Recent Developments on Universal Forms

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Abstract

In this article, a brief history and recent developments on universal forms are introduced. Recent developments include Conway-Schneeberger's fifteen theorem, Bhargava's finiteness theorem on representability of an infinite set of positive integers by positive definite integral quadratic forms, and their generalizations to higher rank representability. Some interesting corollaries and applications are discussed.

1 Introduction

Representation theory of quadratic forms cherishes a long and splendid history since Pythagoras. For example, positive integers that are representable by sums of two, three, and four squares were determined by great names like Fermat-Euler, Gauss, and Lagrange, respectively. Hilbert [H] paid a tribute to this fascinating subject by posting two problems among his famous 23 problems for the 20th century - the 11th to quadratic forms over number fields and their integer rings, and the 17th to sums of squares over rational function fields.

Recall Lagrange's *four square theorem* [L], which states: *the integral quadratic form* $x^2 + y^2 + z^2 + u^2$ *represents all positive integers.* This celebrated statement had been generalized in many different directions such as Waring's problem and Pythagoras numbers, to name a few. One interesting generalization was made by Ramanujan [R] in the early 20th century, who found and listed all 55 positive definite integral quaternary diagonal quadratic forms, up to equivalence, that represent all positive

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integers. Dickson [D] called such forms *universal* and confirmed Ramanujan's list except one form from the list that was included by mistake. Later, Willerding [W] added 124 quaternary non-diagonal universal forms to the list, up to equivalence, and claimed that the list is complete. It is not hard to show that 'quaternary' is the best possible in the sense that there is no ternary universal form. Throughout this paper, by *integral* quadratic forms we mean *classic* ones, that is, quadratic forms with integer coefficients such that coefficients of non-diagonal terms are multiples of 2.

In 1930, Mordell [Mo1] proved *the five square theorem*, which states: *the integral quadratic form* $x^2 + y^2 + z^2 + u^2 + v^2$ *represents all positive definite integral binary quadratic forms.* (There is a very interesting new direction of generalizing Lagrange's four square theorem, called *the quadratic Waring's problem*[1] [Mo2], [K1,2]. See also [KO1-4], [Sa] for recent developments in this direction.) Such a form is called *2-universal*. Recently, all quinary 2-universal forms were determined by the authors [KKO1]. For positive integers $k, 3 \leq k \leq 8$, k-universal forms were also investigated in [KKO1] and [O1]. In 1941, Maass [M] proved *the three square theorem*, which states: *the integral quadratic form* $x^2 + y^2 + z^2$ *is universal over* $\mathbb{Q}(\sqrt{5})$*, i.e., it represents all totally positive integers over the ring of integers of* $\mathbb{Q}(\sqrt{5})$. All positive definite integral ternary universal forms over real quadratic fields were determined in [CKR]. For further development on universal forms over number fields, see [Ki1,2] and [EK].

In 1997, Conway and Schneeberger [Sch] announced so called *the fifteen theorem*, which characterizes the (1-)universality by the representability of a finite set of positive integers, the largest of which is 15. Using this criterion, they corrected several mistakes in the Willerding's list and announced the new and complete list of the 204 quaternary universal forms, up to equivalence. This result was so stunning and beautiful that it was introduced in the Notices of American Mathematical Society [Du]. It shed new light in the global theory of representations of quadratic forms. Motivated by the fifteen theorem, the authors proved 2-universal and 8-universal analogies in [KKO1] and [O2].

Then came Bhargava's generalization (see [C]). It was announced that he proved: for any infinite set S of positive integers there is a finite subset S_0 of S such that any positive definite integral quadratic form that represents every element of S_0 represents all elements of S. As a byproduct, he found S_0 for some interesting sets S, for example, the set of all primes, the set of all positive odd integers, and so on. His result was fully generalized by B.M. Kim, B.-K. Oh and the author as follows: *for any infinite set S of positive definite integral quadratic forms of a given rank, there is a finite subset S_0 of S such that any positive definite integral quadratic form that represents every element of S_0 represents all of S.*

[1] Mordell called it a *new Waring's problem.*

We adopt lattice theoretic language. Let F be a totally real number field, $\mathcal{O} = \mathcal{O}_F$ its ring of integers, and $\mathcal{O}^+$ the set of totally positive integers of F. An $\mathcal{O}$-*lattice* L is a finitely generated free $\mathcal{O}$-module equipped with a non-degenerate symmetric bilinear form B such that $B(L, L) \subseteq \mathcal{O}$. The corresponding quadratic map is denoted by Q. Note that $\mathcal{O}$-lattices naturally correspond to (classic) integral quadratic forms.

For an $\mathcal{O}$-lattice $L = \mathcal{O}\mathbf{x}_1 + \mathcal{O}\mathbf{x}_2 + \cdots + \mathcal{O}\mathbf{x}_n$, where $\{\, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \,\}$ is a fixed $\mathcal{O}$-basis, we write

$$L \cong (B(\mathbf{x}_i, \mathbf{x}_j)).$$

The matrix on the right hand side is called a *matrix presentation* of L, denoted by M_L. For $\mathcal{O}$-sublattices L_1, L_2 of L, we write $L = L_1 \perp L_2$ if $L = L_1 \oplus L_2$ and $B(\mathbf{v}_1, \mathbf{v}_2) = 0$ for all $\mathbf{v}_1 \in L_1, \mathbf{v}_2 \in L_2$. If L admits an orthogonal basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, we call L *diagonal* and simply write

$$L \cong \langle Q(\mathbf{x}_1), Q(\mathbf{x}_2), \ldots, Q(\mathbf{x}_n)\rangle.$$

We call L *non-diagonal*, otherwise. L is called *totally positive definite* or simply *positive* if $Q(\mathbf{v})$ is totally positive for any $\mathbf{v} \in L, \mathbf{v} \neq \mathbf{0}$. We call $Q(\mathbf{v})$ the *quadratic norm* of $\mathbf{v}$. Throughout this article, we assume that:

Every $\mathcal{O}$-lattice is positive

unless stated otherwise. As usual, $d(L) = \det(B(\mathbf{x}_i, \mathbf{x}_j))$ is called the *discriminant* of L, well-defined up to unit squares.

Let $V := FL = F \otimes L$. For any prime spot $\mathfrak{p} \subset \mathcal{O}$, let $L_\mathfrak{p} := \mathcal{O}_\mathfrak{p} L = \mathcal{O}_\mathfrak{p} \otimes L$ and $V_\mathfrak{p} := F_\mathfrak{p} V = F_\mathfrak{p} \otimes V$ be the *localizations* of L and V, respectively, at $\mathfrak{p}$.

Let ℓ, L be $\mathcal{O}$-lattices ($\mathcal{O}_\mathfrak{p}$-lattices, resp.). We say that L *represents* ℓ and write $\ell \to L$ if there is an injective $\mathcal{O}$-linear ($\mathcal{O}_\mathfrak{p}$-linear, resp.) map σ from ℓ into L preserving quadratic norms. Such σ is called a *representation*. If σ is onto, we call it an *isometry*. If there is an isometry between ℓ and L, we say that ℓ is isometric to L and write $\ell \cong L$. The set of all $\mathcal{O}$-lattices K that are isometric to a given lattice L is called the *class* of L, and the set of all $\mathcal{O}$-lattices K such that $K_\mathfrak{p}$ is isometric to $L_\mathfrak{p}$ at every $\mathfrak{p}$ is called the *genus* of L. It is well known that the genus of L contains only finitely many classes, the number of which is called the *class number* of L or of the genus.

A (positive) $\mathcal{O}$-lattice L is called k-*universal* if L represents all k-ary (positive) $\mathcal{O}$-lattices ℓ. A lattice is simply said to be *universal* if it's 1-universal. For a given set S of $\mathcal{O}$-lattices, an $\mathcal{O}$-lattice L is called S-*universal* if L represents all $\mathcal{O}$-lattices in S.

We write for convenience

$$[a, b, c] := \begin{pmatrix} a & c \\ c & b \end{pmatrix} \quad [a, b, c, d, e, f] := \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}.$$

For any unexplained terminologies and basic facts about quadratic lattices and spaces, we refer the readers to O'Meara's book [O'M1] and Conway-Sloane's book [CS].

2 Universal $\mathbb{Z}$-lattices

The famous four square theorem of Lagrange [L] can be rephrased as follows:

The $\mathbb{Z}$-lattice $I_4 \cong \langle 1,1,1,1\rangle$ is universal.

In the 19th century, this theorem was extended by Liouville and Pepin[2]. Liouville [Li] found all pairs (a,b) of integers $0 < a \le b$ for which $\langle 1,a,b,ab\rangle$ are universal. They are:

$$(a,b) \;=\; (1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3)^{\,3},\ (2,4),\ (2,5).$$

Pepin [P] found six more universal $\mathbb{Z}$-lattices:

$$\langle\, 1,1,1,2\,\rangle,\ \langle\, 1,1,1,4\,\rangle,\ \langle\, 1,1,2,4\,\rangle,\ \langle\, 1,2,2,2\,\rangle,\ \langle\, 1,1,2,8\,\rangle,\ \langle\, 1,2,4,4\,\rangle.$$

It was Ramanujan [R] who found all 54 universal quaternary[4] diagonal $\mathbb{Z}$-lattices.

Theorem 2.1 (Ramanujan) *There are exactly* 54 *universal quaternary diagonal $\mathbb{Z}$-lattices, up to isometry. They are:*

$\langle 1,1,1,1\rangle$	$\langle 1,1,1,2\rangle$	$\langle 1,1,2,2\rangle$	$\langle 1,2,2,2\rangle$	$\langle 1,1,1,3\rangle$	$\langle 1,1,2,3\rangle$
$\langle 1,2,2,3\rangle$	$\langle 1,1,3,3\rangle$	$\langle 1,2,3,3\rangle$	$\langle 1,1,1,4\rangle$	$\langle 1,1,2,4\rangle$	$\langle 1,2,2,4\rangle$
$\langle 1,1,3,4\rangle$	$\langle 1,2,3,4\rangle$	$\langle 1,2,4,4\rangle$	$\langle 1,1,1,5\rangle$	$\langle 1,1,2,5\rangle$	$\langle 1,2,2,5\rangle$
$\langle 1,1,3,5\rangle$	$\langle 1,2,3,5\rangle$	$\langle 1,2,4,5\rangle$	$\langle 1,1,1,6\rangle$	$\langle 1,1,2,6\rangle$	$\langle 1,2,2,6\rangle$
$\langle 1,1,3,6\rangle$	$\langle 1,2,3,6\rangle$	$\langle 1,2,4,6\rangle$	$\langle 1,2,5,6\rangle$	$\langle 1,1,1,7\rangle$	$\langle 1,1,2,7\rangle$
$\langle 1,2,2,7\rangle$	$\langle 1,2,3,7\rangle$	$\langle 1,2,4,7\rangle$	$\langle 1,2,5,7\rangle$	$\langle 1,1,2,8\rangle$	$\langle 1,2,3,8\rangle$
$\langle 1,2,4,8\rangle$	$\langle 1,2,5,8\rangle$	$\langle 1,1,2,9\rangle$	$\langle 1,2,3,9\rangle$	$\langle 1,2,4,9\rangle$	$\langle 1,1,5,9\rangle$
$\langle 1,1,2,10\rangle$	$\langle 1,2,3,10\rangle$	$\langle 1,2,4,10\rangle$	$\langle 1,2,5,10\rangle$	$\langle 1,1,2,11\rangle$	$\langle 1,2,4,11\rangle$
$\langle 1,1,2,12\rangle$	$\langle 1,2,4,12\rangle$	$\langle 1,1,2,13\rangle$	$\langle 1,2,4,13\rangle$	$\langle 1,1,2,14\rangle$	$\langle 1,2,4,14\rangle$

[2]Their purpose of study was to find an explicit formula for the number of representations rather than to find universal lattices.

[3]Jacobi [J] also found this. That is, $\langle 1,2,3,6\rangle$ is universal.

[4]It is not hard to show that 'quaternary' is the best possible in the sense that there is no universal ternary $\mathbb{Z}$-lattices.

Ramanujan's idea of proof was as follows: Let $L = \langle a, b, c, d\rangle$, $0 < a \le b \le c \le d$. In order for L to represent 1 and 2, it is necessary to have $a = 1$, $1 \le b \le 2$. If $a = b = 1$, then $1 \le c \le 3$ for L to represent 3. If $a = 1, b = 2$, then $2 \le c \le 5$ for L to represent 5. For each of these seven cases, one can obtain the range of d, for example, if $a = b = c = 1$, then $1 \le d \le 7$ in order for L to represent 7. Only 55 candidates, including $\langle 1, 2, 5, 5\rangle$ [5] that is not in the above list, survives this elimination process. The hard part is to prove that these are all universal except $\langle 1, 2, 5, 5\rangle$. For each of theses candidates, he figured out the numbers that cannot be represented by ternary sublattices and showed that those exceptional numbers are indeed represented by the lattice.

It was Dickson [D] who called such forms *universal*. He confirmed Ramanujan's list and found some universal quaternary non-diagonal $\mathbb{Z}$-lattices. Later, Willerding [W] found 124 universal quaternary non-diagonal $\mathbb{Z}$-lattices, and claimed that these are all up to isometry. Her list, however, turned out to be erroneous and incomplete. She included nine wrong lattices, listed one lattice twice, and missed thirty six. Schneeberger [Sch] fixed her mistakes and completed the list by proving the fifteen theorem (see also [B]).

Theorem 2.2 (The Fifteen Theorem) *A $\mathbb{Z}$-lattice is universal if and only if it represents the following nine positive integers* [6]:

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 14,\ 15.$$

Schneeberger proved the fifteen theorem by introducing the notion of turant and escalation starting from the $\mathbb{Z}$-lattice $\ell_0 = \{0\}$. Let's call ℓ_0 the 0-dimensional escalation lattice. The *turant* of a non-universal $\mathbb{Z}$-lattice L is the smallest positive integer that cannot be represented by L. An *escalation* of non-universal L is a lattice that contains L as a sublattice of codimension 1 and represents the turant of L [7]. An *escalation lattice* is a lattice that can be obtained from a finite sequence of escalations starting from ℓ_0. Then $\ell_1 \cong \langle 1\rangle$ is the unique 1-dimensional escalation lattice. Since the turant of ℓ_1 is 2, the 2-dimensional escalation lattices are $\ell_{21} \cong \langle 1, 1\rangle$ and $\ell_{22} \cong \langle 1, 2\rangle$. There are three escalations of ℓ_{21} and six escalations of ℓ_{22}. These nine $\mathbb{Z}$-lattices are the 3-dimensional escalation lattices. Escalating each of these nine 3-dimensional escalation lattices, one obtains 207 escalation lattices of rank 4, all of which are universal except six. Finally, one obtains 1630 escalation lattices of rank 5 from these six non-universal escalation lattices of rank 4. All of them are universal and the escalation process stops. The fifteen theorem follows immediately from

[5] In the Ramanujan's original list, $\langle 1, 2, 5, 5\rangle$, which fails to represent 15, was included by mistake.

[6] The set of theses nine positive integers is the minimal set in the sense that for each a in the set, there exists a $\mathbb{Z}$-lattice that represents all positive integers but a.

[7] Note that Ramanujan's elimination process is also based on this notion.

the observation that only possible truants of the non-universal escalation lattices are $1, 2, 3, 5, 6, 7, 10, 14, 15$.

Schneeberger also considered *non-classic* universal $\mathbb{Z}$-lattices [8], which correspond to quadratic forms with integer coefficients, and introduced *the* 290-*Conjecture*:

Any non-classic $\mathbb{Z}$-lattice is universal if it represetns $1, 2, \ldots, 290$.

In this article, however, we are not going to discuss non-classic case in detail [9].

The following is the new and complete list of L's, where $\langle 1\rangle \perp L$ covers all 150 universal quaternary non-diagonal $\mathbb{Z}$-lattices [10], up to isometry, by Schneeberger:

[1, 2, 2, 0, 1, 0]	[2, 2, 2, 0, 1, 1]	[1, 2, 3, 0, 1, 0]	[2, 2, 2, 0, 1, 0]	[1, 2, 4, 0, 1, 0]	[2, 2, 3, 1, 1, 0]
[1, 3, 3, 0, 1, 0]	[2, 2, 3, 0, 1, 1]	[1, 2, 5, 0, 1, 0]	[2, 2, 3, 1, 0, 0]	[2, 2, 3, 0, 1, 0]	[2, 2, 4, 1, 1, 0]
[1, 2, 6, 0, 1, 0]	[1, 3, 4, 0, 1, 0]	[2, 2, 4, 1, 0, 0]	[2, 2, 5, 1, 1, 0]	[2, 3, 3, 1, 0, 1]	[2, 3, 3, 0, 1, 1]
[1, 3, 5, 0, 1, 0]	[2, 2, 4, 0, 1, 0]	[1, 2, 8, 0, 1, 0]	[2, 2, 5, 1, 0, 0]	[2, 3, 3, 0, 0, 1]	[2, 3, 3, 0, 1, 0]
[1, 2, 9, 0, 1, 0]	[1, 3, 6, 0, 1, 0]	[2, 3, 4, 1, 0, 1]	[2, 2, 5, 0, 1, 0]	[2, 3, 4, 1, 1, 0]	[1, 2, 10, 0, 1, 0]
[2, 3, 4, 0, 1, 1]	[2, 2, 6, 0, 1, 1]	[2, 3, 4, 1, 0, 0]	[2, 4, 4, 0, 2, 1]	[2, 2, 6, 0, 1, 0]	[2, 3, 4, 0, 1, 0]
[2, 3, 5, 1, 0, 1]	[1, 2, 12, 0, 1, 0]	[2, 3, 5, 1, 1, 0]	[2, 2, 7, 0, 1, 1]	[2, 4, 4, 0, 2, 0]	[2, 4, 4, 1, 0, 1]
[1, 2, 13, 0, 1, 0]	[2, 3, 5, 0, 1, 1]	[2, 3, 5, 1, 0, 0]	[2, 2, 7, 0, 1, 0]	[2, 4, 4, 0, 1, 1]	[1, 2, 14, 0, 1, 0]
[2, 3, 5, 0, 0, 1]	[2, 4, 5, 1, 2, 0]	[2, 3, 5, 0, 1, 0]	[2, 4, 4, 0, 0, 1]	[2, 4, 5, 0, 2, 1]	[2, 4, 4, 0, 1, 0]
[2, 3, 6, 0, 1, 1]	[2, 4, 5, 1, 0, 1]	[2, 4, 5, 0, 2, 0]	[2, 3, 6, 0, 0, 1]	[2, 3, 6, 0, 1, 0]	[2, 4, 5, 0, 1, 1]
[2, 4, 6, 1, 2, 0]	[2, 4, 5, 1, 0, 0]	[2, 4, 5, 0, 0, 1]	[2, 4, 6, 0, 2, 1]	[2, 5, 5, 1, 2, 1]	[2, 5, 5, 0, 2, 1]
[2, 4, 5, 0, 1, 0]	[2, 4, 6, 1, 0, 1]	[2, 3, 7, 0, 0, 1]	[2, 3, 7, 0, 1, 0]	[2, 4, 6, 0, 2, 0]	[2, 4, 6, 1, 1, 0]
[2, 4, 7, 1, 2, 0]	[2, 4, 6, 0, 1, 1]	[2, 4, 6, 1, 0, 0]	[2, 5, 5, 0, 2, 0]	[2, 3, 8, 0, 1, 1]	[2, 4, 6, 0, 0, 1]
[2, 4, 7, 1, 0, 1]	[2, 5, 5, 0, 0, 1]	[2, 5, 6, 1, 2, 1]	[2, 3, 8, 0, 1, 0]	[2, 4, 6, 0, 1, 0]	[2, 5, 6, 1, 2, 0]
[2, 4, 7, 1, 1, 0]	[2, 5, 6, 0, 2, 1]	[2, 5, 5, 0, 1, 0]	[2, 3, 9, 0, 1, 1]	[2, 4, 7, 1, 0, 0]	[2, 5, 6, 1, 0, 1]
[2, 4, 7, 0, 1, 1]	[2, 3, 9, 0, 0, 1]	[2, 3, 9, 0, 1, 0]	[2, 5, 6, 0, 2, 0]	[2, 5, 6, 1, 1, 0]	[2, 5, 6, 0, 1, 1]
[2, 4, 7, 0, 1, 0]	[2, 5, 6, 1, 0, 0]	[2, 5, 7, 1, 2, 1]	[2, 3, 10, 0, 1, 1]	[2, 5, 6, 0, 0, 1]	[2, 5, 7, 1, 2, 0]
[2, 4, 8, 0, 2, 0]	[2, 3, 10, 0, 0, 1]	[2, 3, 10, 0, 1, 0]	[2, 4, 8, 0, 1, 1]	[2, 5, 6, 0, 1, 0]	[2, 5, 7, 1, 0, 1]
[2, 4, 9, 0, 2, 1]	[2, 5, 7, 1, 1, 0]	[2, 4, 8, 0, 1, 0]	[2, 5, 7, 0, 2, 0]	[2, 5, 7, 0, 1, 1]	[2, 5, 7, 1, 0, 0]
[2, 4, 9, 0, 1, 1]	[2, 4, 9, 0, 0, 1]	[2, 4, 10, 0, 2, 1]	[2, 5, 7, 0, 1, 0]	[2, 4, 9, 0, 1, 0]	[2, 4, 10, 0, 2, 0]
[2, 5, 8, 0, 2, 0]	[2, 4, 10, 0, 1, 1]	[2, 4, 10, 0, 0, 1]	[2, 5, 9, 0, 2, 1]	[2, 4, 10, 0, 1, 2]	[2, 5, 8, 0, 1, 0]
[2, 4, 11, 0, 2, 0]	[2, 4, 11, 0, 1, 1]	[2, 5, 9, 0, 2, 0]	[2, 5, 9, 0, 0, 1]	[2, 4, 11, 0, 1, 0]	[2, 5, 10, 0, 2, 1]
[2, 4, 12, 0, 2, 0]	[2, 5, 9, 0, 1, 0]	[2, 4, 12, 0, 1, 1]	[2, 4, 13, 0, 2, 1]	[2, 5, 10, 0, 2, 0]	[2, 5, 10, 0, 1, 1]
[2, 4, 12, 0, 1, 0]	[2, 5, 10, 0, 0, 1]	[2, 4, 13, 0, 2, 0]	[2, 4, 13, 0, 1, 1]	[2, 5, 10, 0, 1, 0]	[2, 4, 13, 0, 0, 1]
[2, 4, 14, 0, 2, 1]	[2, 4, 13, 0, 1, 0]	[2, 4, 14, 0, 2, 0]	[2, 4, 14, 0, 1, 1]	[2, 4, 14, 0, 0, 1]	[2, 4, 14, 0, 1, 0]

[8] Scaling by 2, one may study classic *even-universal* $\mathbb{Z}$-lattices, which represent all positive even integers, instead of non-classic universal $\mathbb{Z}$-lattices.

[9] See also [D], [Mor].

[10] So, the total number of universal quaternary $\mathbb{Z}$-lattices is 204. The largest discriminant is 112, which was known to Willerding.

3 2-universal $\mathbb{Z}$-lattices

Mordell's five square theorem [Mo1] can be rephrased as follows:

The $\mathbb{Z}$-lattice $I_5 \cong \langle 1,1,1,1,1\rangle$ is 2-universal.

B.M. Kim, S. Raghavan and the author [KKR] found all 2-universal quinary [11] diagonal $\mathbb{Z}$-lattices. They are: $I_3 \perp \langle a,b\rangle$ for

$$(a,b) \;=\; (1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3).$$

The elimination process to obtain the above five candidates is quite similar to that of Ramanujan. Each of the five candidates is of class number 1 and represents all binary $\mathbb{Z}$-lattices over $\mathbb{Z}_p$ at every prime p. So, they are all 2-universal. The result then was extended to non-diagonal case in [KKO1].

Theorem 3.1 (Kim-Kim-Oh) *There exists eleven 2-universal quinary $\mathbb{Z}$-lattices, up to isometry. There are five diagonal ones, listed above, and six non-diagonal ones:*

$$\begin{array}{lll} I_2 \perp [1,2,2,0,1,0], & I_2 \perp [1,2,3,0,1,0], & I_2 \perp [2,2,2,0,1,0],\\ I_2 \perp [3,2,2,0,1,0], & I_2 \perp [2,2,2,1,1,0], & I_2 \perp [2,2,3,1,1,0]. \end{array}$$

The sketch of proof: Assume L is a 2-universal quinary non-diagonal $\mathbb{Z}$-lattice. Since L represents I_2, which is unimodular, L is split by I_2, i.e., $L \cong I_2 \perp L_0$ for some ternary non-diagonal $\mathbb{Z}$-lattice L_0.
(Case 1) L_0 represents 1: In this case, $L_0 \cong \langle 1\rangle \perp [a,b,c]$ for some a,b,c such that $[a,b,c]$ is non-diagonal. We may assume that $[a,b,c]$ is Minkowski reduced. If $a \geq 3$, then L cannot represent $[2,3,1]$. So, $a=2$ and $c=1$. Then $2 \leq b \leq 3$ in order for L to represent $\langle 3,3\rangle$. Therefore, $L_0 \;\cong\; [1,2,2,0,1,0]$ or $[1,2,3,0,1,0]$.
(Case 2) L_0 does not represent 1: Since L_0 represents $[2,2,1]$, we may conclude that $L_0 \cong [2,2,c,1,0,0]$ or $[2,2,c,1,1,0]$. Then $2 \leq c \leq 3$ in order for L to represent $\langle 2,3\rangle$, and hence $L_0 \;\cong\; [2,2,2,1,0,0]$, $[2,2,3,1,0,0]$, $[2,2,2,1,1,0]$ or $[2,2,3,1,1,0]$.
So, only six candidates survives the elimination process. Every candidate represents all binary $\mathbb{Z}$-lattices over $\mathbb{Z}_p$ at every prime p. All except $I_2 \perp [2,2,2,1,0,0]$ are of class number 1 and hence are 2-universal. The 2-universality of $I_2 \perp [2,2,2,1,0,0]$, which is of class number 2, follows from a series of technical lemmas using binary representation of the sublattice $I_2 \perp [2,2,1]$ of class number 1 contained in both classes [12] (see [KKO1]).

The following 2-universal version of the fifteen theorem that characterizes 2-universal $\mathbb{Z}$-lattices can also be found in [KKO1]:

[11] No quaternary $\mathbb{Z}$-lattice can be 2-universal.
[12] The other class in the genus is the class of $\langle 1,1,1,1,6\rangle$.

Theorem 3.2 (Criterion for 2-universality) *A $\mathbb{Z}$-lattice is 2-universal if and only if it represents the following six positive binary $\mathbb{Z}$-lattices*[13]*:*

$$\langle 1,1\rangle,\quad \langle 2,3\rangle,\quad \langle 3,3\rangle,\quad [2,2,1],\quad [2,3,1],\quad [2,4,1].$$

This theorem follows from the fact that every $\mathbb{Z}$-lattice representing the six binary $\mathbb{Z}$-lattices listed above contain one of the eleven 2-universal quinary $\mathbb{Z}$-lattices.

4 k-universal $\mathbb{Z}$-lattices for $k \geq 3$

We refer the readers to [KKO1] or [O2] for the details in this section. It is well known [K1] that:

The $\mathbb{Z}$-lattices I_{k+3} is k-universal for $3 \leq k \leq 5$.

Define

$$u[k] = u_{\mathbb{Z}}[k] := \min\{\ \mathrm{rank}(L)\ :\ L\text{'s are } k\text{-universal } \mathbb{Z}\text{-lattices}\ \}$$

and call it the *k-th universal number of* $\mathbb{Z}$, which exists for every $k \geq 1$. Since $u[k] \geq k+3$ for all k, we may conclude $u[k] = k+3$ for $1 \leq k \leq 5$ thanks to Lagrange, Mordell and Ko.

Let $u_{\mathbb{Q}}(k)$ to be the minimal dimension of positive definite quadratic spaces over $\mathbb{Q}$ which represent all positive definite quadratic spaces of rank k. Then it is well known (see [O'M1] for example) that

$$u_{\mathbb{Q}}[k] = k+3 \ \text{ for all } \ k.$$

If we let $u_p[k]$ be the minimal rank of $\mathbb{Z}_p$-lattices which represent all $\mathbb{Z}_p$-lattices of rank k, then: $u_p[k] = k+3$ for all $k \geq 3$, and

$$u_p[1] = \begin{cases} 2 & \text{if } p \neq 2, \\ 3 & \text{if } p = 2, \end{cases} \qquad u_p[2] = \begin{cases} 4 & \text{if } p \neq 2, \\ 5 & \text{if } p = 2. \end{cases}$$

Returning to $u[k]$, one can show:

Theorem 4.1 *For $1 \leq k \leq 10$, we have:*

$$u[k] = \begin{cases} k+3 & \textit{if} \quad 1 \leq n \leq 5, \\ 13 & \textit{if} \quad n = 6, \\ 15 & \textit{if} \quad n = 7, \\ 16 & \textit{if} \quad n = 8, \\ 28 & \textit{if} \quad n = 9, \\ 30 & \textit{if} \quad n = 10. \end{cases}$$

[13] The set of theses six binary $\mathbb{Z}$-lattices is a minimal set in the sense that for each ℓ in the set, there exists a $\mathbb{Z}$-lattice that represents the other five but not ℓ.

The growth of the k-th universal number $u[k]$ is expected to be of the same magnitude as that of $(k+3)h(I_{k+3})$, where $h(I_{k+3})$ is the class number of I_{k+3}. It is not hard to prove that the number of non-isometric k-universal $\mathbb{Z}$-lattices of the minimal rank $u[k]$ is finite. For each k, $3 \le k \le 8$, we list the k-universal $\mathbb{Z}$-lattices of the minimal rank together with and a few candidates, up to isometry, in the next theorem:

Theorem 4.2 (1) *There are nine* 3*-universal* $\mathbb{Z}$*-lattices of rank* $u[3] = 6$ *found so far, and there can be at most two more. They are*[14] :

$$I_6, \quad I_5 \perp A_1, \quad I_5 \perp \langle 3 \rangle, \quad I_4 \perp A_1 \perp A_1, \quad I_4 \perp A_2, \quad I_4 \perp A_1 10[1\tfrac{1}{2}],$$
$$I_3 \perp A_3, \quad I_3 \perp A_2 21[1\tfrac{1}{3}], \quad I_4 \perp A_1 \perp \langle 3 \rangle, \quad I_3 \perp A_1 \perp {A_2}^*, \quad I_3 \perp A_2 \perp \langle 3 \rangle^*.$$

(2) *There are exactly three* 4*-universal* $\mathbb{Z}$*-lattices of rank* $u[4] = 7$*. They are*:

$$I_7, \quad I_6 \perp A_1, \quad I_5 \perp A_2.$$

(3) *There are two* 5*-universal* $\mathbb{Z}$*-lattices of rank* $u[5] = 8$ *found so far, and there can be at most one more. They are*:

$$I_8, \quad I_7 \perp A_1, \quad I_6 \perp {A_2}^*.$$

(4) *There are exactly two* 6*-universal* $\mathbb{Z}$*-lattices of rank* $u[6] = 13$*. They are*:

$$I_6 \perp E_7, \quad I_7 \perp E_6.$$

(5) *There are exactly three* 7*-universal* $\mathbb{Z}$*-lattices of rank* $u[7] = 15$*. They are*:

$$I_7 \perp E_8, \quad I_8 \perp E_7, \quad I_7 \perp E_7 6[1\tfrac{1}{2}].$$

(6) $I_8 \perp E_8$ *is the only* 8*-universal* $\mathbb{Z}$*-lattice of rank* $u[8] = 16$.

Observe that if $k \ge 6$, then no diagonal $\mathbb{Z}$-lattice can be k-universal. This is simply because E_6 cannot be represented by a sum of of squares. The following is an 8-universal version of the fifteen theorem that characterizes 8-universal $\mathbb{Z}$-lattices:

Theorem 4.3 (Criterion for 8-universality) *A positive* $\mathbb{Z}$*-lattice is* 8*-universal if and only if it represents both* I_8 *and* E_8*.*

[14] Those with $*$-mark in (1) and (3) are candidates. Refer [CS] for $\mathbb{Z}$-lattice notations.

5 Universal $\mathcal{O}$-lattices

Let F be a totally real number field and $\mathcal{O} = \mathcal{O}_F$ be its ring of algebraic integers. Maass' three square theorem [M] says:

The $\mathcal{O}_F$-lattice I_3 is universal, where $F = \mathbb{Q}(\sqrt{5})$.

No $\mathcal{O}$-lattice corresponding to a sum of squares can be universal over any totally real number field F other than $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$ according to Siegel [S1], who proved:

The only totally real number fields, where every totally positive integer can be written as a sum of squares, are $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$.

The following theorem extends Maass' three square theorem [CKR]:

Theorem 5.1 (Chan-Kim-Raghavan) *There exist exactly five universal ternary $\mathcal{O}_F$-lattices, up to isometry, when $F = \mathbb{Q}(\sqrt{5})$. They are*[15]:

$$\langle 1,1,1\rangle \quad \langle 1,1,2\rangle \quad \langle 1,1,2+\epsilon_5\rangle \quad \langle 1\rangle \perp [2,2+\epsilon_5,1] \quad \langle 1\rangle \perp [2,2+\epsilon_5',1],$$

where $\epsilon_5 = (1+\sqrt{5}\,)/2$ is the fundamental unit of F and $\epsilon_5' = (1-\sqrt{5}\,)/2$.

If L is universal, then $L \cong \langle 1\rangle \perp L_0$ for some binary sublattice L_0. In order to prove the theorem, one eliminates all other possibilities for L except the five listed by finding upper bounds of norms of successive minima of L_0. Then prove that each of the five represents all totally positive algebraic integers of F over $(\mathcal{O}_F)_\mathfrak{p}$ at every prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ (see [O'M2]). Furthermore, one can show by using Siegel's mass formula that they all have class number 1, which proves their universality (see [Kör]).

Totally real number fields may well admit universal $\mathcal{O}$-lattices that are not sums of squares. It is not hard to show that there exist no universal binary $\mathcal{O}$-lattices. Hence, the minimal rank of universal $\mathcal{O}$-lattices is at least 3. So, if we define

$$u(\mathcal{O}_F) := \min\{\ \mathrm{rank}(L)\ :\ L\text{'s are universal } \mathcal{O}_F\text{-lattices}\ \},$$

then $u(\mathcal{O}_F) \geq 3$. We call $u(\mathcal{O}_F)$ *the universal number* of $\mathcal{O}_F$. More generally, we can define *the k-th universal number* of $\mathcal{O}$ by

$$u_\mathcal{O}[k] := \min\{\ \mathrm{rank}(L)\ :\ L\text{'s are } k\text{-universal } \mathcal{O}_F\text{-lattices}\ \}.$$

But almost nothing is known about this number when $k \geq 2$ and $F \neq \mathbb{Q}$. Note that $u(\mathcal{O}) = u_\mathcal{O}[1]$ and, in particular, the universal number of $\mathbb{Z}$ is $u(\mathbb{Z}) = u[1] = 4$. Kitaoka conjectured, in a private communication, that there may be only finitely many totally real number fields F with $u(\mathcal{O}_F) = 3$. His conjecture holds if we restrict F to be real quadratic fields. Indeed, (see [CKR]):

[15] All five have class number 1.

Theorem 5.2 *There are exactly three real quadratic fields F with $u(\mathcal{O}_F) = 3$. They are:*

$$F = \mathbb{Q}(\sqrt{2}), \quad \mathbb{Q}(\sqrt{3}), \quad \mathbb{Q}(\sqrt{5}).$$

We give a sketchy of proof: Let $F = \mathbb{Q}(\sqrt{m})$, where $m > 5$ is a square-free integer. Suppose L is a ternary universal $\mathcal{O}_F$-lattice. Then $L \cong \langle 1 \rangle \perp L_0$ for some binary sublattice L_0. Using norms of properly chosen totally positive algebraic integers to be represented by L_0, one may rule out all possibilities for L_0 except $\mathcal{O}_F\mathbf{x} + \mathcal{O}_F\mathbf{y}$ and $\mathfrak{p}^{-1}\mathbf{x} + \mathfrak{p}\mathbf{y}$, where $Q(\mathbf{x}) = 2$ and $\mathfrak{p}$ is the unique dyadic prime ideal of $\mathcal{O}_F$ when $2\mathcal{O}_F = \mathfrak{p}^2$. For each square-free integer $m > 5$ in both remaining cases, one can explicitly write a totally positive algebraic integer in terms of m that cannot be represented by L_0.
Then prove that $\mathbb{Q}(\sqrt{2})$ admits four universal ternary $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$-lattices[16]:

$$\langle 1, 1, 1+\epsilon_2 \rangle, \quad \langle 1 \rangle \perp [1+\epsilon_2, 1+\epsilon_2', 1], \quad \langle 1 \rangle \perp [1+\epsilon_2, 3, 1], \quad \langle 1 \rangle \perp [1+\epsilon_2', 3, 1],$$

where $\epsilon_2 = 1+\sqrt{2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{2})$ and $\epsilon_2' = 1-\sqrt{2}$, and that $\mathbb{Q}(\sqrt{3})$ admits two universal ternary $\mathcal{O}_{\mathbb{Q}(\sqrt{3})}$-lattices[17]:

$$\langle 1, 1, \epsilon_3 \rangle \quad \text{and} \quad \langle 1, \epsilon_3, \epsilon_3 \rangle,$$

where $\epsilon_3 = 2+\sqrt{3}$ is the fundamental unit of $\mathbb{Q}(\sqrt{3})$. These together with Theorem 5.1 complete the proof.

It is now clear that $u(\mathcal{O}_F) \geq 4$ for all real quadratic fields $F = \mathbb{Q}(\sqrt{m})$ with square-free integer $m > 5$. One can easily prove by using product formula of Hasse symbols that $u(\mathcal{O}_F) \geq 4$ for all totally real fields F with $[F : \mathbb{Q}] =$ odd.

Concerning Kitaoka's conjecture, B.M. Kim [Ki1] proved that for any given n, there exist finitely many totally real fields F with $[F : \mathbb{Q}] = n$ such that $u(\mathcal{O}_F) = 3$. He further proved that if a given n is 2 or odd, then there exist finitely many totally real fields F with $[F : \mathbb{Q}] = n$ such that $u(\mathcal{O}_F) = 4$. See [Ki2] for higher rank universal $\mathcal{O}$-lattices over real quadratic fields.

We close this section with an easy but interesting characterization of $\mathbb{Q}$. Let F be a totally real number field such that every binary $\mathcal{O}_F$-lattice can be represented by a sum of squares. Then by Siegel's result mentioned in the beginning of this section, F is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. But in the latter case, the binary $\mathcal{O}_F$-lattice $[2, 4+3\epsilon, 1]$ cannot be represented by a sum of squares. Therefore,

The only totally real number field F, where every binary $\mathcal{O}_F$-lattice can be represented by a sum of squares, is $\mathbb{Q}$.

[16] Proof is quite similar to that of Theorem 5.1. All four have class number 1.

[17] Both have class number 1. So, all universal ternary $\mathcal{O}$-lattices over real quadratic fields are of class number 1.

6 S-universal $\mathbb{Z}$-lattices

Bhargava's generalization (see [C]) of Schneeberger's fifteen theorem is as follows:

> *For any given infinite set S of positive integers, there exists a finite subset S_0 of S such that every S_0-universal $\mathbb{Z}$-lattice is S-universal.*

It is known that he found S_0's explicitly for some interesting sets S. For examples, if S is the set of all primes, then

$$S_0 = \{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 67, 73 \},$$

and if S is the set of all positive odd integers, then

$$S_0 = \{ 1, 3, 5, 7, 11, 15, 33 \},$$

and so on. His result was fully generalized as follows [18] in [KKO2]:

Theorem 6.1 (Kim-Kim-Oh) *For any given infinite set S of $\mathbb{Z}$-lattices of a given rank, there exists a finite subset S_0 of S such that every S_0-universal $\mathbb{Z}$-lattice is S-universal.*

We sketchy the proof: Let $S = \{\ell_1, \ell_2, \ldots, \ell_t, \ldots\}$ be a given infinite set of $\mathbb{Z}$-lattices of rank k. For any $\mathbb{Z}$-lattice L, define ℓ_L as follows:

$$\ell_L := \begin{cases} \ell_j & \text{if } \ell_i \to L \text{ for all } i < j \text{ and } \ell_j \not\to L, \\ 0 & \text{if such an } j \text{ does not exist.} \end{cases}$$

Define T_i's inductively as follows: Let T_1 be the set of all $\mathbb{Z}$-lattices of rank k that represent ℓ_1. Then T_1 is clearly finite. For $i \geq 1$, let

$$U_i := \{L \in T_i \,:\, \ell_L \neq 0\},$$

and for each $L \in U_i$ let $U(L)$ be the set of $\mathbb{Z}$-lattices M satisfying the following two conditions: (i) M represents both L and ℓ_L, and (ii) no sublattice of M of rank less than rank(M) can represent both L and ℓ_L. Then define

$$T_{i+1} := \bigcup_{L \in U_i} U(L).$$

[18] In recent private communication with Bhargava, the author learned that he also obtained the same result as ours. But unfortunately we haven't seen his proof yet.

Note that $\text{rank}(M) \leq n + \text{rank}(L)$. Let $L \in U_{i-1}$ for $i \geq 2$ and $M \in U(L)$. Suppose that $\overline{\mu}(M) > \max(\overline{\mu}(\ell_L), \overline{\mu}(L))$, where $\overline{\mu}$(a lattice) denotes the last successive minimum of the lattice. Let K be the sublattice of M generated by the vectors $\mathbf{x} \in M$ with $Q(\mathbf{x}) < \overline{\mu}(M)$ and define $\hat{K} := \mathbb{Q}K \cap M$. Then $\hat{K}$ is a sublattice with rank less than $\text{rank}(M)$ and represents both L and ℓ_L, which is a contradiction. So the last successive minimum of M is bounded above. From this follows:

For every $i = 1, 2, 3, \ldots$, T_i is a finite set.

Note that every $\mathbb{Z}$-lattice in T_i represents ℓ_j for all $j \leq i$.
Assume that there exists an r such that every $\mathbb{Z}$-lattice in T_r is S-universal. Let

$$S_0 := \{\ell_1\} \cup \left\{ \ell_L \,:\, L \in \bigcup_{i=1}^{r-1} U_i \right\}.$$

Let M be any $\mathbb{Z}$-lattice that represents all $\mathbb{Z}$-lattices in S_0. If $K \to M$ for some $K \in \bigcup_{i=1}^{r-1}(T_i - U_i)$, then M is S-universal. So, one many assume that $K \not\to M$ for any $K \in \bigcup_{i=1}^{r-1}(T_i - U_i)$. Then for all $1 \leq j \leq r-1$, one can find $\mathbb{Z}$-sublattices M_j of M such that $M_j \in U_j$ and $M_j \subseteq M_{j+1}$, $1 \leq j \leq r-2$. Since $M_{r-1} \in U_{r-1}$ and $\ell_{M_{r-1}} \to M$, there exists at least one $\mathbb{Z}$-lattice in T_r that is represented by M. So, M is again S-universal. Therefore, every S_0-universal $\mathbb{Z}$-lattice is S-universal.
It only remains to show the existence r such that every $\mathbb{Z}$-lattice in T_r is S-universal. This can be proved by applying Hsia-Kitaoka-Kneser's theorem [HKK] repeatedly (see [KKO2] for detail). In the process, one needs the following local version of Theorem 6.1, whose proof is quite similar to that of Lemma 1.5 [HKK].

Theorem 6.2 *For any given infinite set $S(p)$ of $\mathbb{Z}_p$-lattices of a given rank, there exists a finite subset $S_0(p)$ of $S(p)$ such that every $S_0(p)$-universal $\mathbb{Z}_p$-lattice is $S(p)$-universal.*

The following corollaries are immediate consequences of Theorem 6.1:

Corollary 6.1 *Let S be an infinite set of $\mathbb{Z}$-lattices of bounded rank. Then there exists a finite subset S_0 of S such that every S_0-universal $\mathbb{Z}$-lattice is S-universal.*

An S-universal $\mathbb{Z}$-lattice is said to be *new* if it does not contain a sublattice of lower rank that is also S-universal and *old*, otherwise.

Corollary 6.2 *Let S be an infinite set of $\mathbb{Z}$-lattices of rank n. Then there exist positive integers $N_1(S)$ and $N_2(S)$ such that*
(1) *there is no S-universal $\mathbb{Z}$-lattice of rank less than $N_1(S)$,*
(2) *there exist S-universal $\mathbb{Z}$-lattices of rank N for every $N \geq N_1(S)$, and*
(3) *there is no new S-universal $\mathbb{Z}$-lattice of rank greater than $N_2(S)$.*
In particular, the total number of new S-universal $\mathbb{Z}$-lattices is finite.

Let $\mathcal{P}_k$ denote the set of all $\mathbb{Z}$-lattices of rank k. It is known (see [B] and [KKO1]) that

$$N_1(\mathcal{P}_1) = 4, N_2(\mathcal{P}_1) = 5 \quad \text{and} \quad N_1(\mathcal{P}_2) = N_2(\mathcal{P}_2) = 5.$$

Note that $N_1(\mathcal{P}_k) = u[k]$ (see Theorem 4.1). Define

$$\Gamma(S) := \{ S' \subset S \,:\, S' \text{ is finite such that every } S'\text{-universal } \mathbb{Z}\text{-lattice is } S\text{-universal} \}.$$

Schneeberger [Sch] proved that $S_0 = \{ 1, 2, 3, 5, 6, 7, 10, 14, 15 \}$ is a unique minimal set in $\Gamma(\mathcal{P}_1)$. The uniqueness of a minimal S_0, however, is not guaranteed in general. For example, if we let $S = \{ \langle 2^i, 2^j, 2^k \rangle \,:\, i, j, k \geq 0 \}$, then

$$S_0 = \{ \langle 1,1,1\rangle,\ \langle 1,1,2\rangle \} \text{ and } S_0' = \{ \langle 1,1,1\rangle,\ \langle 2,2,2\rangle \}$$

are both minimal in $\Gamma(S)$. The following open questions seem to be quite interesting but difficult:

(1) *For which S is there a unique minimal $S_0 \in \Gamma(S)$?*
(2) *Is $|S_0| = \gamma(S)$ for every minimal $S_0 \in \Gamma(S)$? If not, when?*

Here $\gamma(S) := \min\{ |S_0| \,:\, S_0 \in \Gamma(S) \}$, which is finite by Theorem 6.1. One can show both $N_1(\mathcal{P}_k)$ and $\gamma(\mathcal{P}_k)$ have exponential lower bounds (see [KKO2]).

Remarks: (1) Theorem 6.1, Corollaries 6.1 and 6.2 should hold for totally positive $\mathcal{O}_F$-lattices, where F is any totally real number field because all ingredients (reduction theory, local and global theory, and Hsia-Kitaoka-Kneser theorem, *etc.*) in the proofs are available over totally real number fields.

(2) For a given $\mathbb{Z}$-lattice L, let S be the set of all $\mathbb{Z}$-lattices of rank k that are represented over $\mathbb{Z}_p$ by L at all p. Then by Theorem 6.1, there is a finite subset S_0 of S such that if L represents every element of S_0 then L represents all of S, that is, L is *k-regular*. The existence of such S_0, when L is k-regular, can also be deduced from finiteness of class numbers.

7 Fourier coefficients of theta-series

Consider the Siegel theta-series $\Theta^k(Z; L)$ attached to a $\mathbb{Z}$-lattice L of rank m defined by

$$\begin{aligned}\Theta^k(Z;L) : &= \sum_{X \in M_{m\times k}(\mathbb{Z})} \exp(2\pi i \mathrm{Tr}({}^t X M_L X Z)) \\ &= \sum_{N \in \mathcal{N}_k} r(N;L) \exp(2\pi i \mathrm{Tr}(NZ)), \quad Z \in \mathfrak{H}_k\,,\end{aligned}$$

where M_L is a matrix presentation of L, Tr is the trace, $\mathcal{N}_k$ is the set of all $k \times k$ semi positive definite symmetric matrices over $\mathbb{Z}$,

$$r(N; L) := \#\{\, X \in M_{m\times k}(\mathbb{Z}) \,:\, {}^tXM_LX = N \,\} < \infty$$

is the *representation number* of N by L, and finally

$$\mathfrak{H}_k := \{\, Z = X + iY \,:\, Z = {}^tZ,\ X, Y \in M_k(\mathbb{R}),\ Y \text{ is positive definite} \,\}$$

is the *Siegel upper half plane* of degree k.

Let $q = q(L)$ be the *level* of L, the smallest positive integer such that $q(2M_L)^{-1}$ is integral with even diagonal entries. Define the character $\chi = \chi(L)$ associated to L by

$$\chi(d) = (\mathrm{sign}(d))^{m/2}\left(\frac{(-1)^{m/2}\det(2M_L)}{|\,d\,|}\right)$$

when m is even (see [AM]), and

$$\chi(d) = \chi^*(d)\chi_2(d)^{(1-m)/2} = \left(\frac{2\det(2M_L)}{|\,d\,|}\right)$$

when m is odd (see [Kim]), where $(-)$ is the Jacobi symbol and $\chi^* = \chi(L \perp I_3), \chi_2 = \chi(I_2)$. Then χ is a Dirichlet character (mod q) and $\Theta^k(Z; L)$ is a Siegel modular form of degree k, weight $m/2$, level q and character χ. (See [S2] Siegel modular forms and their applications.)

Let S be a given infinite set of rank less than or equal to k. Since $r(N, L)$'s are the same for all N's that are matrix presentations of a given $\ell \in S$, we may write it by $r(\ell, L)$. Then Theorem 6.1 is equivalent to the following:

There exists a finite subset S_0 of S such that if $r(\ell_0; M) > 0$ for all $\ell_0 \in S_0$, then $r(N; M) > 0$ for every N that is a matrix presentation of some $\ell \in S$.

Examples. (1) Let $k = 1$ and $S = \mathcal{P}_1$. For a $\mathbb{Z}$-lattice L of rank m, let

$$\Theta(z; L) := \sum_{X\in M_{m\times 1}(\mathbb{Z})} \exp(2\pi i({}^tXM_LX)z) = \sum_{n\geq 0} r(n; L)\exp(2\pi i n z),\ z \in \mathfrak{H}_1$$

be the elliptic theta-series attached to L. Then $r(n; L) > 0$ for every non-negative integer n if $r(n_0; L) > 0$ for $n_0 \in S_0 = \{\, 1, 2, 3, 5, 6, 7, 10, 14, 15 \,\}$ by the fifteen theorem.

Furthermore, according to Bhargava, if we take $S = P$, the set of all primes, then $r(p_0; L) > 0$ for $p_0 = 2, 3, \ldots, 47, 67, 73$ implies $r(p\,; L) > 0$ for all $p \in P$, and similarly, if we take $S = O$, the set of all positive odd integers, then $r(n_0; L) > 0$ for $n_0 = 1, 3, 5, 7, 11, 15, 33$ implies $r(n\,; L) > 0$ for all $n \in O$.

(2) Let $k = 2$ and $S = \mathcal{P}_1 \cup \mathcal{P}_2$. For a $\mathbb{Z}$-lattice L of rank m, let

$$\Theta^2(Z; L) := \sum_{N \in \mathcal{N}_2} r(N; L) \exp(2\pi i \mathrm{Tr}(NZ)),\ \ Z \in \mathfrak{H}_2$$

be the Siegel theta-series of degree 2 attached to L. Then $r(N; L) > 0$ for all $N \in \mathcal{N}_2$ if $r(N_0; L) > 0$ for all N_0 in

$$S_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \right\}.$$

In other words, if L represents six $\mathbb{Z}$-lattices in S_0, then L represents all unary and binary $\mathbb{Z}$-lattices (see Theorem 3.2).

(3) Let $k = 8$ and let $S = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_8$. For a $\mathbb{Z}$-lattice L of rank m, let

$$\Theta^8(Z; L) := \sum_{N \in \mathcal{N}_8} r(N; L) \exp(2\pi i \mathrm{Tr}(NZ)),\ \ Z \in \mathfrak{H}_8$$

be the Siegel theta-series of degree 8 attached to L. Then $r(N; L) > 0$ for all $N \in \mathcal{N}_8$ if both $r(I_8; L)$ and $r(E_8; L)$ are positive. In other words, if L represents I_8 and E_8, then L represents all positive $\mathbb{Z}$-lattices of rank 8 or less (see Theorem 4.3).

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