Review for Midterm Exam

1. Remarks

The exam covers material discussed in Weeks 1 through 6 of the course. The lecture notes available on the course website describe the contents of these lectures.

The exam will ask you to:

- provide precise definitions of terms used in the course. Possibilities include:

  the quotient of a group, an action of a group on a set, an action of a group on a group by automorphisms, normal and characteristic subgroups, p-sylow subgroup for a prime p

  inner automorphisms, commmutator, abelianization of a group, $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ for a field F

  a category, an initial and terminal object of a category, the notion of a product and a co-product in a category

  free modules, direct sum of R-modules

  algebras, integral domains, prime ideals, maximal ideals, fields, polynomial ring

- statements of important theorems which might include:

  Cayley's theorem (and its linear analogue), Lagrange's Theorem, Sylow's Theorem, the first isomorphism theorem (for groups, rings, or modules)

  any p-group has a non-trivial center

  characterize Sylow subgroups of a finite group whose order is a product pq of distinct primes.

  characterize groups of order 60 having no normal Sylow 5-subgroup.

  uniqueness of initial and terminal objects of a category, the relationship between free R-modules and direct sums of copies of R

  Zorn's Lemma, existence of maximal ideals, characterizations of maximal and prime ideals

  Of these, be prepared to reproduce the proofs of Lagrange's Theorem, the existence of maximal ideals, and the characterizations of maximal and prime ideals.

2. Sample problems

Problem 1: Suppose that A is a commutative ring, let M and N be A-modules and let $\varphi : M \to N$ be an isomorphism of A-modules.

Using φ, define an isomorphism of groups $\Phi : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ where $\mathrm{Aut}(M)$ is the group of all R-module automorphisms of M. Recall that an R-module homomorhpism $f : M \to M$ is an automorphism if it is an isomorphism in the category R-mod.

(compare with assignment 1 problem 2)

Problem 2: Let R be a ring and let $f : M \to N$ be an R-module homomorphism. Suppose that $X \subseteq M$ is an R-submodule and that $X \subseteq \ker f$. Prove that f induces an R-module homomorphism $\overline{f} : M/X \to N$.

Problem 3: see assignment 2 problem 2.

Let $a \in \mathbb{Z}$ and let $\sigma_a \in S(\mathbb{Z}/14\mathbb{Z})$ be defined by $\sigma_a(i) = i + a$.

a. If $\gcd(a, 14) = 1$, what is the order of σ?
b. Let $a = 2$ and let $\tau \in S(\mathbb{Z}/14\mathbb{Z})$ be defined by $\tau(i) = 3i$. Prove that $\langle \sigma_2, \tau \rangle$ is a group of order 42.

Problem 4: Let F be a field. Prove that the group $\mathrm{GL}_2(F)$ acts transitively on the set of non-zero vectors in the vector space $V = F^2$.

(Transitively means: for non-zero vectors v, w there is $g \in \mathrm{GL}_2(F)$ such that $gv = w$.)

Compare assignment 2 problem 3.

Problem 5: see assignment 4 problem 5

Let G be a finite group, let p be a prime number, and let $P \in \mathrm{Syl}_p(G)$.

Let $H = N_G(P) = \{g \in G \mid \mathrm{Inn}_g P = P\}$ be the normalizer of P in G. Prove that $N_G(H) = H$. (In words: the normalizer of a Sylow p- subgroup is self-normalizing).

Problem 6:

a. Let R be a commutative ring, and let $A = R \oplus R$. Fix an element $a \in R$ and consider the submodule $X_a \subseteq A$ generated by $(1, a)$ and the submodule $Y \subseteq A$ generated by $(0, 1)$.

   Prove that A is the direct sum of X_a and Y. More precisely, show that $A = X_a + Y$ and that $X_a \cap Y = \{0\}$.
b. Let $A = \mathbb{Z} \oplus \mathbb{Z}$ and let $B = \langle (2, 1), (0, 3) \rangle$. What is the order of A/B? Is A/B cyclic?