March 15, 2025

PHYSICS 2b – Final

This exam covers all of the readings in the text and the lecture notes for the term.

This is an **OPEN BOOK** exam with the following limitations: You may consult only your textbook (Griffiths & Schroeter, Intro to QM), notes that you have taken in recitation or lecture, online lecture notes, and HW solns (posted as well as your own). No other references are allowed. You may use a calculator and symbolic manipulation programs (eg. Mathematica) for integrals or algebra, although they are not required.

In your studies, do not look at quizzes or exams from previous years of Phys 2 or 12.

The time limit is **4 HOURS**, in one continuous sitting. A 30 minute break beyond the 4 hours is permitted during this period, as long as you are not working on the exam during this time. No credit for overtime work.

This Exam is **DUE** Wed., March 19 at 11:59 PM and should be turned in via Gradescope.
Late Exams will not be accepted for credit except by prior arrangement with the Head TA.

There are 4 problems on pages 2–5 for a total of 60 points

To get partial credit, show as much work as you can!

FYI!
If you fill out the TQFR before March 23, you get 5 (that's right 5) extra credit points on the final.

Problem 1 - Short"ish" Problems

(a) (4 points) An electron in the first excited state of the hydrogen atom can drop down to the ground state by emitting a photon. Determine the wavelength of this photon and identify it as being either infrared, visible or ultraviolet.

(b) Using the eigenstates of $\hat{S}_z$ for a spin $\frac{1}{2}$ particle we can form superposition states from the two-particle eigenstates $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ and $\downarrow\downarrow\rangle$ where $|m_1 m_2\rangle$ are the spin $\frac{1}{2}$ m quantum numbers for each particle. Label the states as E for entangled and N for not entangled and provide a brief explanation for your choices.

- **(i) (2 points)** $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$
- **(ii) (2 points)** $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\downarrow\rangle)$
- **(iii) (2 points)** $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$
- **(iv) (2 points)** $\frac{1}{\sqrt{3}}(|\uparrow\uparrow\rangle - |\downarrow\uparrow\rangle)$

(c) An electron is in the following spin state:

$$\chi = A \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$$

- **(i) (1 point)** Determine the normalization constant A.
- **(ii) (3 points)** Find the expectation values of S_x, S_y, and S_z for this state.

(d) (3 points) Consider a Hamiltonian operator that can be written as a *sum* of terms, each of which depends upon a different set of coordinates that represent different particles, e.g.

$$\hat{H}_{tot} = \hat{H}_a(1) + \hat{H}_b(2) + \hat{H}_c(3) + ...$$

where eg. $\hat{H}_a(1) = \hat{H}_a(p_1, x_1)$.

Show that the energy eigenfunctions for this system can be written as a *product* of eigenfunctions of the separate terms $\psi_{tot} = \psi_a(1)\psi_b(2)\psi_c(3)...$ with the corresponding energy eigenvalues $E_{tot} = E_a + E_b + E_c + ...$

(e) (3 points) Consider the "exchange operator", as defined by $\hat{X}_{12}\psi(x_1, x_2) = \psi(x_2, x_1)$, where $\psi(x_1, x_2)$ is an arbitrary two-particle wavefunction in one dimension. Explicitly show that $\hat{X}_{12}$ is Hermitian by showing that $< \phi|\hat{X}_{12}\psi > = < \hat{X}_{12}\phi|\psi >$.

Problem 2: The Quantum Mechanics of Microwave Ovens

Some molecules can have internal excitations associated with rotations (similar to a symmetrical top except now quantized). In the molecular rest frame (where θ and ϕ are the only independent variables) we can define moments of inertia I_x, I_y, I_z which are constants, related to the particle's mass. For the symmetric top we have $I_x = I_y = I_0$ and $I_z \neq I_0$ with a Hamiltonian, in the molecule's rest frame, given by:

$$\hat{H} = \frac{1}{2I_0}(\hat{L}_x^2 + \hat{L}_y^2) + \frac{1}{2I_z}\hat{L}_z^2$$

(a) (4 points) By re-writing the Hamiltonian show that the energy eigenstates are simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$ and thus show that the eigenstates are the spherical harmonics - Y_l^m.

(b) (4 points) Determine the energy eigenvalues in terms of I_0 and I_z and the appropriate quantum numbers.

(c) (3 points) Now assume your rotor is a water molecule (H_2O) and that $I_0 << I_z$. Then, estimating I_0 by recalling from Ph1a that $I = \sum M_i R_i^2$ with $R_i \sim$ the size of the hydrogen atom and M_i is the proton mass, calculate the energy of the first excited rotational state of H_2O. Use this energy to calculate the corresponding frequency and compare it to the typical operating frequency of a microwave oven of 2.45 GHz.

[Aside: Note that the frequency of microwave ovens has more to do with both the FCC, and that water molecules in food are not free particles but a liquid - which significantly modifies the Hamiltonian].

Problem 3 - The Flip Operator

A quantum system has only two normalized energy eigenstates, $|0\rangle, |1\rangle$, with the corresponding energy eigenvalues E_0, E_1 ($E_1 > E_0$). Apart from the energy, the system is also characterized by a physical observable called "Flip" whose operator $\hat{\mathcal{F}}$ acts on the energy eigenstates as follows:

$$\hat{\mathcal{F}}|0\rangle = |1\rangle, \quad \hat{\mathcal{F}}|1\rangle = |0\rangle$$

The operator $\hat{\mathcal{F}}$ can be regarded as an operator allowing you flip from one state to the other. Thus, clearly,

$$\hat{\mathcal{F}}\hat{\mathcal{F}}|0\rangle = |0\rangle \text{and} \quad \hat{\mathcal{F}}\hat{\mathcal{F}}|1\rangle = |1\rangle$$

(a) (3 points) Show that you can use the above relations to find the normalized eigenstates of $\hat{\mathcal{F}}$ (call them $|+\rangle$ and $|-\rangle$) in terms of $|0\rangle$ and $|1\rangle$. Find the corresponding eigenvalues of $\hat{\mathcal{F}}$.

(b) (4 points) Now assume that the system is initially ($t = 0$) in a $|+\rangle$ eigenstate of $\hat{\mathcal{F}}$. Determine an expression for the wave function of the system at any later time t.

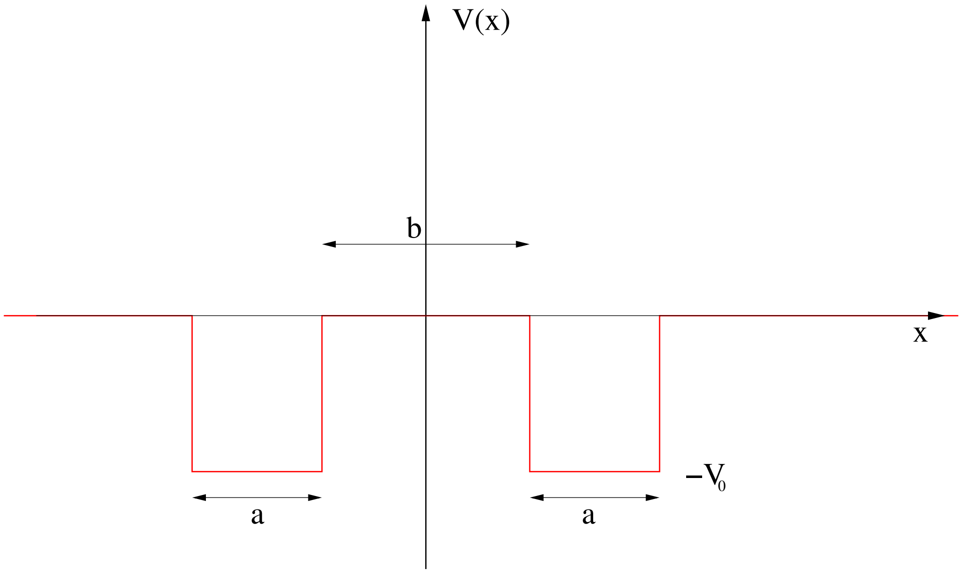
(c) (4 points) At some later time t, a $\hat{\mathcal{F}}$ measurement is made on the system. What is the probability of finding the system with positive flip (i.e. in the $|+\rangle$ state)?

(d) (3 points) Again starting in the $|+\rangle$ state, imagine that you make a series of flip measurements at the times $\Delta t, 2\Delta t, \ldots, N\Delta t = T$. What is the probability of finding the system in the positive flip state (i.e. in the $|+\rangle$ state) at time T? Show that, in the limit that $\Delta t \to 0$ this probability $\to 1$.

[This last result is an example of the so-called Quantum Zeno Effect].

Problem 4: The Double Square Well

Consider the double finite square-well potential shown below for a single particle of mass m.



For this problem assume that the well depth (V_0) and the width a are fixed and large enough so that several bound states exist. Also all of the following questions are qualitative questions - **no detailed calculations needed!**.

(a) (3 points) Sketch the ground state ψ_1 and first excited state ψ_2 wave functions for a barrier separation $b = 0$ (essentially a single well).

(b) (4 points) Sketch the ground state ψ_1 and first excited state ψ_2 wave functions for a barrier separation $b >> a$. This is a little tricky. We would expect to find the particle equally likely in either well and the math should give a solution that looks the same for both wells (up to a minus sign). Is the ground state energy lower or higher than the case in part (a)? [Hint: remember that there's only one particle]. Is the first excited state energy lower or higher than the case in part (a)? [Hint #2 - you might want to do part (c) before answering this].

(c) (3 points) Sketch the ground state ψ_1 and first excited state ψ_2 wave functions for a barrier separation $b \sim a$. This case should be between the previous two cases.

(d) (3 points) Ignoring the Coulomb repulsion of the two nuclei, this problem is a simple model for an electron bound in a diatomic molecule (the two wells represent the attractive force on the electron from the nuclei). One expects that the relative location of the nuclei would be a configuration with the lowest energy. Using your results from above would you expect the ground state to have the nuclei close together or far apart?