AP} = 3\vec{AQ}$,; i.e., $\vec{AP} = \frac{3}{4}\vec{AQ}$; i.e., A, P, and Q lie on one line.

EXERCISES

1. Draw a rhombus $ABCD$ and construct vectors $\vec{AM}$ and $\vec{CN}$ equal to vector $\vec{DB}$. Prove that $AMNC$ is a rectangle.

2. Point S divides segment AB into two parts so that $AS = 12$ cm and $SB = 42$ cm. If O is any point not lying on AB, express $\vec{OS}$ in terms of $\vec{OA}$ and $\vec{OB}$.

3. Points L, M, and N on sides BC, CA, and AB of ΔABC are such that $BL : LC = 1 : 3$, $CM : MA = 2 : 5$, and $AN : NB = 3 : 2$. Express $\vec{LM}$, $\vec{MN}$, and $\vec{NL}$ in terms of vectors $\vec{CA} = \vec{a}$ and $\vec{CB} = \vec{b}$.

4. Let M and N be the midpoints of sides AB and CD of quadrilateral $ABCD$. If S is the midpoint of MN, and O is an arbitrary point, prove that $\vec{OS} = \frac{1}{4}(\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD})$.

5. In ΔABC, A_1, B_1, and C_1 are the midpoints of sides BC, AC, and AB. Prove that segments CC_1 and A_1B_1 have a common midpoint.

6. In ΔABC, prove that $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$. Is the converse statement below also true?

"If $\vec{u} + \vec{v} + \vec{w} = \vec{0}$, then there exists some triangle ABC in which $\vec{AB} = \vec{u}$, $\vec{BC} = \vec{v}$, and $\vec{CA} = \vec{w}$."

7. Consider ΔABC. Prove that there exists a triangle whose sides are equal and parallel to the medians of ΔABC.

(*Hint*: The median-vectors of ΔABC are $\vec{AA_1} = \frac{1}{2}(\vec{AB} + \vec{AC})$, $\vec{BB_1} = \frac{1}{2}(\vec{BA} + \vec{BC})$, and $\vec{CC_1} = \frac{1}{2}(\vec{CA} + \vec{CB})$. Use Exercise 6.)

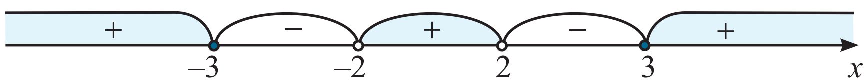
76. THE METHOD OF INTERVALS. EXERCISES

EXAMPLE 1 Solve the inequality $\dfrac{2x^2-13}{x^2-4} \ge 1.$

Solution:

We rewrite the inequality $\dfrac{2x^2-13}{x^2-4} \ge 1$ in the form

$$\frac{2x^2-13}{x^2-4}-1\ge 0 \Leftrightarrow \frac{2x^2-13-x^2+4}{x^2-4}\ge 0 \Leftrightarrow \frac{x^2-9}{x^2-4}\ge 0 \Leftrightarrow \frac{(x+3)(x-3)}{(x+2)(x-2)}\ge 0.$$



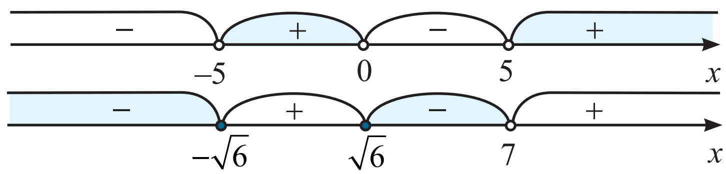
Answer: $x \in (-\infty, -3] \cup (-2, 2) \cup [3, +\infty)$

EXAMPLE 2 Solve the system of inequalities $\begin{cases} x^3-25x>0 \\ \dfrac{x^2-6}{x-7}\le 0. \end{cases}$

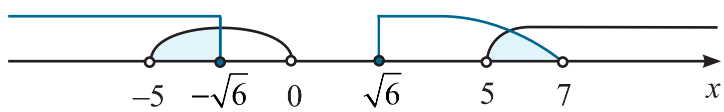
Solution:

1. We rewrite the system in the form

$$\begin{cases} x(x+5)(x-5)>0 \\ \dfrac{(x+\sqrt{6})(x-\sqrt{6})}{x-7}\le 0. \end{cases}$$



2. We find the common solutions to the two inequalities.



Answer: $x \in \left(-5, -\sqrt{6}\right] \cup (5, 7)$

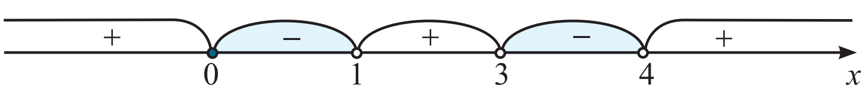
EXAMPLE 3 Solve the inequality $\dfrac{3}{x^2-4x+3} \ge \dfrac{4}{x^2-5x+4}.$

Solution:

$$\frac{3}{x^2-4x+3}-\frac{4}{x^2-5x+4}\ge 0 \Leftrightarrow \frac{3}{(x-1)(x-3)}-\frac{4}{(x-1)(x-4)}\ge 0$$

$$\Leftrightarrow \frac{3(x-4)-4(x-3)}{(x-1)(x-3)(x-4)}\ge 0 \Leftrightarrow \frac{-x}{(x-1)(x-3)(x-4)}\ge 0 \quad |\cdot(-1)$$

$$\Leftrightarrow \frac{x}{(x-1)(x-3)(x-4)}\le 0$$

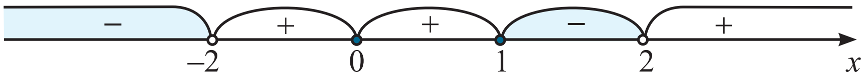


Answer: $x \in [0, 1) \cup (3, 4)$

EXAMPLE 4 Solve the inequality $\dfrac{x^3-4}{x^2-4}\le 1.$

Solution:

$$\frac{x^3-4}{x^2-4}-1\le 0 \Leftrightarrow \frac{x^3-4-x^2+4}{x^2-4}\le 0 \Leftrightarrow \frac{x^2(x-1)}{(x+2)(x-2)}\le 0$$



Answer: $x\in(-\infty,-2)\cup\{0\}\cup[1,2)$

EXAMPLE 5 A motorboat must travel from port A to port B and back in no more than 3 h. What should the motorboat's speed (in calm waters) be if the speed of the current of the river is 5 km/h, the distance between A and B is 28 km, and the boat rests at port B for 40 min?

Solution:

1. We denote by x the motorboat's speed in calm waters, AV: $x > 5$.
Then $V_{\text{with the current}} = x + 5$ km/h, and $V_{\text{against the current}} = x - 5$ km/h.

2. We make a table:

	S (km)	V (km/h)	t (h)
with current	28	$x+5$	$\dfrac{28}{x+5}$
against current	28	$x-5$	$\dfrac{28}{x-5}$

3. The relationship is

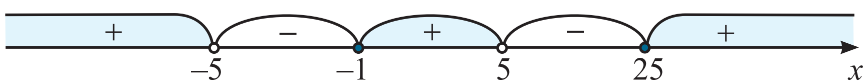
$$t_{\text{w/ cur.}}+t_{\text{against cur.}}+t_{\text{break}}\le 3.$$

$$\frac{28}{x+5}+\frac{28}{x-5}+\frac{2}{3}\le 3$$

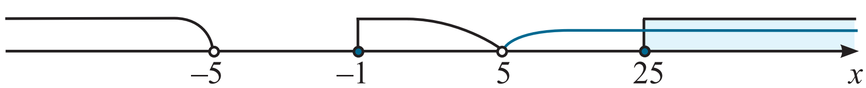
$$\frac{-x^2+24x+25}{3(x+5)(x-5)}\le 0 \quad |\cdot(-1)$$

$$\frac{(x-25)(x+1)}{(x+5)(x-5)}\ge 0$$

4. We solve the inequality using the method of intervals.



5. We take into account that AV: $x > 5$.



Answer: The speed of the motorboat must be at least 25 km/h.

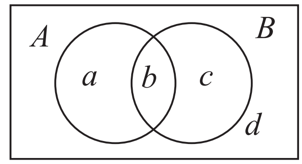
EXERCISES Solve the inequalities:

1. $3-\dfrac{2}{x}<x;$
2. $\dfrac{x-4}{3}>\dfrac{4}{x};$
3. $\dfrac{7x-5}{x+1}>x;$
4. $\dfrac{1}{x-3}\le\dfrac{1}{2x};$
5. $\dfrac{x-11}{x^2-4x-5}\le 0;$
6. $\dfrac{5x^2+x-3}{x^2-3x-4}<1;$
7. $\dfrac{x^2-13x+6}{x^2-4x+3}\le 2;$
8. $\dfrac{2x}{x-5}\le\dfrac{x+5}{x-3};$
9. $\dfrac{(x-2)^4(x-5)}{(x+3)(x+7)^2}<0;$
10. $\dfrac{(x+4)^4(x-7)}{(x+9)(x-1)^2}>0.$

89. PROBABILITY OF THE SUM OF INTERSECTING EVENTS

EXAMPLE 1 Using the figure, express the following probabilities in terms of a, b, c and d:

a) $P(A)$; b) $P(B)$;
c) $P(A \cap B)$; d) $P(A \cup B)$;
e) $P(A) + P(B) - P(A \cap B)$.



Solution:

a) $P(A) = \dfrac{a+b}{a+b+c+d}$; b) $P(B) = \dfrac{b+c}{a+b+c+d}$;

c) $P(A \cap B) = \dfrac{b}{a+b+c+d}$; d) $P(A \cup B) = \dfrac{a+b+c}{a+b+c+d}$;

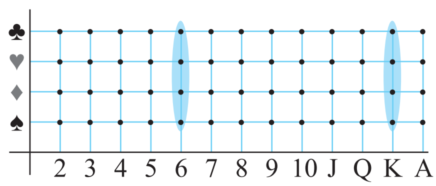
e) $P(A) + P(B) - P(A \cap B) = \dfrac{a+b}{a+b+c+d} + \dfrac{b+c}{a+b+c+d} - \dfrac{b}{a+b+c+d}$

$$= \frac{a+b+c}{a+b+c+d} = P(A \cup B).$$

EXAMPLE 2 We have a deck of 52 cards. We randomly draw one of them. Represent graphically the set of all elementary outcomes and find the probability that the card drawn is:

a) 6 or a King; b) 6 or hearts.

Solution: a)



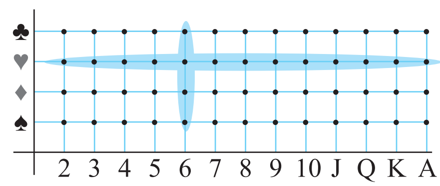
- $A = \{$the card is a 6$\}$
- $B = \{$the card is a King (K)$\}$
- $A \cap B = \varnothing$
- $A \cup B = \{$the card is 6 or a King$\}$

$\Rightarrow P(A) = \dfrac{4}{52} = \dfrac{1}{13}$; $P(B) = \dfrac{4}{52} = \dfrac{1}{13}$;

$\Rightarrow P(A \cap B) = 0$; $P(A \cup B) = \dfrac{8}{52} = \dfrac{2}{13}$;

$\Rightarrow P(A \cup B) = P(A) + P(B)$.

b)



- $A = \{$the card is 6$\}$
- $B = \{$the card is hearts (♥)$\}$
- $A \cap B = \{$the card is 6 of hearts (6♥)$\}$
- $A \cup B = \{$the card is 6 or hearts$\}$

$\Rightarrow P(A) = \dfrac{4}{52}$; $P(B) = \dfrac{13}{52}$;

$\Rightarrow P(A \cap B) = \dfrac{1}{52}$; $P(A \cup B) = \dfrac{16}{52}$;

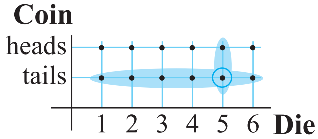
$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

D23 Events A and B are called **compatible (overlapping)** when they *can* occur at the same time or together; i.e., if $A \cap B \neq \varnothing$.

T10 If the events A and B are **compatible**, then the probability of their union is the sum of the probabilities of these events minus the probability of their intersection; i.e., $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

EXAMPLE 3 We simultaneously roll a fair coin and a die. Represent graphically the set of all elementary outcomes and find the probability that the die will show 5 or the coin will show tails.

Solution:



- $A = \{\text{the coin shows tails}\} \Rightarrow P(A) = \frac{1}{2}$.
- $B = \{\text{the die shows 5}\} \Rightarrow P(B) = \frac{1}{6}$.
- $A \cap B = \{\text{the coin shows tails and the die show 5}\}$,

$$\Rightarrow P(A \cap B) = \frac{1}{12}.$$

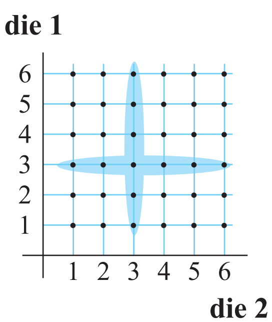
- $A \cup B = \{\text{the coin shows tails or the die shows 5, or both}\}$,

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = \frac{1}{2} + \frac{1}{6} - \frac{1}{12} = \frac{7}{12}.$$

EXAMPLE 4 We roll two fair dice. Find the probability that at least one die shows 3.

Solution:



- $A = \{\text{die 1 shows 3}\},\ P(A) = \frac{1}{6}$.
- $B = \{\text{die 2 shows 3}\},\ P(B) = \frac{1}{6}$.
- $A \cap B = \{\text{both dice show 3}\},\ P(A \cap B) = \frac{1}{36}$.
- $A \cup B = \{\text{die 1 shows 3 or die 2 shows 3, or both show 3}\}$.

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$$\Rightarrow P(A \cup B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}.$$

EXERCISES

1. We have a deck of 52 cards. Without looking, we draw one of them. Represent graphically the set of all elementary outcomes and find the probability the card drawn is diamonds or an Ace.

2. We have a deck of 32 cards. Without looking, we draw one of them. Represent graphically the set of all elementary outcomes and find the probability the card drawn is red or a face card.

3. We have a deck of 52 cards. Without looking, we draw one of them. Represent graphically the set of all elementary outcomes and find the probability the card drawn is black or 5.

4. We roll two fair dice. Find the probability that at least one die shows a number divisible by 3.

5. We roll two fair dice. Find the probability that at least one die shows an even number.