ME 323 – Mechanics of Materials
Final Examination
December 12th, 2022

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Name (Print) SOLUTION
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PROBLEM # 1 (25 points)

A thin-walled pressure vessel shown on Figure 1.a is subjected to an internal pressure which results in normal stresses $\sigma_a = \sigma_y = 50\ MPa$ and $\sigma_h = \sigma_x = 100\ MPa$ on its outer surface. An accidental event causes a torque to be applied on the entire structure resulting in the in-plane stress state at point A on the structure shown in Figure 1.b. The added unknown shear stress causes point A shown to experience a *maximum in-plane shear stress* of $\tau_{max,in\text{-}plane} = 40\ MPa$. Use this information to provide answers to the following questions.

(a) Calculate σ_{avg}.
(b) Calculate R, the radius of the Mohr's circle.
(c) On the grid provided, indicate the center of the Mohr's circle and draw **the Mohr's circle corresponding to the state of stress**.
(d) On Mohr's circle draw and label the x and y axes.
(e) Use this **Mohr's circle** to calculate:
  i. The principal stresses in the x-y plane.
  ii. The absolute maximum shear stress.
  iii. The shear stress τ_{xy} acting in the x-y plane.
  iv. Calculate the angle of rotation between the x-axis and the direction of the positive principal stress. *Draw a stress element to show the in-plane principal stresses correctly oriented with respect to the x-axis.*
  v. Calculate the angle of rotation between the x-axis and the direction of the positive in plane maximum shear stress. *State whether this rotation angle is measured clockwise or counterclockwise from the x-direction. Draw a stress element to show the in-plane maximum shear stress and the normal stresses correctly oriented with respect to the x-axis.*

(f) What would be the values of the principal stresses if the shear stress applied could be avoided, i.e., if $\tau_{xy} = 0$?

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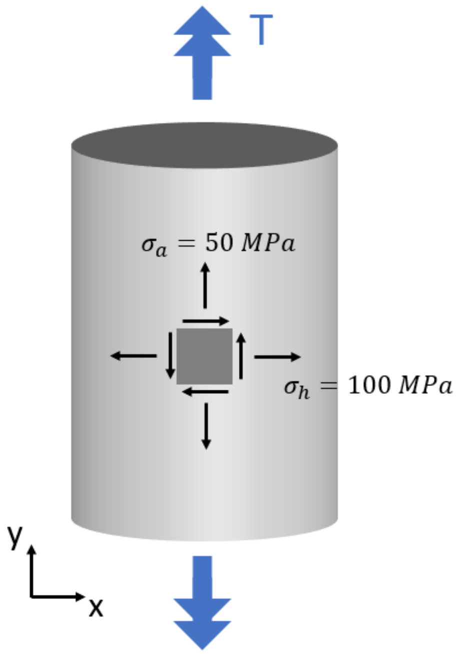

Figure 1.a

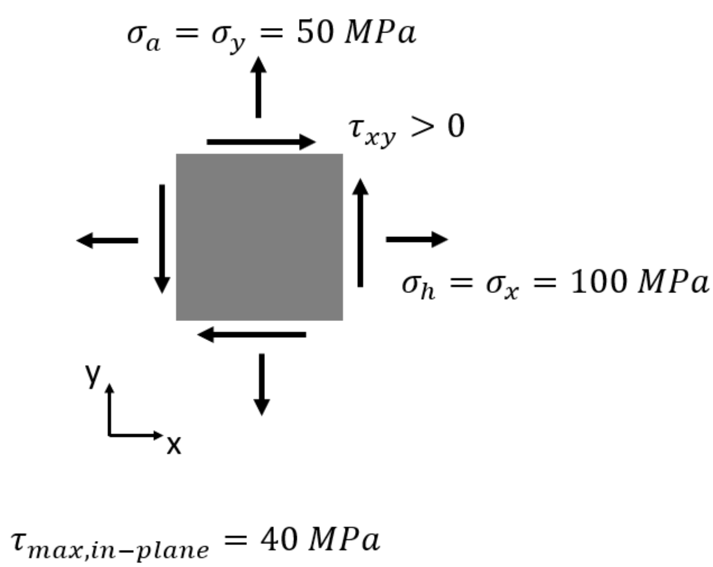

Figure 1.b

(a) $\sigma_{avg} = \dfrac{100+50}{2} = 75\,MPa$

(b) $R = \tau_{max,\,in\text{-}plane} = 40\,MPa$

(c) i.) $\sigma_{p_1} = \sigma_{avg} + R = 75 + 40 = 115\,MPa$

$\sigma_{p_2} = \sigma_{avg} - R = 75 - 40 = 35\,MPa$

ii) $\tau_{max,abs} = \dfrac{\sigma_{p_1}}{2} = \dfrac{115}{2} = 57.5\,MPa$

iii) $\tau_{xy} = \sqrt{R^2 - (\sigma_x - \sigma_{avg})^2} = \sqrt{(40)^2 - (25)^2}$
$= 31.22\,MPa$

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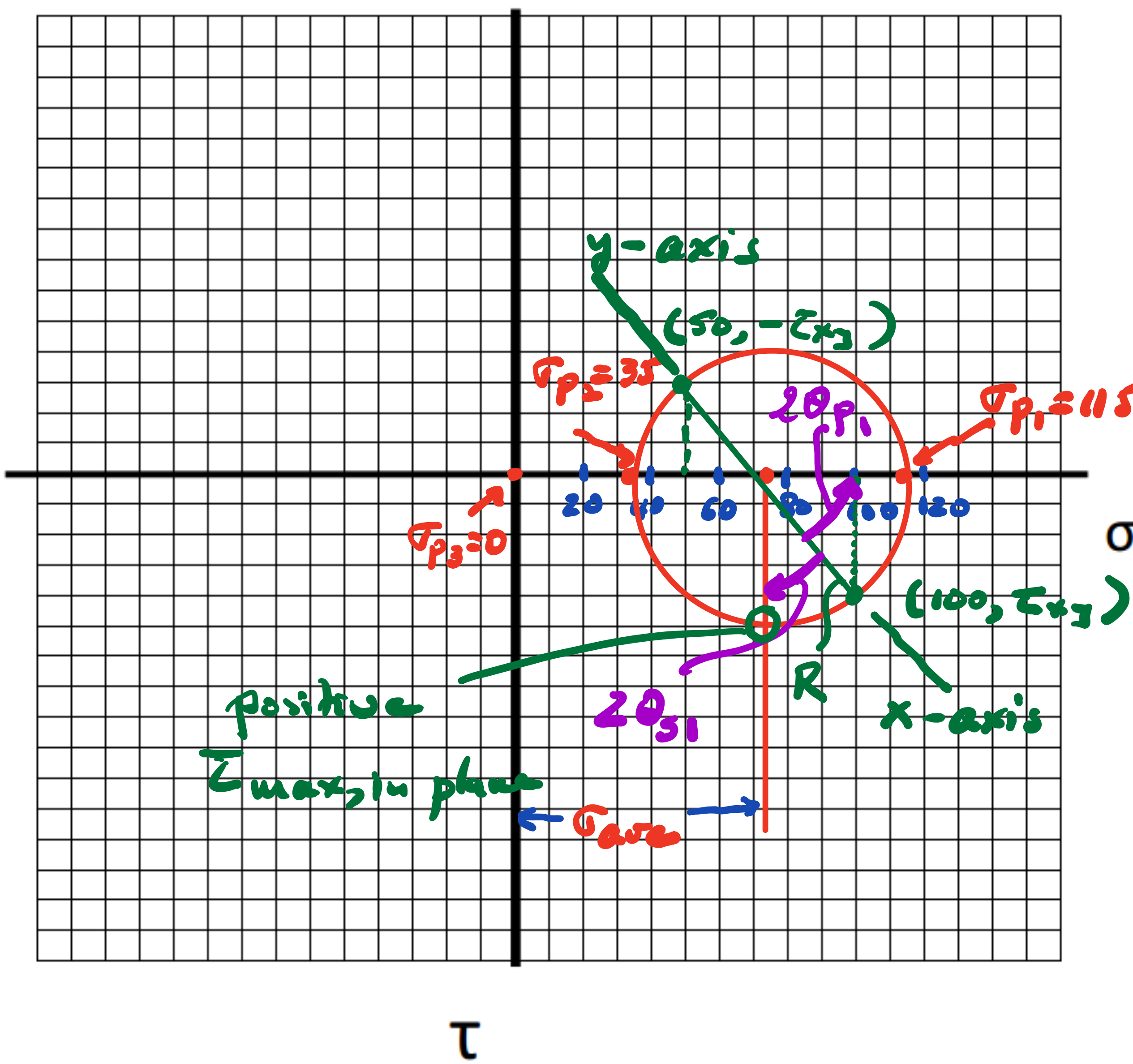
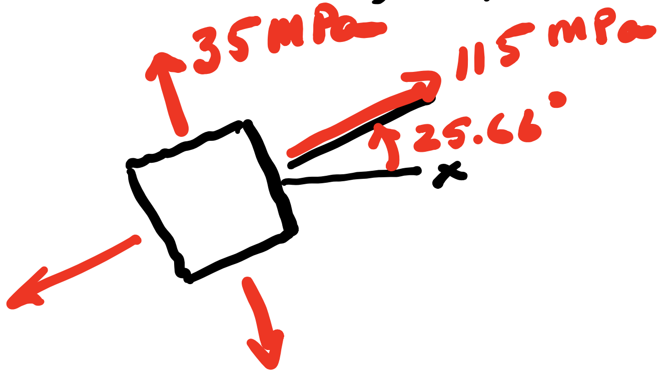


iv) $2\theta_{p_1} = \cos^{-1}\frac{(100-75)}{40} = 51.32^\circ$

$\theta_{p_1} = \frac{51.32}{2} = 25.66^\circ$ CCW from x-axis

stress element for principal stresses

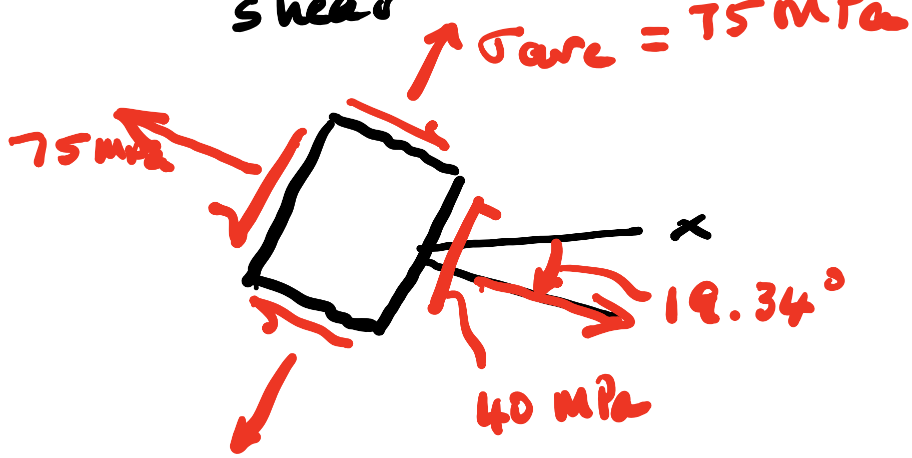


v) $2\theta_{s_1} = 90^\circ - 2\theta_{p_1} = 90^\circ - 51.32^\circ = 38.68^\circ$

$\theta_{s_1} = \frac{38.68}{2} = 19.34^\circ$ CW from x-axis

OR $= 160.66^\circ$ CCW from x-axis

stress element for the POSITIVE maximum shear stress direction



(f) If $\tau_{xy} = 0$, then $\sigma_{p_1} = \sigma_h = 100$ MPa

$\sigma_{p_2} = \sigma_a = 50$ MPa

ME 323 – Mechanics of Materials
Final Examination
December 6th, 2022

PURDUE UNIVERSITY | Mechanical Engineering

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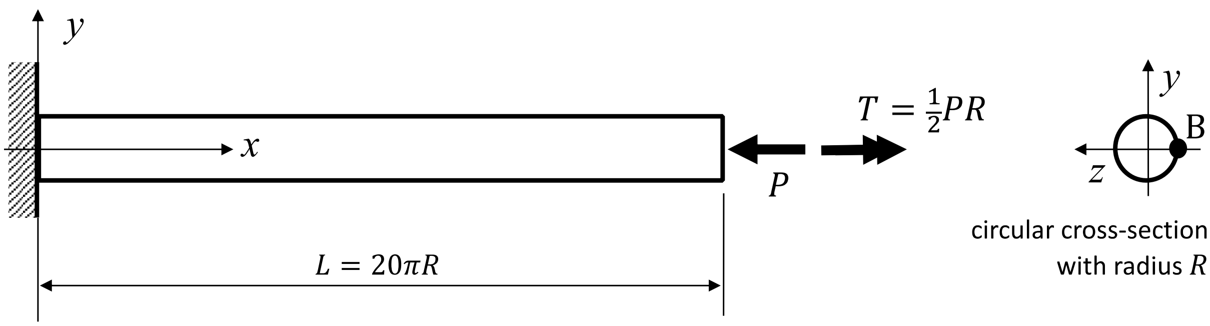
PROBLEM # 2 (25 points)



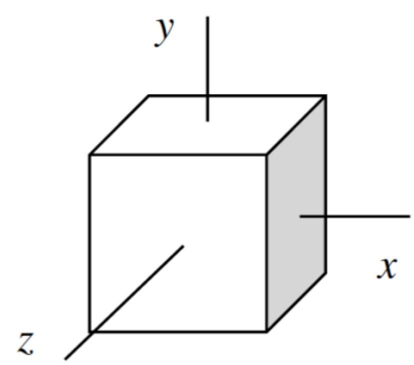
A cylindrical deformable body has a radius R and a length $L = 20\pi R$, and it made of a material with a yield strength equal to σ_Y and a Young's modulus equal to $E = 1000\sigma_Y$. The cylinder is fixed to a wall at one end, and it is loaded axially by a compressive force P and by a positive torque $T = \frac{1}{2}PR$ on the other end. The figure illustrates the configuration.

Follow these steps to determine the smallest radius R that will prevent yielding and buckling of the cylindrical body.

a) Determine the state of stress at point B located at a radial distance R from the axis (see figure) on the cross-section located at $x = L$. Express the stress components in terms of P and R.

b) Draw the state of stress at point B on the stress element provided.

c) Determine the Mises equivalent stress at point B. Express the stress in terms of P and R.



d) Is there any other point on the deformable body with a Mises equivalent stress larger than the one at point B? Answer Yes/No and justify your answer.

e) Utilizing the maximum-distortion-energy failure theory, determine the smallest radius R that will prevent yielding of the deformable body. Express the smallest radius in terms of P and σ_Y.

f) Utilizing Euler elastic buckling theory, determine the smallest radius R that will prevent buckling of the deformable body. Express the smallest radius in terms of P and σ_Y.

g) Utilizing the result from (d) and (e), determine the smallest radius that will prevent both yielding and buckling of the cylindrical body.

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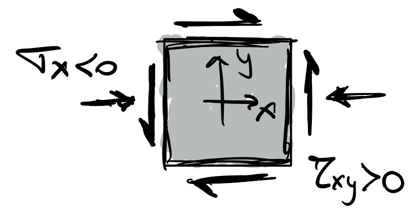
PROBLEM # 1 (cont.)

(a) with $A = \pi R^2$ and $I_p = \dfrac{\pi R^4}{2}$



Axial force $P \Rightarrow \sigma = \dfrac{P}{A} = \dfrac{P}{\pi R^2}$

Torque $T = \frac{1}{2} PR \Rightarrow \tau = \dfrac{TR}{I_p} = \dfrac{P}{\pi R^2}$

State of stress: $\sigma_x = -\dfrac{P}{\pi R^2}$ $\quad \tau_{xy} = +\dfrac{P}{\pi R^2}$ $\quad \sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$

(c)
$$\sigma_M = \frac{\sqrt{2}}{2}\left[\sigma_x^2 + \sigma_x^2 + 6\tau_{xy}^2\right]^{1/2}$$

$$\sigma_M = \frac{\sqrt{2}}{2}\left[2\left(\frac{P}{\pi R^2}\right)^2 + 6\left(\frac{P}{\pi R^2}\right)^2\right]^{1/2} = \frac{\sqrt{2}}{2}\left[2+6\right]^{1/2}\frac{P}{\pi R^2} \Rightarrow \sigma_M = 2\frac{P}{\pi R^2}$$

(d) NO ... (σ is uniform and τ is maximum at the outer radius)

(e) $\sigma_Y > \sigma_M = 2\dfrac{P}{\pi R^2} \Rightarrow \hat{R} > \left(\dfrac{2P}{\pi \sigma_Y}\right)^{1/2}$ to prevent yielding

(f) $P < \dfrac{\pi^2 E I}{L_{eff}^2}$ with $I = \dfrac{\pi R^4}{4}$ $\quad L_{eff} = 2L = 40\pi R$

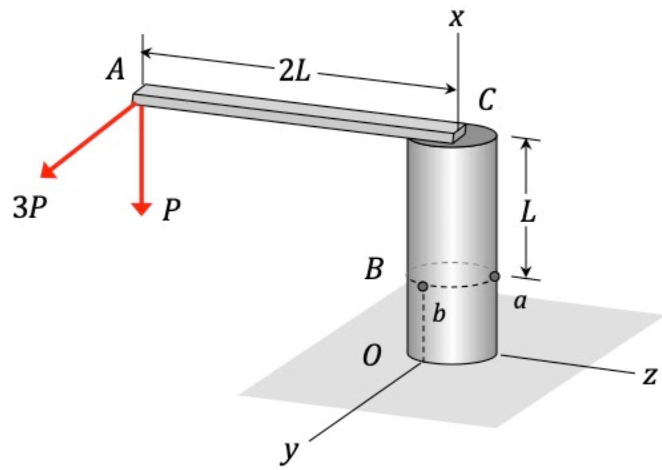
$E = 1000\sigma_Y$

$$P < \frac{\pi^2\, 1000\sigma_Y\, \pi R^4}{1600\,\pi^2 R^2\, 4} = \frac{5}{32}\pi\sigma_Y R^2 \Rightarrow \hat{R} > \left(\frac{32P}{5\pi\sigma_Y}\right)^{1/2}$$ to prevent buckling

(g) $R > \max\left[\left(\dfrac{2P}{\pi\sigma_Y}\right)^{1/2}; \left(\dfrac{32P}{5\pi\sigma_Y}\right)^{1/2}\right] = \left(\dfrac{32P}{5\pi\sigma_Y}\right)^{1/2}$ to prevent both yielding and buckling

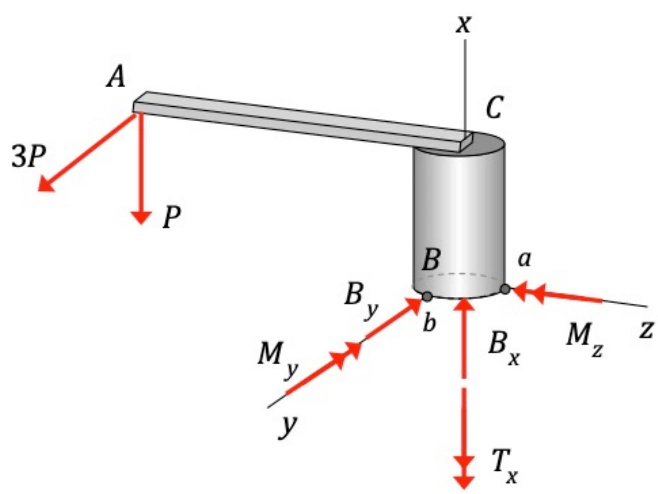
SOLUTION

PROBLEM #3 (25 points)



Rigid bar AC (having a length of $2L$ and aligned with the z-axis)) is welded to the top end of the elastic shaft OC, where shaft OC has a radius of r. Loads of P and $3P$ act at end A of the bar, where P acts in the negative x-direction and $3P$ acts in the positive y-direction. In this problem, we are asked to determine the state of stress at points "a" and "b" on the shaft at location B (located a distance of L below end C). To this end, complete the following steps:

Part (a): Draw a free body diagram (FBD) of the bar and upper portion of the shaft using the figure below.



SOLUTION

Problem #3 - continued

Part (b): Using your FBD above, determine the internal resultants acting on this upper portion of the shaft at B.

$$\sum F_x = B_x - P = 0 \quad \Rightarrow \quad B_x = P$$

$$\sum F_y = -B_y + 3P = 0 \quad \Rightarrow \quad B_y = 3P$$

$$\sum \vec{M}_B = -T_x\hat{i} - M_y\hat{j} - M_z\hat{k} + \left(L\hat{i} - 2L\hat{k}\right)\times\left(-P\hat{i} + 3P\hat{j}\right)$$

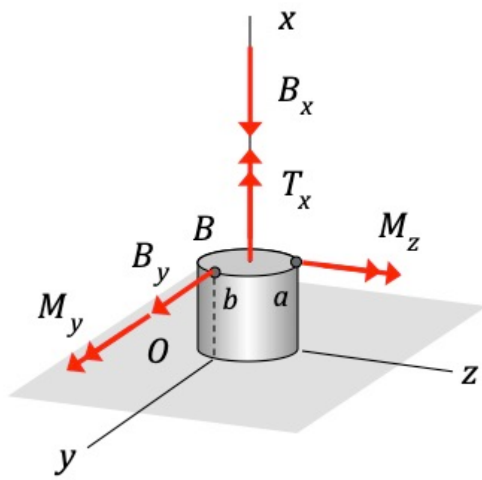
$$\vec{0} = \left(-T_x + 6PL\right)\hat{i} + \left(-M_y + 2PL\right)\hat{j} + \left(-M_z + 3PL\right)\hat{k} \quad \Rightarrow$$

$$\hat{i}: \quad -T_x + 6PL = 0 \quad \Rightarrow \quad T_x = 6PL = 24Pr$$

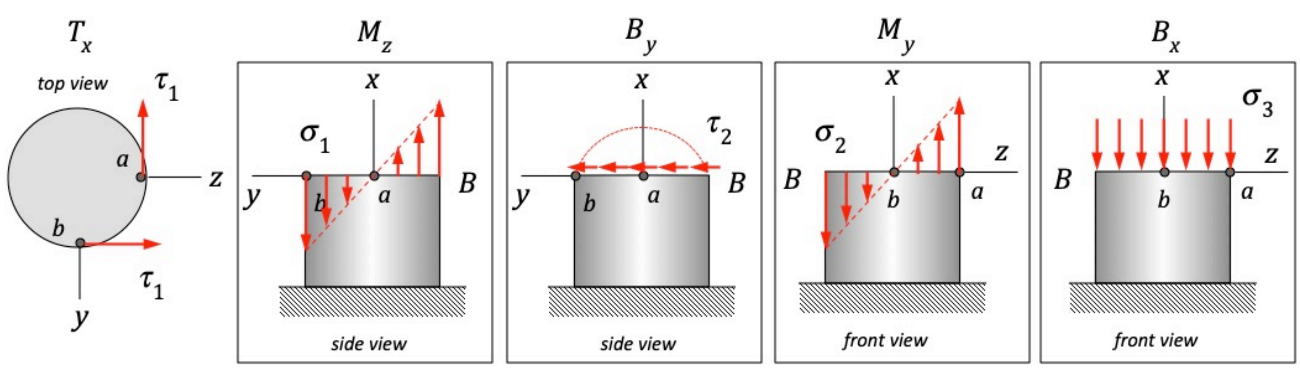
$$\hat{j}: \quad -M_y + 2PL = 0 \quad \Rightarrow \quad M_y = 2PL = 8Pr$$

$$\hat{k}: \quad -M_z + 3PL = 0 \quad \Rightarrow \quad M_z = 3PL = 12Pr$$

Part (c): Transcribe the internal resultants found above to the figure of the lower portion of the shaft provided below.



Part (d): Show the resulting stress distributions for the internal resultants on the figures of the lower portion of the shaft provided below.



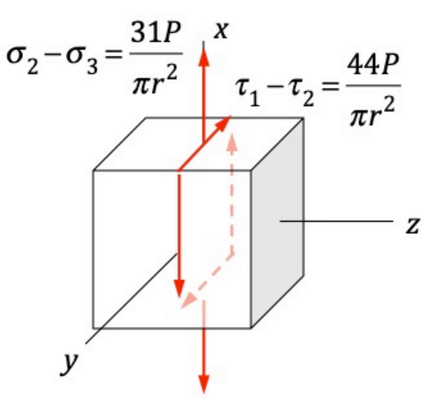
Problem #3 - continued

Part (e): Complete the table below providing the stress components at "a" and "b" for each of the internal resultants shown in your figure in Part (c). Express the components in terms of *P* and *r*.

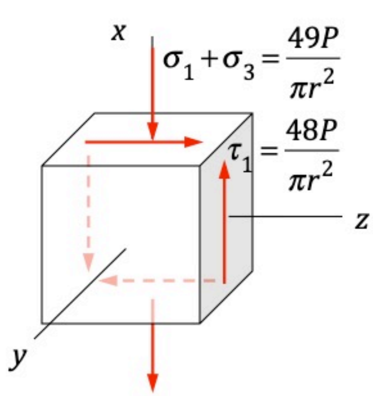
resultant	***stress @ "a"***	***stress @ "b"***
B_x	$\sigma_3 = \dfrac{B_x}{A} = \dfrac{P}{\pi r^2}$	$\sigma_3 = \dfrac{B_x}{A} = \dfrac{P}{\pi r^2}$
B_y	$\tau_2 = \dfrac{4}{3}\dfrac{B_y}{A} = \dfrac{4}{3}\left(\dfrac{3P}{\pi r^2}\right) = \dfrac{4P}{\pi r^2}$	0
T_x	$\tau_1 = \dfrac{T_x r}{I_P} = \dfrac{(24Pr)r}{\pi r^4/2} = \dfrac{48P}{\pi r^2}$	$\tau_1 = \dfrac{T_x r}{I_P} = \dfrac{(24Pr)r}{\pi r^4/2} = \dfrac{48P}{\pi r^2}$
M_y	$\sigma_2 = \dfrac{M_y r}{I} = \dfrac{(8Pr)r}{\pi r^4/4} = \dfrac{32P}{\pi r^2}$	0
M_z	0	$\sigma_1 = \dfrac{M_z r}{I} = \dfrac{(12Pr)r}{\pi r^4/4} = \dfrac{48P}{\pi r^2}$

where***:*** $I = \pi r^4/4$, $I_P = \pi r^4/2$ and $A = \pi r^2$.

Part (f): Show the components of the state of stress on elements "a" and "b" provided below. Be sure to include the non-zero components for ALL faces of each element in your sketches.



stress element "a"



stress element "b"

ME 323 – Mechanics of Materials
Final Examination
December 12th, 2022

PURDUE UNIVERSITY | Mechanical Engineering

Name (Print) ____
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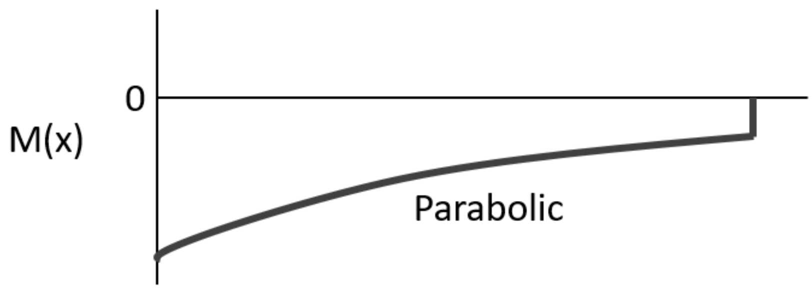
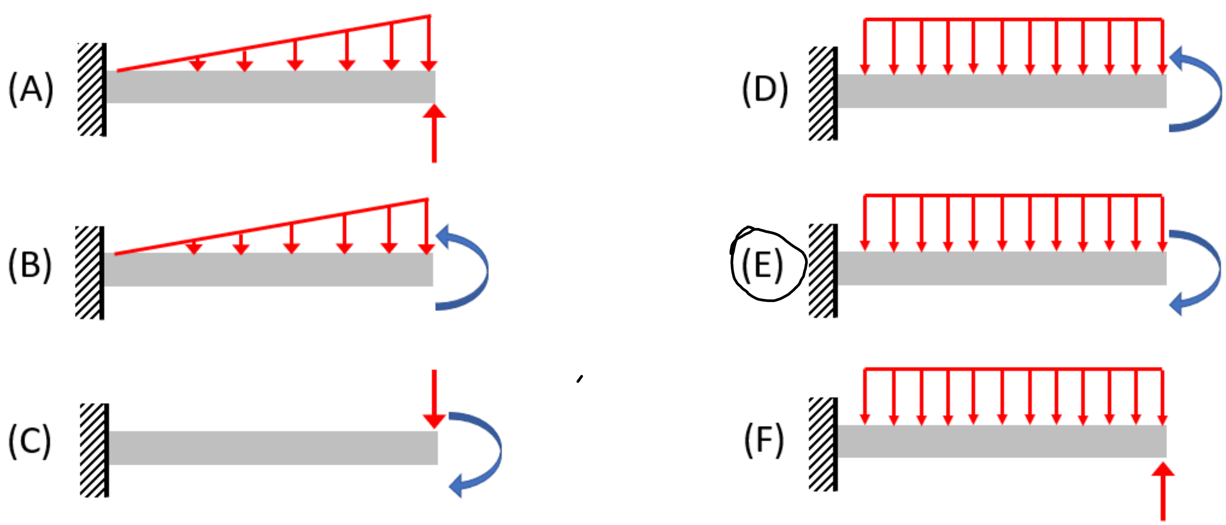
<u>PROBLEM # 4 (25 points)</u>

Part A (4 Points):
Circle the loading conditions that produce the given bending moment diagram.





Parabolic $\Rightarrow M = f(x^2)$

$V = \frac{dM}{dx} \Rightarrow V = f(x)$

$p = \frac{dV}{dx} \Rightarrow p = \text{constant}$

$\Rightarrow$ eliminate (A), (B), (C)

Positive vertical jump in M(x) caused by negative applied couple

$\Rightarrow$ Choose (E)

ME 323 – Mechanics of Materials
Final Examination
December 12th, 2022

PURDUE UNIVERSITY | Mechanical Engineering

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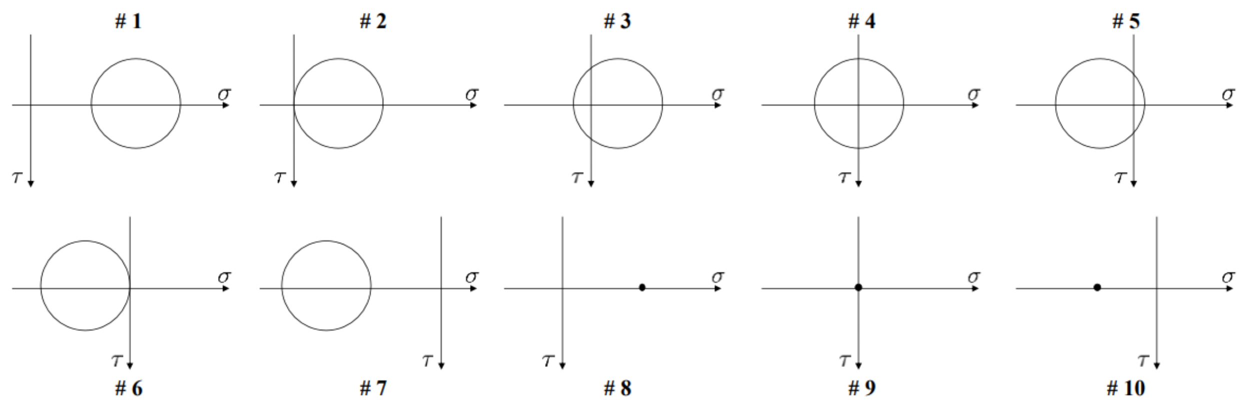
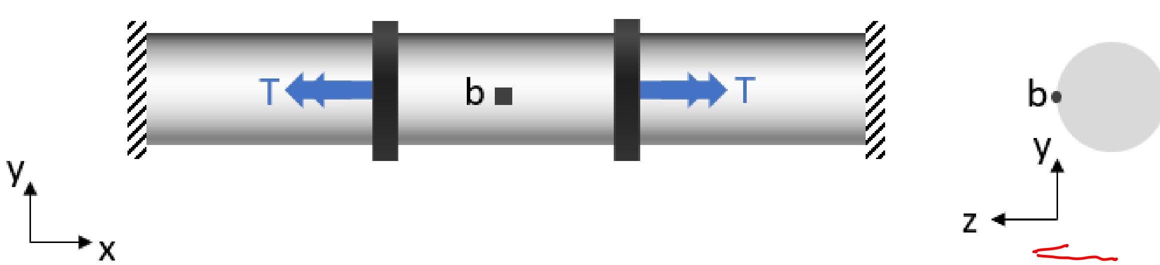
PROBLEM # 4 (cont.)

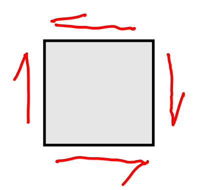
Part B (6 Points):

B(i) and **B(ii)** refer to these in-plane Mohr's circles:



An axial torsion assembly is subjected to torques as shown below:





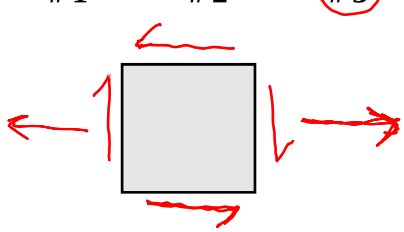
Pure shear

B(i):
Choose the Mohr's circle that corresponds to the loading at point b. (3 Points)

 # 1 **# 2** **# 3** **# 4** **# 5** **# 6** **# 7** **# 8** **# 9** **# 1 0**

B(ii):
The torsion assembly is subjected to a ***uniform temperature decrease of ΔT*** (the temperature of all members in the assembly decrease by ΔT). Choose the Mohr's circle that corresponds to the loading at point b after this temperature decrease. (3 Points)

 # 1 **# 2** **# 3** **# 4** **# 5** **# 6** **# 7** **# 8** **# 9** **# 1 0**



Same shear + ΔT causes tensile stress

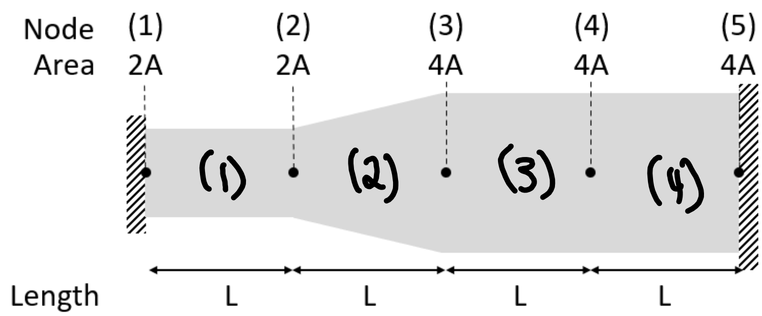
$\sigma_{avg} = \frac{\sigma_x}{2}$ $R = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} > \frac{\sigma_x}{2}$

$\Rightarrow \sigma_{p1} > 0$ $\sigma_{p2} < 0$

Name (Print) ____
(Last) (First)

PROBLEM # 4 (cont.)

Part C (4 points):
The axial structure shown below is analyzed using the finite element approach with four elements. All elements have a Young's modulus of E, and the lengths and areas are shown in the figure.



What is the value of the matrix element at the second row and second column of the reduced stiffness matrix?

(a) $-\frac{2AE}{L}$

(b) $-\frac{4AE}{L}$

(c) $-\frac{8AE}{L}$

(d) $\frac{6AE}{L}$

(e) $\frac{7AE}{L}$

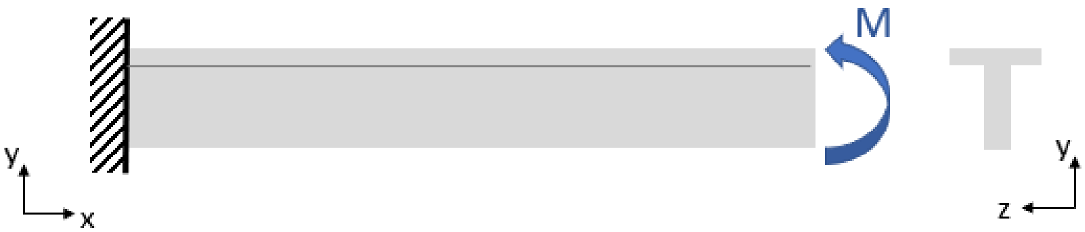
(f) $\frac{8AE}{L}$

$$\begin{array}{cc} & \begin{array}{ccccc} \quad & \quad & 2 & \quad & \quad \end{array} \\ \begin{array}{c} \\ \\ 2 \rightarrow \\ \\ \\ \end{array} & \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & \boxed{k_2+k_3} & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \end{array}$$

reduced matrix with 2 BC's (node 1 and 5)

$$k_2+k_3 = \frac{(2A+4A)}{2}\left(\frac{E}{L}\right) + \frac{4AE}{L} = \frac{3AE}{L} + \frac{4AE}{L} = \frac{7AE}{L}$$

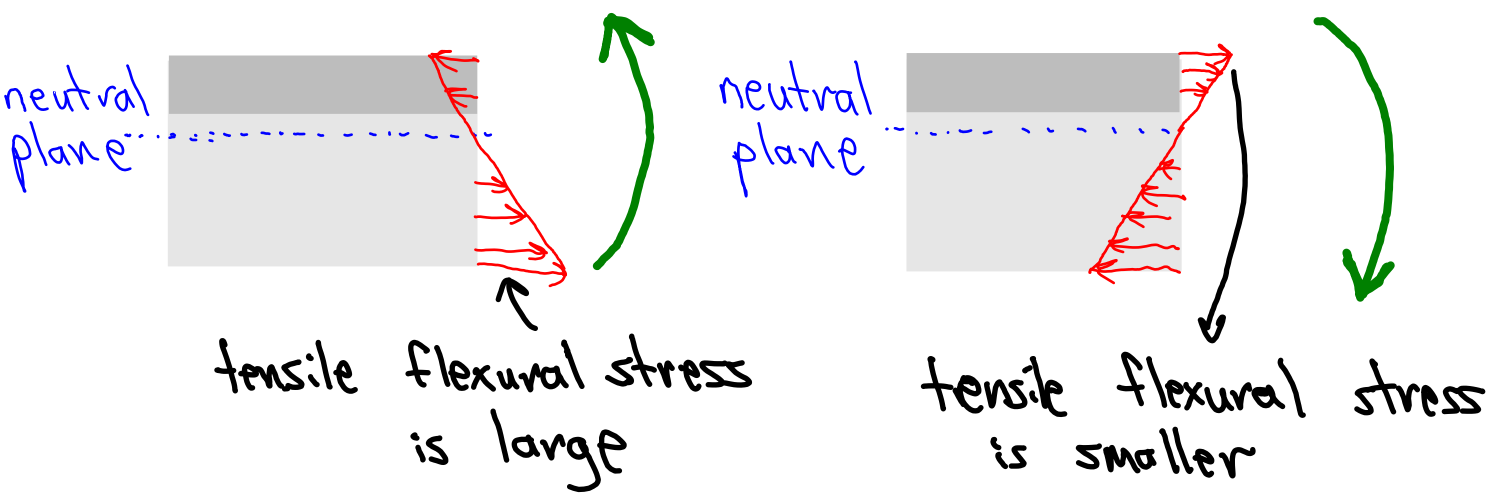
Part D (3 points):
A beam with a "T-shaped" cross-section is composed of a material that exhibits brittle fracture and has an ultimate compressive stress that is 4 times larger than its ultimate tensile stress.
($\sigma_{UC} = 4\sigma,\ \sigma_{UT} = \sigma$).



When the moment is positive (as shown), the moment at which the beam will fail is referred to as $M_{fail,pos}$. When the moment is negative (opposite to shown), the moment at which the beam will fail is referred to as $M_{fail,neg}$. How are the magnitudes of these two failure moments related?

(a) $M_{fail,pos} > M_{fail,neg}$
(b) $M_{fail,pos} = M_{fail,neg}$
(c) $M_{fail,pos} < M_{fail,neg}$

$\sigma_{UC} \gg \sigma_{UT} \Rightarrow$ will fail in tension.



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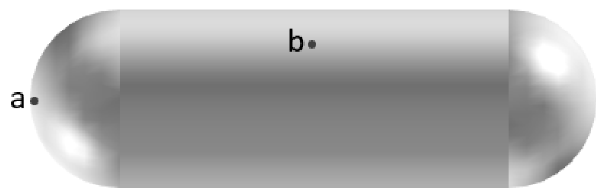
PROBLEM # 4 (cont.)

Part E (6 Points):
A pressure vessel with hemispherical caps is subjected to a positive pressure of P. P_{MSS} refers to the pressure at failure predicted by the maximum shear stress theory. P_{MDE} refers to the pressure at failure predicted by the maximum distortional energy theory.

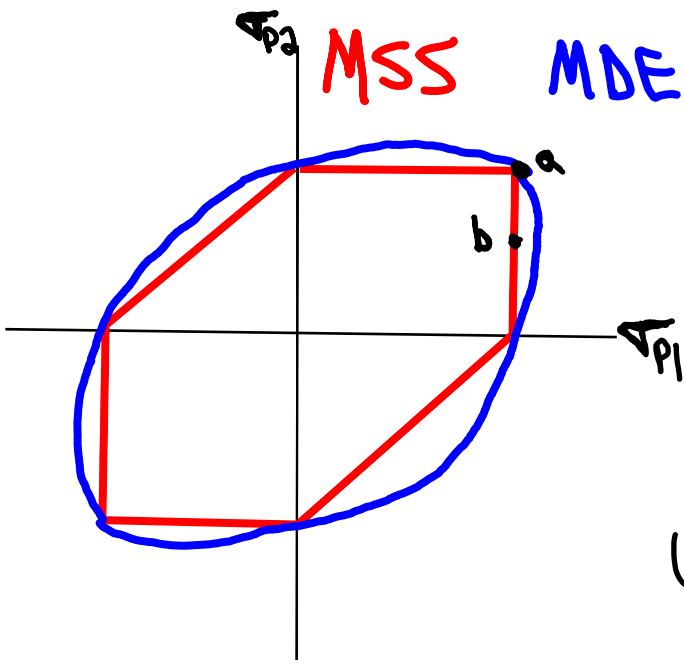


(i) At point a, circle the answer that describes the relative sizes of P_{MSS} and P_{MDE}. (3 points)

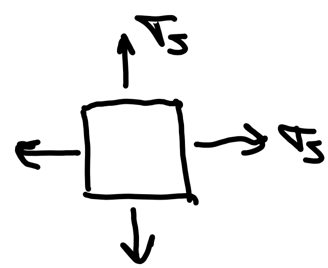
$P_{MSS} < P_{MDE}$ $\quad$ $P_{MSS} = P_{MDE}$ $\quad$ $P_{MSS} > P_{MDE}$

(ii) At point b, circle the answer that describes the relative sizes of P_{MSS} and P_{MDE}. (3 points)

$P_{MSS} < P_{MDE}$ $\quad$ $P_{MSS} = P_{MDE}$ $\quad$ $P_{MSS} > P_{MDE}$

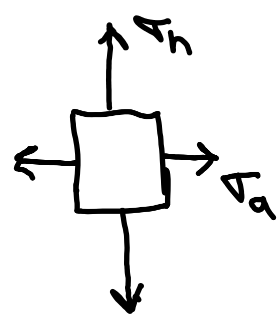


(i) a



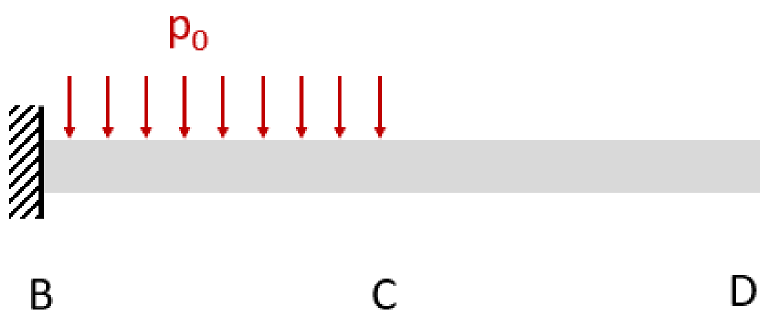
$\sigma_{p1} = \sigma_{p2} = \sigma_s$

(ii) b



$\sigma_{p1} = \sigma_h > \sigma_a = \sigma_{p2}$

Part F (2 points):



In the structure shown above, using Castigliano's second theorem to solve for the slope (also referred to as angular change, or θ) at D would require:

(a) A dummy load at C
(b) A dummy moment at C
(c) A dummy load at D
(d) A dummy moment at D

$$\frac{\partial U}{\partial M} = \theta$$