IMO Selection 2023
Solutions

Bern

May 13, 2023

First Exam

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

1. In a garden, there are 2023 rose bushes planted in a row. Each bush contains either red or blue roses. Vicky is taking a walk and wants to pick some of the flowers. She starts at a bush of her choice, and picks a rose from it to add to her basket. She then continues walking down the row and picks a single flower from each bush she visits. Vicky can skip some bushes, but she cannot skip two adjacent bushes. She can leave the garden at any point. Let r and b be the number of red and blue roses she picked, respectively. Determine the maximal value of $|r-b|$ Vicky can achieve, irrespective of the configuration of bushes.

Solution: Solution: 507
WLOG, let there be more red than blue bushes, with $R > B$ total bushes of each colour, respectively. If $r = R$, Vicky is only forced to stop by at most $\lfloor \frac{B}{2} \rfloor$, which gives a total of $R - \lfloor \frac{B}{2} \rfloor \geq 507$. This bound cannot be surpassed in the construction

$$B, R, R, B, B, \ldots, B, B, R, R$$

To see this, call the groups of consecutive equal bushes "pairs"; we have a singleton followed by 1011 pairs. Vicky must pass through at least one element of each pair, so if her path intersects k pairs then she has a score of at most $2 \cdot \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{2} \rfloor$ if she wants more reds, or $1 - \lceil \frac{k}{2} \rceil + 2 \cdot \lfloor \frac{k}{2} \rfloor$ if she wants more blues. It is easy to verify that this is an increasing function of k and is maximised when $k = 1011$ and we take the first function, giving 507.

Marking Scheme: The following points are additive:

- 2P: Any construction with maximal value 507 **with claim that 508 cannot be attained for this specific construction.**
- 1P: Accompanying proof of the claim.
- 3P: Any proof that 507 can always be attained.
- 1P: Finishing.

If the student obtains none of the previous points, they may obtain up to 2 points for any of the following, which are nonadditive:

- 1P: Claiming that Vicky should visit every instance of the more common coloured bush.
- 2P: Any proof of the prior statement.
- 1P: Claiming that given Vicky's start and end point, she could visit every instance of one colour and most half the instances of the other colour between those two points.
- 1P: Claiming the value is 507 or any equivalent formulation (e.g. $\lceil \frac{2023}{2} \rceil - \lfloor \frac{2023}{4} \rfloor$.
- 1P: Claiming the value is $\lceil \frac{2023}{4} \rceil$ (note that this value is **incorrect** but demonstrates understanding).

Minor numerical errors with the correct method (for example, a perfect proof except the contestant incorrectly computes the final value to be 506 instead of 507) will not be penalized. Minor flaws (for example, a proof where the contestant incorrectly states a value like $\lceil \frac{2023}{2} \rceil - \lceil \frac{2023}{4} \rceil$) will be penalized 1 point. Major flaws are penalized 2 or more points.

2. Let S be a non-empty set of positive integers such that for any $n \in S$, all positive divisors of $2^n + 1$ are also in S. Prove that S contains an integer of the form

$$(p_1 p_2 \dots p_{2023})^{2023},$$

where $p_1, p_2, \dots, p_{2023}$ are distinct prime numbers, all greater than 2023.

Solution: Since S is non-empty, it contains some integer a. Since $1 \mid 2^a + 1$, it follows that $1 \in S$ and thus $2^1 + 1 = 3 \in S$. We call a pair (m, n) of positive integers k-*valid* if we have $m, n \in S$, $n \mid m \mid 2^n + 1$ and m/n is divisible by k distinct primes. We will show inductively that for every $k \geq 1$ there is a k-valid pair.

Lemma. If (m, n) is k-valid and p is prime divisor of m/n, then $(p^\alpha m, p^\alpha n)$ is k-valid for all integers $\alpha \geq 0$.

Proof. Using LTE, we have $v_p(2^{pm}+1) = v_p(2^n+1)+1$ and hence $pn \mid pm \mid 2^{pn}+1$. Also, since $pn \in S$, we have $pm \in S$. Because also $pm/pn = m/n$, it follows that (pm, pn) is k-valid. Repeating α times gives the desired result. $\square$

For $k = 1$, take $(m, n) = (3, 1)$. Now assume (m, n) is k-valid for some $k \geq 1$ and let p be a prime dividing m/n. For any integer $t \geq 1$ we have

$$0 < v_p(2^{tn} + 1) = v_p(2^n + 1) + v_p(t) < n + \log_p(t).$$

Thus, the k primes we know to divide m/n make up at most a fraction

$$\prod_p p^{n+\log_p(t)} < (m/n)^n t^k$$

of $2^{tn} + 1$. Using the lemma, we can pick t arbitrarily large such that (tm, tn) is still k-valid. It follows that $2^{tn} + 1$ is divisible by a prime q that does not already divide m/n, hence (tmq, tn) is $k + 1$-valid. We conclude the solution by picking a N-valid pair (m, n) where N is large enough to ensure m/n is divisible by at least 2023 primes greater than 2023. We then use the lemma to make sure the powers of all these primes are in the prime factorisation of m are at least 2023.

Solution 2 (Without LTE):
We start by defining the following sequence: $a_1 = 1$ and $a_{n+1} = 2^{a_n} + 1$. For the same reason as in the first solution we have $1 \in S$ and therefore $a_n \in S$ for all n, so it's enough to show that an integer as in the problem statement will divide one of the a_n. Now we

also define $r_n = a_{n+1}/a_n$. If r_{n-1} is an integer it's odd (because a_n is odd) and we can factorise:

$$a_{n+1} = 2^{a_n} + 1 = (2^{a_{n-1}})^{r_{n-1}} + 1^{r_{n-1}} = (2^{a_{n-1}+1}) \sum_{i=0}^{r_{n-1}-1} (-1)^n (2^{a_{n-1}})^i = a_n \sum_{i=0}^{r_{n-1}-1} (1 - a_n)^i$$

and therefore:

$$r_n = \sum_{i=0}^{r_{n-1}-1} (1 - a_n)^i$$

is an integer as well. Since $r_1 = 3$ is an integer we therefore get by induction that all r_n are integers and the equation above holds for all n. Now assume we have a divisor d of n and look at the equation mod d, we get $r_n \equiv r_{n-1}$, therefore:

$$d \mid r_n \Longleftrightarrow d \mid r_{n-1}$$

If we apply this to some prime p we realise that as soon as p divides a_n it divides not only r_{n-1} but also r_m for all $m \geq n$, so $v_p(a_n)$ is increasing and will eventually be bigger than 2023. Now let $p \mid a_n$ and $k = v_p(r_n)$; since $v_p(a_{n+1}) = v_p(a_n) + k > k$ and because $p^k \mid r_n$ but $p^{k+1} \nmid r_n$ we also get: $p^k \mid r_{n+1}$ but $p^{k+1} \nmid r_{n+1}$ so $v_p(r_{n+1}) = k = v_p(r_n)$. Apart of $n+1$ this is also true (again by induction) for all $m > n$. So if there was only a finite amount of primes among the divisors of a_n, we would get that r_n is constant for all n bigger than some n_0, which would imply exponential growth, but a_n grows much faster than exponential, so we can conclude that at some point a new prime will appear, which completes the proof.

Marking Scheme: For (partially) complete solutions the following **additive** scheme applies.

- 1P: Proving that $3^k \in S$ for all k
- 2P: Proving that p prime, $n \in S$ and $pn \mid 2^n + 1$ implies $p^k n \in S$ for all k (or a similar result that increases the exponent of a prime dividing $n \in S$)
- 2P: Proving that $2^{p^k n} + 1$ eventually has a prime divisor that does not divide $2^n + 1$ (or a similar result that gives arbitrarily many prime divisors of $2^n + 1$ with $n \in S$)
- 2P: Concluding

For incomplete solution the following **non-additive** point can be awarded

- 1P: Correctly applying (general) LTE somewhere
- 1P: Showing that $1, 3, 9, 27 \in S$

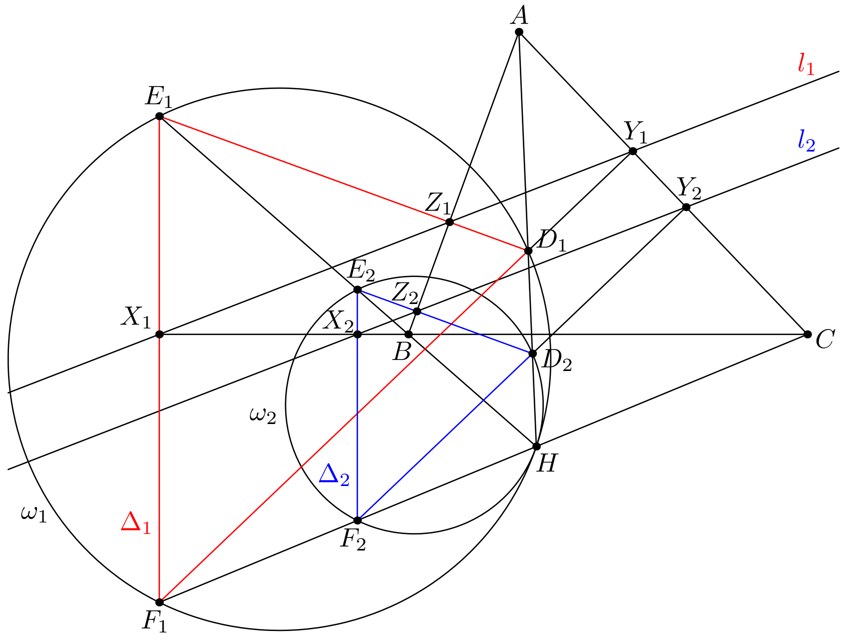
For the second solution (Not cumulative with the first solution):

- 1P: Show that $a_n \mid a_{n+1}$

- 1P: Find the recursive Formula for r_n
- 3P: Show that you get arbitrarily large prime powers
- 2P: Show that you get arbitrarily many primes
  - 1 partial point can be achieved here by using Zsigmondy to show that S contains infinitely many distinct primes.

3. Let ABC be a triangle, and let l_1 and l_2 be two parallel lines. For $i = 1, 2$, assume l_i meets the lines BC, CA and AB at X_i, Y_i and Z_i respectively. Suppose that the line through X_i perpendicular to BC, the line through Y_i perpendicular to CA, and finally the line through Z_i perpendicular to AB, determine a non-degenerate triangle Δ_i. Show that the circumcircles of Δ_1 and Δ_2 are tangent to each other.

Solution:



Let D_i, E_i, F_i be the vertices of Δ_i, such that lines E_iF_i, F_iD_i and D_iE_i are the perpendiculars through X_i, Y_i and Z_i. In triangles $D_1Y_1Z_1$ and $D_2Y_2Z_2$ we have $Y_1Z_1 \parallel Y_2Z_2$ because they are parts of l_1 and l_2. Moreover, $D_1Y_1 \parallel D_2Y_2$ are perpendicular to AC and $D_1Z_1 \parallel D_2Z_2$ are perpendicular to AB, so the two triangles are homothetic and the centre of homothety is $Y_1Y_2 \cap Z_1Z_2 = A$. Hence, line D_1D_2 passes through A. Analogously, line E_1E_2 passes through B and F_1F_2 passes through C.

Let $\measuredangle(p, q)$ denote the directed angle between lines p and q, taken modulo $180°$. The corresponding sides of Δ_1 and Δ_2 are parallel, because they are perpendicular to the respective sides of triangle ABC. Hence Δ_1 and Δ_2 are homothetic, or they can be translated to each other. Using that B, X_2, Z_2 and E_2 are concyclic, C, X_2, Y_2 and F_2 are concyclic, $Z_2E_2 \perp AB$ and $Y_2F_2 \perp AC$, we can calculate

$$\begin{aligned}\measuredangle(E_1E_2, F_1F_2) &= \measuredangle(E_1E_2, X_1X_2) + \measuredangle(X_1X_2, F_1F_2) = \measuredangle(BE_2, BX_2) + \measuredangle(CX_2, CF_2) \\ &= \measuredangle(Z_2E_2, Z_2X_2) + \measuredangle(Y_2X_2, Y_2F_2) = \measuredangle(Z_2E_2, l_2) + \measuredangle(l_2, Y_2F_2) \\ &= \measuredangle(Z_2E_2, Y_2F_2) = \measuredangle(AB, AC) \not\equiv 0°,\end{aligned}$$

and conclude that lines E_1E_2 and F_1F_2 are not parallel. Hence, Δ_1 and Δ_2 are homothetic: the lines D_1D_2, E_1E_2, and F_1F_2 are concurrent at the homothetic centre of the two triangles, denoted by H.

For $i = 1, 2$, let ω_i be the circumcircle of Δ_i. Then, by the computation above, we know that

$$\begin{aligned} \measuredangle(HE_i, HF_i) &= \measuredangle(E_1E_2, F_1F_2) = \measuredangle(AB, AC) \\ &= \measuredangle(Z_iE_i, Y_iF_i) = \measuredangle(D_iE_i, D_iF_i), \end{aligned}$$

so H lies on circle ω_i. We now see that the homothety that maps Δ_1 to Δ_2 sends ω_1 to ω_2, and its centre H is a common point of the two circles. This proves that ω_1 and ω_2 are tangent to each other.

Marking Scheme: The following marking scheme is additive. A student gets the mark for a similar statement, for example proving that B, E_1, E_2 are collinear instead of A, D_1, D_2.

- 1P: Proving that A, D_1, D_2 are collinear
- 1P: Proving that $AY_iD_iZ_i$ is cyclic for some i
- 2P: Proving that $\measuredangle(AD_i, BE_i) = \measuredangle(CA, CB)$
- 1P: Proving that $D_1D_2 \cap E_1E_2 \in (D_iE_iF_i)$ for some i
- 1P: Proving that there is homothety sending Δ_1 to Δ_2, or proving that the center of the spiral similarity sending ABC to $D_iE_iF_i$ lies on (ABC) and $(D_iE_iF_i)$ for some i.
- 1P: Concluding

If none of the above points are given, the student may get at most 1P for any of the following:

- 1P: Claiming that the tangency point of the circles is the intersection of two lines, for example $AD_1 \cap BE_1$.
- 1P: Proving that D_1D_2, E_1E_2, F_1F_2 (or AD_i, BE_i, CF_i for some i) are concurrent or parallel.

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Solutions

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May 14, 2023

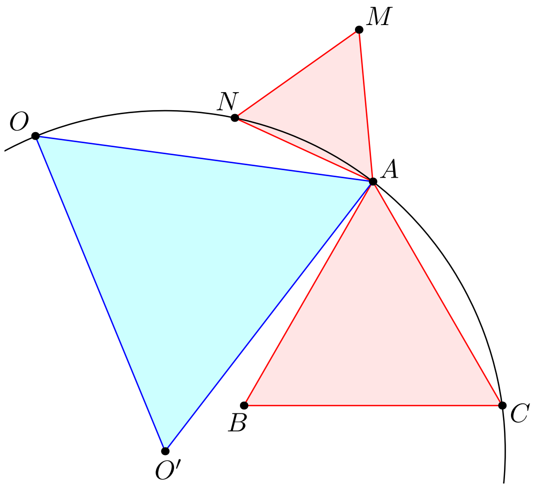
Second Exam

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

4. Let ABC and AMN be two similar, non-overlapping triangles with the same orientation, such that $AB = AC$ and $AM = AN$. Let O be the circumcentre of the triangle MAB. Prove that the points O, C, N and A lie on a circle if and only if the triangle ABC is equilateral.

Solution:



Let O' be the circumcentre of triangle NAC. Consider the rotation of center A and angle $\angle BAC$: the conditions of the problem imply that it maps B to C and M to N. Moreover, because it fixes A we know that it maps O to O' and triangle AOO' is also similar to triangle ABC. Then

$$O,\ C,\ N,\ A \text{ on a circle} \iff O'O = O'A \iff CB = CA \iff \triangle ABC \text{ equilateral},$$

as desired.

Marking Scheme: The following points are non-additive:

- 1P: Proving that O, C, N, A lie on a circle if triangle ABC is equilateral.
- 1P: Considering a rotation/spiral similarity with center A and angle $\angle BAC$.
- 4P: Proving that $\triangle AOO' \sim \triangle ABC$.
- 7P: Complete solution

5. The Tokyo Metro system is one of the most efficient in the world. There is some odd positive integer k such that each metro line passes through exactly k stations, and each station is serviced by exactly k metro lines. One can get from any station to any other station using only one metro line - but this connection is unique. Furthermore, any two metro lines must share exactly one station. David is planning an excursion for the IMO team, and wants to visit a set S of k stations. He remarks that no three of the stations in S are on a common metro line. Show that there is some station not in S, which is connected to every station in S by a different metro line.

Solution: Solution 1 (David, Tanish):

Call a metro line *charming* if it contains exactly one of the stations David wants to visit, and *breathtaking* if it contains two such stations. Every train station in S has k metro lines passing through it, of which $k-1$ link it to other members of S and are breathtaking, and the last one is charming. In particular, we have a one-to-one correspondence between charming metro lines and stations in S and so there are k charming metro lines.

Consider now an arbitrary breathtaking metro line M and an arbitrary station s on that line. We claim s must lie on exactly one charming metro line. Note that this has already been proven for stations in S, so suppose $s \notin S$. Firstly, it has to lie on a charming metro line, because every breathtaking metro line connects it to two stations in S but this is a disjoint partition of S and S has an odd number of elements. However, there must now be at least one charming metro line for every element of M and these metro lines must be distinct, meaning there is necessarily a one-to-one correspondence and there is exactly one charming metro line passing through every station on M.

Consider now the intersection of two charming lines. This station cannot lie on a breathtaking line by the converse of the previous claim. It follows that it must lie on all the charming lines, in order for it to be possible to go from it to stations in S without changing metro line. This intersection is therefore the desired station.

Solution 2 (Felix):

As before, consider charming and breathtaking metro lines, and call a metro line *dull* if it contains no elements of S. It is easy to see there is at least one dull metro line, as it is not difficult to compute that we have k charming metro lines, $\frac{k(k-1)}{2}$ breathtaking metro lines but $k^2 - k + 1$ metro lines in total (the last computation follows from the fact that there are as many metro lines as stations by double counting, and you can count the number of stations as $k(k-1)+1$ by considering an arbitrary station and looking at the distinct stations on each metro line it is on). Furthermore, these three definitions give a trichotomy on the metro lines.

Consider an arbitrary dull metro line D. Every station on D needs to lie on at least $\frac{k+1}{2}$ non-dull metro lines, to connect it to the k (odd) stations in S, since each non-dull metro line can connect it to at most 2 of these. Additionally, there is a bijection between non-dull metro lines and stations in D by the unique intersection of metro lines condition. As there are $\frac{k(k+1)}{2}$ non-dull metro lines in total, every station on a dull metro line must lie on exactly $\frac{k+1}{2}$ non-dull metro lines, of which exactly one is charming.

Now consider the intersection of two charming metro lines. If this station lies on a dull metro line, this contradicts what we have just proven. So it cannot lie on a dull metro line and since it lies on k metro lines each of which connect it to at least one station in S, every metro line it is on must connect it to a different station in S.

Solution 3 (Mathys, Anna):
As before, consider charming and breathtaking metro lines. In particular, it is not too hard to show every station not in S is on at least one charming line, by a simple parity argument.

Now, we place an upper bound on the number of stations not in S. As shown in previous solutions, we know there are k charming lines, so order this arbitrarily. The first charming line contains exactly $k-1$ stations not in S. The second charming line contains at most $k-2$ new stations not in S, as it must intersect with the previous charming line in at least one station. The third charming line contains at most $k-2$ new stations, as it must intersect with the previous charming lines in at least one station. Continuing in this manner, we obtain a total upper bound of $k-1+(k-2)(k-1) = k^2-2k+1$ stations not in S.

However, as in previous solutions, one can show there are k^2-k+1 stations in total, meaning this bound is tight and during our bounding process, each charming metro line had to provide $k-2$ new stations. This implies that all the charming metro lines intersect in one station, which is the desired station.

Remark: One can also complete Solution 3 using Karamata [**Bora**].

Marking Scheme: The following three schemes are for the three known approaches to the problem. Each is additive. Contestants should be awarded the maximum of their points from the three schemes.

Solution 1:

- 1P: Considering dull, charming and breathtaking metro lines (if two of the three groups are considered, this is sufficient).
- 1P: Claiming that the desired station is the intersection of the charming metro lines.
- 2P: Showing that every station on a breathtaking metro line belongs to at least one charming metro line.
- 2P: Showing that every station on a breathtaking metro line belongs to at most one charming metro line.
- 1P: Concluding by considering the intersection of two charming metro lines.

Solution 2:

- 1P: Considering dull, charming and breathtaking metro lines (if two of the three groups are considered, this is sufficient).

- 0P: Stating any station not in S must lie on at least $\frac{k+1}{2}$ non-dull metro lines, even if they consider a station on a dull metro line.
- 1P: Remarking that there is a bijection between non-dull metro lines and stations on a dull metro line.
- 2P: Proving that every station on a dull metro line lies on exactly $\frac{k+1}{2}$ non-dull metro lines.
- 1P: Concluding that every station on a dull metro line lies on exactly one charming metro line.
- 1P: Claiming that the desired station is the intersection of the charming metro lines.
- 1P: Showing that lying on two charming metro lines implies you cannot lie on a dull metro line, implying in turn you must lie on only charming metro lines and concluding.

Solution 3:

- 1P: Considering dull, charming and breathtaking metro lines (if two of the three groups are considered, this is sufficient).
- 1P: Proving any station not in S must lie on a charming metro line.
- 3P: Bounding the total number of stations not in S by $k^2 - 2k + 1$ in such a manner that the tight case is one in all which charming metro lines intersect in one station.
- 1P: Remarking that **this bound** is tight (i.e., no points are awarded for showing there are $k^2 - k + 1$ total stations without also having obtained the previous bound).
- 1P: Concluding that all charming lines must intersect in one station.

Counting the number of dull/charming/breathtaking metro lines is worth **0 points**. If a contestant provides the second solution but fails to prove the existence of at least one dull line, they are penalized **1 point**.

6. Determine all positive integers $n \geq 2$ for which there exist n distinct real numbers $a_1, a_2, \ldots, a_n$ and a real number $r > 0$ such that

$$\{a_j - a_i \mid 1 \leq i < j \leq n\} = \{r, r^2, \ldots, r^{\binom{n}{2}}\}.$$

Solution: Solution 1 (IMOSL):
The solution is $n = 2, 3, 4$.
For $n = 2$, take $(a_1, a_2) = (0, r)$.
For $n = 3$, take $(a_1, a_2, a_3) = (0, r, r + r^2)$, where r is a non-zero root of $r^3 - r^2 - r = 0$. More precisely, take $r = \frac{1+\sqrt{5}}{2}$.
For $n = 4$, take r to be a positive root of $x^3 - x - 1 = 0$. Such a root exists, as $p(1) = -1$ and $p(2) = 5$. The set $(a_1, a_2, a_3, a_4) = (0, r, r + r^2, r + r^2 + r^3)$ is good, as $a_3 - a_1 = r^2 + r = r^4$, $a_4 - a_2 = r^3 + r^2 = r^5$ and $a_4 - a_1 = r^3 + r^2 + r = r^3 + r^4 = r^6$.

For $n \geq 5$ we proceed by contradiction. Suppose there exist numbers $a_1 < \cdots < a_n$ and $r > 1$ satisfying the conditions of the problem. We start with a lemma:
Lemma We have $r^{n-1} > 2$.
Proof. There are only $n-1$ differences $a_j - a_i$ with $j = i+1$. So there exists an exponent $l \leq n$ and a difference $a_j - a_i$, with $j \geq i+2$ such that $a_j - a_i = r^l$. This implies

$$r^n \geq r^l = a_j - a_i = (a_j - a_{a_j-1}) + (a_{j-1} - a_i) > r + r = 2r.$$

Thus $r^{n-1} > 2$ as desired.

Denote $b = \frac{n(n-1)}{2}$. Clearly we have $a_n - a_1 = r^b$. Consider the $n-2$ equations of the form:

$$a_n - a_1 = (a_n - a_i) + (a_i - a_1) \text{ for } i \in \{2, \ldots, n-1\}.$$

In each equation, one of the two terms on the right hand-side must be at least $\frac{1}{2}(a_n - a_1)$. But from the lemma we have $r^{b-(n-1)} = r^b/r^{n-1} < \frac{1}{2}(a_n - a_1)$, so there are at most $n-2$ sufficiently large elements in $\{r^k \mid 1 \leq k < b\}$, namely $r^{b-1}, \ldots, r^{b-(n-2)}$ (note that r^b is already used for $a_n - a_1$). Thus, the "large" terms must be, in some order, precisely equal to elements in

$$L = \{r^{b-1}, \ldots, r^{b-(n-2)}\}$$

Next we claim that the "small" terms in the $n-2$ equations must be equal to the elements in

$$S = \{r^{b-(n-2)-\frac{1}{2}i(i+1)} \mid 1 \leq i \leq n-2\},$$

in the corresponding order (the largest "large" term with the smallest "small" term, etc.). Indeed, suppose that

$$r^b = a_n - a_1 = r^{b-1} + r_i^{\alpha} \text{ for } i \in \{1, \ldots, n-2\},$$

where $1 \leq \alpha_1 < \cdots < \alpha_{n-2} \leq b-(n-1)$. Since $r > 1$ and $f(r) = r^n$ is convex, we have

$$r^{b-1} - r^{b-2} > r^{b-2} - r^{b-3} > \cdots > r^{b-(n-3)} - r^{b-(n-2)},$$

implying

$$r^{\alpha_2} - r^{\alpha_1} > \cdots > r^{\alpha_{n-2}} - r^{\alpha_{n-3}}.$$

Convexity of $f(r) = r^n$ further implies

$$\alpha_2 - \alpha_1 > \alpha_3 - \alpha_2 > \cdots > \alpha_{n-2} - \alpha_{n-3}.$$

Note that $\alpha_{n-2} - \alpha_{n-3} \geq 2$: Otherwise we would have $\alpha_{n-2} - \alpha_{n-3} = 1$ and thus

$$r^{\alpha_{n-3}}(r-1) = r^{\alpha_{n-2}} - r^{\alpha_{n-3}} = r^{b-(n-3)} - r^{b-(n-2)} = r^{b-(n-2)} \cdot (r-1),$$

implying that $\alpha_{n-3} = b - (n-2)$, a contradiction. Therefore, we have

$$\begin{aligned}\alpha_{n-2} - \alpha_1 &= (\alpha_{n-2} - \alpha_{n-3}) + \cdots + (\alpha_2 - \alpha_1) \\ &\geq 2 + 3 + \cdots + (n-2) \\ &= \frac{1}{2}(n-2)(n-1) - 1 = \frac{n(n-3)}{2}.\end{aligned}$$

On the other hand, from $\alpha_{n-2} \leq b - (n-1)$ and $\alpha_1 \geq 1$ we get

$$\alpha_{n-2} - \alpha_1 \leq b - n = \frac{1}{2}n(n-1) - n = \frac{n(n-3)}{2},$$

implying that equalities must occur everywhere and the claim about small terms follows. Now, assuming $n - 2 \geq 2$, we have two different equations:

$$r^b = r^{b-(n-2)} + r^{b-(n-2)-1} \text{ and } r^b = r^{b-(n-3)} + r^{b-(n-2)-3},$$

which can be written as

$$r^{n-1} = r + 1 \text{ and } r^{n+1} = r^4 + 1. \tag{1}$$

Simply algebra now gives

$$r^4 + 1 = r^{n+1} = r^{n-1} \cdot r^2 = r^3 + r^2 \Longrightarrow (r-1)(r^3 - r - 1) = 0.$$

Since $r \neq 1$, using Equation (1) we conclude $r^3 = r + 1 = r^{n-1}$, thus $n = 4$, which gives a contradiction.

Solution 2 (Raphi):
The construction is the same as in solution 1.

We will encounter the equation $r^a + r^b = r^c \longleftrightarrow r^{a'} - r^{b'} - 1 = 0$ a lot. (note that $a' > b' > 0$). So first let's study this equation a bit. Let's call this equation basic. Let $a_1 \ldots a_n$ be numbers fulfilling the condition for the constant $r > 0$. We call a basic equation good if it is solved for the fixed value r.

Lemma 1 Two different good basic equations will have completely distinct pairs of exponents. That means for $r^a - r^b - 1 = r^c - r^d - 1 = 0$, we have $a \neq c \wedge b \neq d$. Also even further we have $a - b \neq c - d$.
Proof: Just compare the good equations if one of the inequalities doesn't hold.

Lemma 2 If we have two good basic equations with exponents $a > b$ and $c > d$. Then $a \geq c \Leftrightarrow a - b \leq c - d \Leftrightarrow b \geq d$.
Proof: multiply the second basic equation by $a - c$ (or by $b - d$) and then compare the two equations.

It's clear that $a_n - a_1 = r^{\binom{n}{2}}$. Wlog let $a_{n-1} - a_1 = r^{\binom{n}{2}-1}$. Then $r^{\binom{n}{2}-2}$ either appears at $a_{n-2} - a_1$ or as $a_n - a_2$. From this we get that the basic equations $r^{\binom{n}{2}} = r^{\binom{n}{2}-1} + r^{x_1} = r^{\binom{n}{2}-2} + r^{x_2}$. So we have

$$r^\alpha - r^{\alpha-1} = r^\beta - r^{\beta-2} = 1$$

for $\alpha = \binom{n}{2} - x_1$, $\beta = \binom{n}{2} - x_2$. Dividing the equation by $(r-1)r^{\beta-2}$, we get that

$$r^{\alpha-\beta+1} - r - 1 = 0$$

From Lemma 2 we follow that there exists at most $\alpha - \beta$ good basic equations. as the difference $a - b$ in a good basic equation must be in the set $\{1, 2, \ldots, \alpha - \beta\}$.

Bound 1:
Now let's go back to the actual problem. Call a difference between two consecutive a_{i+1}, a_i a small gap and any other difference a big gap. Note that the smallest big gap is less or equal to r^n. As it can be written as a sum of two small gaps it must be greater of equal to $\alpha - \beta + 1 + 1$, as $r + r^2 = r^{\alpha-\beta+2}$. So $\alpha - \beta + 2 \leq n$.

Bound 2:
Looking at the basic equations formed by the difference of a_i, a_j, a_k for the triples $(1, 2, k)$ where k ranges over all values $3 \leq k \leq n$. We see that there exist at least $n - 2$ different good equations (using Lemma 1 and 2). So $\alpha - \beta \geq n - 2$

Together we get $n - 2 = \alpha - \beta = x_1 - x_2$.

Corollary 1
Looking back at the first bound we realise that the small gaps are exactly the values $r, r^2, \ldots, r^{n-1}$.

Corollary 2
All the good basic equations are of the form $r^{c_k+k} - r^{c_k} - 1 = 0$, where $1 \leq k \leq \alpha - \beta$. Call this the k-th basic equation. From the second bound, every k-th basic equation is good (for an appropriate c_k). Note that the c_i are decreasing and actually $c_i + i \geq c_j + j$ for $i \leq j$. (By Lemma 2) Also $c_1 = \alpha - 1$, $c_2 = \beta - 2$ and $c_n = \alpha - \beta + 1 = n$.

Corollary 3
If we have $r^a + r^b = r^c$ with $a < b < c$, then $c_n + n = \alpha - \beta + 1 \leq c - a \leq \alpha = c_1$, $1 \leq c - b \leq \alpha - \beta = n$ and $1 = c_n \leq b - a \leq c_1$. Call the exponent a the "smaller exponent", b the "bigger exponent" and the exponent c the "resulting exponent".

Back to using basic equations:
Note that in the second bound we could replace $(1,2)$ with any pair $(i, i+1)$ and let k range over all values unequal $i, i+1$. So for each fixed $(i, i+1)$ there exist $n-2$ associated good basic equations. So also for the small gap with value $n-1$. It can be the bigger exponent at most once. So it is in at least $n-3$ distinct equations (all the equations where it is the smaller exponent). But then the biggest resulting exponent must be at least $(n-1)+(c_2+2)$, as we use all but one different types of basic equations and $c_1+1 \geq c_2+2$. So $c_2+n+1 \leq \binom{n}{2} = c_2+2+x_2$. So $x_2 \geq n-1$.

Lemma 3
We have $a_2 - a_1 = r^{x_2}$ and thereby $x_2 \leq n-1$.
Proof
Write $a_2 - a_1 = r^k$. In the triple $(1,2,n)$, r^k must be the small exponent. So the resulting exponent $\binom{n}{2}$ must be equal to $c_i + i + k$ for some $i \in [n-2]$. But now if $i > 2$ we have

$$c_i + i + k \leq c_2 + 2 + k \leq c_2 + 2 + n - 1 \leq c_2 + 2 + x_2 = \binom{n}{2}$$

which clearly is a contradiction. If $i = 1$ then we would have to have that $k = \alpha_1$, which also can't be true, as $r^{x_1} = a_n - a_{n-1}$. So $i = 2$, which leads to $x_2 = k \leq n-1$.

We now have $x_2 = n-1$ and $x_2 - x_1 = \alpha - \beta = n-2$, leading to $x_1 = 1$. So no other value other than r^1 can be used in an equation of 1st type. So every value for a small gap appears as the bigger summand at least once. So every small gap is neighbouring to a small gap with a lesser value. So the small gaps must be ordered in size starting at $a_2 - a_1 = r^{n-1}$ going all the way to $a_n - a_{n-1} = r^1$.

Finishing the proof
Now it's easy to see that all the other values for the big gaps are forced. If $g_{s,i}$ is a gap of size s starting at a_i, then $g_{s,i} > g_{t,j}$ if $s > t$ and $g_{s,i} = g_{s,j} + j - i$. Drawing a pyramid diagram with this starting from the longest row, the row of small gaps, going from right to left we will fill in the values 1 all the way up to $\binom{n}{2}$.

$$g_{s,i} = sn - \binom{s}{2} - i$$

But now the gap $g_{1,2}$ should be part of the 2nd basic equation to create $r^{\binom{n}{2}-n+3}$. But actually it's part of the gap $g_{n-1,2}$ which has size $r^{\binom{n}{2}-1}$ which is bigger than that. Contradiction.

Solution 3 (inspired by Felix:) Well Felix had a fun Lemma, and funnily enough I found a very nice and quick solution from this.

Denote by $[r^k]$ the closed interval, which is the interval spanned by the two a_i, a_j, with distance r^k.
Felix Lemma:
The two intervals $[r^\alpha]$ and $[r^{\alpha+1}]$ are not disjoint. Where $1 \leq \alpha \leq \binom{n}{2} - 1$.

Proof:
Assume the two intervals are disjoint. Let the endpoints of these two intervals be $a_{\xi_1} <$

$a_{\xi_2} < a_{\zeta_1} < a_{\zeta_2}$. Write $r^m = a_{\zeta_1} - a_{\xi_2}$ and $r^\beta = a_{\zeta_1} - a_{\xi_1}$. Since the inequality $r^{\beta+1} > r^{\alpha+1} + r^m > r^\beta$ holds, the difference $a_{\zeta_2} - a_{\xi_1} = r^{\alpha+1} + r^m$ can't be a power of r. Contradiction.

From the Lemma it's clear that the two Intervals $[r]$ and r^2 share a common endpoint.

Finishing Lemma:
Either $r^2 = r + 1$ or $r^3 = r + 1$.

Proof: Let k be the biggest value such that all the intervals $[r^i], i \leq k$ appear in order, where two consecutive intervals share exactly one point. From the comment above, we know that k exists and is at least 2.
By the assumption on k (and the nature of the intervals), we must have that either $[r^k] \subset [r^{k+1}]$ or $[r^{k-1}] \subset [r^{k+1}]$. This leads to these cases:

Case 1: $[r^{k+1}]$ contains $[r^k]$
We must have that $r^{k+1} = r^k + r^{k-1}$. (A drawing would help here) Because in all other cases we would have another interval $[r^C]$ in $[r^{k+1}]$, where $C > k + 1$. Clearly a contradiction.

Case 2: $[r^{k+1}]$ contains $[r^{k-1}]$ and not $[r^k]$
We must have $r^{k+1} = r^{k-1} + r^{k-2}$. If $[r^{k+1}]$ also contained $[r^{k-3}]$, we would have the interval $[r^{k-3}] \cup [r^{k-2}]$ in $[r^{k+1}]$. This interval must be a power of r, where the exponent is greater than $k + 1$ (We already used all the smaller exponents somewhere else). Clearly a contradiction.

We now distinguish between the two cases $r^2 - r - 1 = 0$ and $r^3 - r - 1 = 0$. Now starting with the two neighbouring intervals $[r]$ and $[r^2]$, we can easily figure out where the intervals up to $[r^6]$ must appear, simply by checking where the next biggest interval could go, where the equation quite often forces its position. We can then also by case bashing lead this to a contradiction if $n > 4$.

Marking Scheme:

Solution 1 The following points are additive:

(a) 1P: Have a correct construction for $n = 2, 3, 4$. Explicitly showing the case $n = 4$.

(b) 1P: Claim that r solves the $n-2$ equations: $r^b = r^{b-k} + r^{\binom{k+1}{2}}$ for $k \in \{1, \ldots, n-2\}$.

(c) 4P: Show that the equations related with the triples $(a_1, a_i, a_n), 2 \leq i \leq n - 1$ are exactly the $n - 2$ equations above.

   i. 1P: Show that in the equations associated with (a_1, a_i, a_n) the "larger" terms in the sum are exactly $\{r^{b-1}, \ldots r^{b-(n-2)}\}$.

   ii. 2P: Show that $a_{i+1} - a_i \geq n - i$

(d) 1P: Finish.

Solution 2 The following points are additive:

(a) 1P: Have a correct construction for $n = 2, 3, 4$. Explicitly showing the case $n = 4$.

(b) 1P: Show that the amount of good basic equations is bounded by d. Where d is the exponent in the good basic equation $r^d - r - 1 = 0$. (or equivalent)

(c) 1P: Prove that $d \leq n - 1$.

(d) 1P: Prove that $d \geq n - 1$

(e) 2P: Show that the small gaps are $(r, r^2, \ldots, r^{n-1})$ and in that order.

(f) 1P: Finish.

Solution 3 The following points are additive:

(a) 1P: Have a correct construction for $n = 2, 3, 4$. Explicitly showing the case $n = 4$.

(b) 2P: Felix Lemma.

(c) 3P: Finishing Lemma.

(d) 1P: Finish.

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IMO Selection 2023
Solutions

Bern

May 27, 2023

Third Exam

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

7. Determine all monic polynomials $P(x) = x^{2023} + a_{2022}x^{2022} + \cdots + a_1x + a_0$ with real coefficients such that $a_{2022} = 0$, $P(1) = 1$, and all roots of P are real and less than 1.

Solution: Write $P(x) = (x - z_1)(x - z_2) \ldots (x - z_{2023})$, where $z_1, \ldots, z_{2023}$ are the roots of P. Note that $P(1) = 1$ is equivalent to $(1-z_1)(1-z_2)\ldots(1-z_{2023}) = 1$. Furthermore, by Vieta, $z_1 + z_2 + \ldots + z_{2023} = 0$. This gives us the two equations

$$\begin{aligned}(1 - z_1) + (1 - z_2) + \ldots + (1 - z_{2023}) &= 2023,\\ (1 - z_1) \cdot (1 - z_2) \cdot \ldots \cdot (1 - z_{2023}) &= 1.\end{aligned}$$

By observing that

$$\frac{1}{2023} \cdot \sum_{i=1}^{2023}(1 - z_i) = \left(\prod_{i=1}^{2023}(1 - z_i)\right)^{\frac{1}{2023}}$$

and using the equality case of AM-GM (which can be applied since $1 - z_i > 0$ for all i), we deduce that $z_1 = z_2 = \ldots = z_{2023} = 0$. Therefore, the only solution is $P(x) = x^{2023}$. This polynomial obviously satisfies the desired properties.

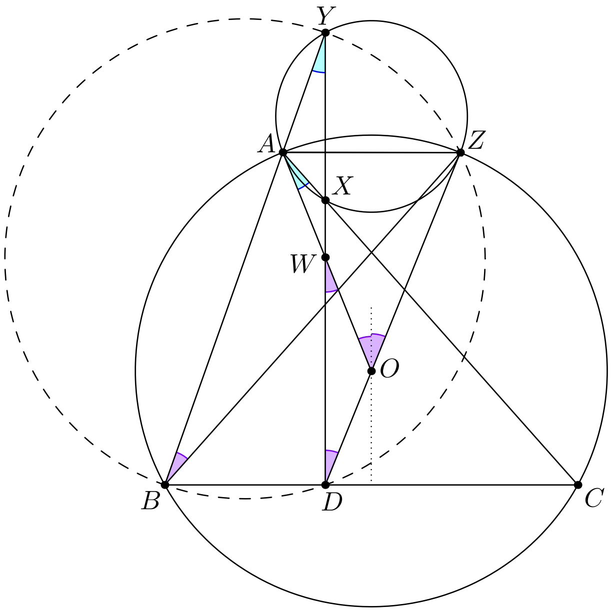
Marking Scheme:

- 1P: Noting that $P(x) = x^{2023}$ is a solution.
- 1P: Introducing the roots and observing $(1 - z_1)(1 - z_2) \ldots (1 - z_{2023}) = 1$.
- 1P: Finding that $z_1 + \ldots + z_{2023} = 0$.
- 1P: Rewriting the former in terms of $(1 - z_i)$
- 2P: Applying a suitable inequality relating the two equations
- 1P: Concluding

If an inequality (like AM-GM) is used in a solution without explicitly mentioning that the variables used are non-negative, at most 6 points can be awarded for that solution.

8. Let ABC be an acute triangle with $AC > AB$, let O be its circumcentre, and let D be a point on the segment BC. The line through D perpendicular to BC intersects the lines AO, AC and AB at W, X and Y, respectively. The circumcircles of triangles AXY and ABC intersect again at $Z \neq A$. Prove that if $OW = OD$, then the line DZ is tangent to the circle AXY.

Solution: *Solution 1:*



First, note that

$$\angle AYX = \angle BYD = 90^\circ - \angle B = \angle OAC = \angle OAX,$$

so by the converse of the tangent-chord theorem, OA is tangent to the circle AXY. As $OA = OZ$, the other tangent to the circle AXY through O is line OZ. Hence, it remains to prove that D, O, Z are collinear.

Moreover, we know that $BDZY$ is cyclic from

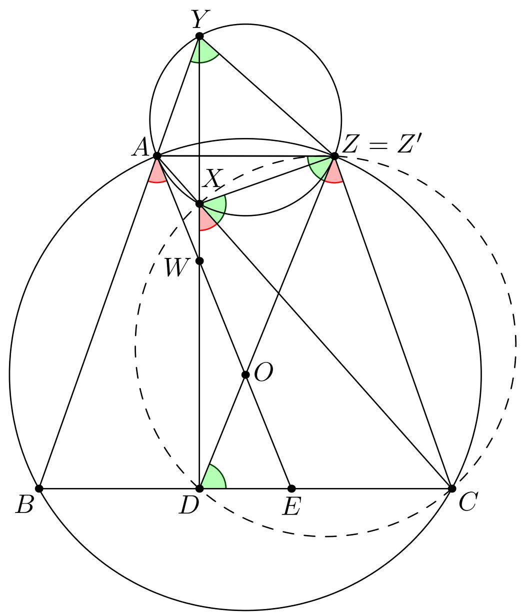
$$\angle DBZ = \angle CBZ = \angle CAZ = \angle XAZ = \angle XYZ = \angle DYZ.$$

Finally, we use the condition that $OW = OD$ to conclude, as

$$\begin{aligned}\angle ZOW = \angle ZOA = 2 \cdot \angle ZBA = 2 \cdot \angle ZBY = 2 \cdot \angle ZDY \\ = \angle ODW + \angle DWO = 180^\circ - \angle WOD\end{aligned}$$

proves that D, O and Z are collinear.

Solution 2:



Let AO intersect BC at E. As EDW is a right-angled triangle and O is on WE, the condition $OW = OD$ means O is the circumcentre of this triangle. So $OD = OE$ which establishes that D, E are reflections in the perpendicular bisector of BC.

We also note that

$$\angle ZCD = \angle ZAY = \angle ZXY = 180^\circ - \angle DXZ$$

shows that $CDXZ$ is cyclic.

Next, we show that $AZ \parallel BC$. To do this, introduce point Z' on circle ABC such that $AZ' \parallel BC$. By the previous result, it suffices to prove that $CDXZ'$ is cyclic. Notice that triangles BAE and $CZ'D$ are reflections in the perpendicular bisector of BC. Using this and that A, O, E and collinear:

$$\angle DZ'C = \angle BAE = \angle BAO = 90^\circ - \frac{1}{2}\angle AOB = 90^\circ - \angle C = \angle DXC,$$

so $DXZ'C$ is cyclic, giving $Z = Z'$ as desired.

Using $AZ \parallel BC$ and $CDXZ$ cyclic we get:

$$\angle AZD = \angle CDZ = \angle CXZ = \angle AYZ,$$

which by the converse of the tangent-chord theorem shows that DZ is tangent to circle AXY.

Marking Scheme: The following marking schemes are additive, and a subitem indicating partial progress towards an item can be chosen instead of that item. Only one marking scheme can be used per student.

Marking Scheme 1:

- 2P: Proving that OZ is tangent to (AXY)
  - 1P: Proving that OA is tangent to (AXY)
- 4P: Proving that D, O, Z are collinear
  - 2P: Proving that $BDZY$ or $CDXZ$ is cyclic
  - 3P: Proving that $AZ \parallel BC$, or that Z is the reflection of A over the perpendicular bisector of BC.
- 1P: Concluding

Marking Scheme 2:

- 2P: Proving that $BDZY$ or $CDXZ$ is cyclic.
- 3P: Proving that $AZ \parallel BC$, or that Z is the reflection of A over the perpendicular bisector of BC.
  - 2P: Proving that D, O, Z are collinear
- 2P: Concluding

9. Let G be a graph whose vertices are the integers. Assume that any two integers are connected by a finite path in G. For two integers x and y, we denote by $d(x, y)$ the length of the shortest path from x to y, where the length of a path is the number of edges in it. Assume that $d(x, y) \mid x - y$ for all $x, y \in \mathbb{Z}$ and define $S(G) = \{d(x, y) \mid x, y \in \mathbb{Z}\}$. Find all possible sets $S(G)$.

Solution: The possible sets are $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 2, 3\}$ and $\mathbb{Z}_{\geq 0}$.

Since d is defined as a distance, we also have the triangle inequality, stating that

$$d(x, y) \leq d(x, z) + d(z, y)$$

for all $x, y, z \in \mathbb{Z}$. Now note that for all $x \in \mathbb{Z}$ we must have $d(x, x+1) \mid 1$ and thus $d(x, x+1) = 1$. Iterating this result with the triangle inequality we find that $d(x, y) \leq |x - y|$ for all $x, y \in \mathbb{Z}$. If we consider the graph on $\mathbb{Z}$ where x, y are contested if and only if $|x - y| = 1$, then have $d(x, y) = |x - y|$ which satisfies the condition. Excluding this case, we can assume without loss of generality that $d(0, a) < a$ for some positive integer a, after shifting d accordingly. Moreover, we can assume that a is the smallest positive integer with this property. If $d(0, a) < a - 2$, then we would have

$$d(0, a-1) \leq d(0, a) + d(a, a-1) < (a-2) + 1 < a - 1,$$

contradicting minimality of a. Hence, $a - 2 \leq d(0, a) < a$. But we must also have $d(0, a) \mid a$, which together with $d(0, a) < a$ implies that $2 \cdot d(0, a) \leq a$. Putting the two inequalities together we find that $2(a-2) \leq 2 \cdot d(0, a) \leq a$, which is equivalent to $a \leq 4$. In particular, we have

$$d(0, 5) \leq d(0, a) + d(a, 5) < a + (5 - a) = 5$$

and hence $d(0, 5) = 1$. Since for all integers k we have

$$d(k+1, k+6) \leq d(k+1, k) + d(k, k+5) + d(k+5, k+6) = d(k, k+5) + 2$$

and similarly $d(k, k+5) \leq d(k+1, k+6) + 2$, it follows that $d(k, k+5) = 1$ for all $k \in \mathbb{Z}$. Now note that for all k, t we have

$$d(k, t+6) \leq d(k, t) + d(t, t+5) + d(t+5, t+6) = d(k, t) + 2$$

ans similarly $d(k, t) \leq d(k, t+6) + 2$. It follows by induction that $d(k, k + 6t \pm 1) = 1$, where the base case $t = 0$ is trivial and we use the fact that $6t \pm 1$ is coprime to 2 and 3. Now since each integer is at most distance 2 from an integer of the form $6t \pm 1$, it follows that $d(x, y) \leq 3$ for all $x, y \in \mathbb{Z}$.

Now set $S = \{d(x, y) \mid x, y \in \mathbb{Z}\}$ and note that certainly $0, 1 \in S$. If $3 \in S$, then we must also have $2 \in S$ since any shortest path of length 3 can be shortened to a shortest path of length 2. The only remaining possibilities for S are

$$\{0, 1\}, \quad \{0, 1, 2\} \quad \text{and} \quad \{0, 1, 2, 3\}.$$

Now consider the graph where x and y are connected if and only if

$$x - y \equiv \pm 1 \pmod 6$$

and quick check shows that this graph satisfies the condition and $S = \{0, 1, 2, 3\}$. Considering instead the graph where x and y are connected if they are of different parity, we also check that the condition is met and $S = \{0, 1, 2\}$. Finally considering the complete graph on $\mathbb{Z}$ also shows that $S = \{0, 1\}$ is possible.

Marking Scheme:

- 1P: Noting that $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{Z}$
- 2P: Proving that $S = \mathbb{Z}_{\geq 0}$ or $d(x, x + k) < k$ for some integers x and $k \leq 5$
- 1P: Proving that $S = \mathbb{Z}_{\geq 0}$ or $d(x, x + 5) = 1$ for all $x \in \mathbb{Z}$
- 2P: Proving that $S = \mathbb{Z}_{\geq 0}$ or $d(x, y) = 1$ for all $x, y \in \mathbb{Z}$ with $x - y \equiv \pm 1 \pmod 6$
- 1P: Giving examples of graphs with $S = \{0, 1\},\ \{0, 1, 2\},\ \{0, 1, 2, 3\},\ \mathbb{Z}_{\geq 0}$
- -1P: Not mentioning that $S(G)$ has to be downwards closed

IMO Selection 2023
Solutions

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Bern
May 28, 2023
Fourth Exam

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

10. Let $a, d > 1$ be two coprime integers. Define the sequence $(x_i)_{i \in \mathbb{N}}$ by setting $x_1 = 1$ and

$$x_{k+1} = \begin{cases} x_k/a & \text{if } a \text{ divides } x_k \\ x_k + d & \text{otherwise} \end{cases}$$

for all $k \geq 1$. Determine the largest non-negative integer n such that a^n divides at least one term of the sequence, or prove that no such n exists.

Solution:

The answer is $n = 1 + \lfloor \log_a(d) \rfloor$.

We first prove that $x_k \leq ad$. Assume for a contradiction that there is some least index k with $x_k > ad$. Let x_t be the last term before x_k that was not obtained by the second of the two cases. It follows that $x_t \leq d$, otherwise $x_{t-1} > ad$ already. Thus,

$$x_k - x_t \geq ad + 1 - d > (a-1)d$$

and hence the a term starting from x_t are of the form $x_t + id < x_k$ for $0 \leq i \leq a - 1$. However, at least one of these terms must be divisible by a, contradiction the minimality of t.

We now prove that the sequence is periodic. Let us consider some term x_k with $k \geq 2$. If $x_k > d$, then we cannot have $x_{k-1} = ax_k > ad$, hence we must have $x_{k-1} = x_k - d$. If instead $x_k \leq d$, then we cannot possibly have $x_{k-1} = x_k - d \leq 0$, hence we must have $x_{k-1} = ax_k$. Thus, the sequence is uniquely defined in both directions. Together with the fact that the sequence is bounded, it follows that it is periodic.

Since for any $k \geq 1$ we have $x_k \leq ad$, it follows that $n \leq 1 + \lfloor \log_a(d) \rfloor$. Conversely, since $x_1 = 1$ and the sequence is periodic, there is some $k \geq 2$ with $x_k = 1$ also. It follows by the above analysis that $x_{k-i} = a^i$ for all $1 \leq i$ with $a^{i-1} < d$. Since a and d are coprime, $\log_a(d)$ is not an integer and we can take $i = 1 + \lfloor \log_a(d) \rfloor$.

Marking Scheme:

- 1P: Stating the correct answer
- 2P: Proving that the sequence is bounded by ad (only 1P for worse bounds)
- 1P: Proving that the sequence is uniquely defined backwards
- 1P: Deducing that the sequence is periodic
- 1P: Proving that any later 1 is preceded by all powers of a that are smaller than d
- 1P: Concluding

11. Denote by $\mathcal{F}$ the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(y)) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Determine all rational numbers q such that for every function $f \in \mathcal{F}$, there exists some $z \in \mathbb{R}$ with $f(z) = qz$.

Solution:

Answer: The desired set is $\left\{\frac{n+1}{n} : n \in \mathbb{Z}, n \neq 0\right\}$.

Solution: Let S be the set of all rational numbers q such that for every function $f \in \mathcal{F}$, there exists some $z \in \mathbb{R}$ satisfying $f(z) = qz$. Let further

$$T = \left\{\frac{n+1}{n} : n \in \mathbb{Z}, n \neq 0\right\}.$$

We prove that $T = S$ by showing the two inclusions: $S \subseteq T$ and $T \subseteq S$. We first prove that $T \subseteq S$. Let $f \in \mathcal{F}$ and let $P(x, y)$ be the relation $f(x + f(y)) = f(x) + f(y)$. First note that $P(0, 0)$ gives $f(f(0)) = 2f(0)$. Then, $P(0, f(0))$ gives $f(2f(0)) = 3f(0)$. We claim that

$$f(kf(0)) = (k + 1)f(0)$$

for every integer $k \geq 1$. The claim can be proved by induction. The cases $k = 1$ and $k = 2$ have already been established. Assume that $f(kf(0)) = (k + 1)f(0)$ and consider $P(0, kf(0))$ which gives

$$f\big((k + 1)f(0)\big) = f(0) + f(kf(0)) = (k + 2)f(0).$$

This proves the claim. We conclude that $\frac{k+1}{k} \in S$ for every integer $k \geq 1$. Note that $P(-f(0), 0)$ gives $f(-f(0)) = 0$. We now claim that

$$f(-kf(0)) = (-k + 1)f(0)$$

for every integer $k \geq 1$. The proof by induction is similar to the one above. We conclude that $\frac{-k+1}{-k} \in S$ for every integer $k \geq 1$. This shows that $T \subseteq S$.

We now prove that $S \subseteq T$. Let p be a rational number outside the set T. We want to prove that p does not belong to S. To that end, we construct a function $f \in \mathcal{F}$ such that $f(z) \neq pz$ for every $z \in \mathbb{R}$. The strategy is to first construct a function

$$g\colon [0, 1) \to \mathbb{Z}$$

and then define f as $f(x) = g(\{x\}) + \lfloor x \rfloor$. This function f belongs to $\mathcal{F}$. Indeed,

$$\begin{aligned}
f(x + f(y)) &= g(\{x + f(y)\}) + \lfloor x + f(y) \rfloor \\
&= g(\{x + g(\{y\}) + \lfloor y \rfloor\}) + \lfloor x + g(\{y\}) + \lfloor y \rfloor \rfloor \\
&= g(\{x\}) + \lfloor x \rfloor + g(\{y\}) + \lfloor y \rfloor \\
&= f(x) + f(y),
\end{aligned}$$

where we used that g only takes integer values.

Lemma. For every $\alpha \in [0,1)$, there exists $m \in \mathbb{Z}$ such that

$$m + n \neq p(\alpha + n)$$

for every $n \in \mathbb{Z}$.

Proof. Note that if $p = 1$ the claim is trivial. If $p \neq 1$, then the claim is equivalent to the existence of an integer m such that

$$\frac{m - p\alpha}{p - 1}$$

is not an integer. Assume the contrary. That would mean that both

$$\frac{m - p\alpha}{p - 1} \quad \text{and} \quad \frac{(m+1) - p\alpha}{p - 1}$$

are integers, and so is their difference. The latter is equal to

$$\frac{1}{p - 1}.$$

Since we assumed $p \notin S$, $1/(p-1)$ is never an integer. This is a contradiction. $\square$

Define $g\colon [0,1) \to \mathbb{Z}$ by $g(\alpha) = m$ for any integer m that satisfies the conclusion of the Lemma. Note that $f(z) \neq pz$ if and and only if

$$g(\{z\}) + \lfloor z \rfloor \neq p(\{z\} + \lfloor z \rfloor)$$

The latter is guaranteed by the construction of the function g. We conclude that $p \notin S$ as desired. This shows that $S \subset T$.

Marking Scheme:

- 2P: Proving that $T \subset S$
  - 1P: Proving that $\frac{n+1}{n} \in S$ for all $n \in \mathbb{N}$
  - 1P: Proving that $\frac{-n+1}{-n} \in S$ for all $n \in \mathbb{N}$
- 5P: Constructing f to show $S \subset T$
  - 1P: Observing that one can construct a function to be in $\mathcal{F}$, using linear functions and/or equivalence classes. In other words, exploit that, if $f(z)$ is defined, then so is $f(z + \mathrm{Im}(f))$.

Appendix: Alternative constructions

Construction 1 (Johann): Instead of constructing a function f_q such that $f_q(x) \neq qx$, we shall construct a more general function f with the property that $f(x) = qx$ has a solution in x for some rational q if and only if $q \in T$. However, one will need to assume the axiom of choice. As in the previous solutions, observe that the functional equation can be rewritten as, for all $x \in \mathbb{R}$ and $c \in \mathrm{Im}(f)$, we have

$$f(x+c) = f(x) + c.$$

Pick $f(0) = 1$, to get $f(z) = z + 1$ where z is an integer. Let α be an arbitrary irrational number and let's pick f such that $f(q) = \alpha$ for **all** rationals $q \in (0, 1)$. From these assumptions, we get that

$$f(\alpha z + r) = f(\alpha z + \{r\}) + \lfloor r \rfloor = f(\{r\}) + \alpha z + \lfloor r \rfloor = \alpha(z+1) + \lfloor r \rfloor$$

for any rational $r \in \mathbb{Q}$. Therefore, this implies that f is uniquely determined on the set

$$K := \{r + z\alpha : r \in \mathbb{Q} \backslash \mathbb{Z}, z \in \mathbb{Z}\}$$

Now, define an equivalence relation on $\mathbb{R}$ where $x \sim y$ if and only if $x - y = z_1 + \alpha z_2$ for integers z_1, z_2. Let

$$[x] = \{y \in \mathbb{R} \mid y \sim x\}$$

and call these equivalence classes. Notice that the function f is already defined on $[0]$ and $[q]$ for any rational q. This forms a partition of the real numbers. Now, use the axiom of choice and take an element in each of the equivalence classes $[x]$. We obtain a set $Z \subseteq \mathbb{R}$ such that for any class, there is a unique element in Z that lies in that class. Hence, mathematically, we have

$$\bigcup_{z \in Z} [z] = \mathbb{R}$$

Let $f(z) = 0$ for any element in $Z \backslash K$. Indeed, imposing $f(z) = 0$ determines the function uniquely on $[z]$. Indeed, this is because for any $z \in \mathbb{R}$, $m \in \mathbb{Z}$ and $n \in \mathbb{Q} \backslash \mathbb{Z}$

$$f(z + m + \alpha n) = f(z) + m + \alpha n = m + \alpha n$$

Thus, the function is defined on each of the classes $[z]$. The partitions form a class, so the function is defined on $\mathbb{R}$. By construction, this function in fact satisfies the functional equation.

Now, let's look for which q we can find an $x \in \mathbb{R}$ such that $f(x) = qx$ for this specific function. Looking at z a nonzero integer implies that any $q = \frac{z+1}{z}$ has a solution in x. If we look at $x \in K \backslash \mathbb{Z}$, write $x = \alpha z + q$ we have that

$$\frac{f(\alpha z + q)}{\alpha z + q} = \frac{f(\alpha z + \{q\}) + \lfloor q \rfloor}{\alpha z + q} = \frac{\alpha(z+1) + \lfloor q \rfloor}{\alpha z + q}$$

If this expression is rational, then

$$\frac{z+1}{z} \cdot \frac{\alpha z + \frac{z}{z+1}\lfloor q \rfloor}{\alpha z + q} \in \mathbb{Q}$$

which is only possible if and only if $\frac{z}{z+1}\lfloor q \rfloor = q$ or $z = -1$, implying that either $\frac{f(x)}{x} \in T$ or $\frac{f(x)}{x} \notin \mathbb{Q}$. Finally, if $x \in \mathbb{R}\backslash(K \cup \mathbb{Z})$, then $\frac{f(x)}{x}$ is irrational because $f(x) \in K$ but $x \notin K \cup \mathbb{Z}$. Hence, one must have $S \subseteq T$, which yields $S = T$, as required.

Construction 2 (Raphael): Once again we work with equivalence classes and use the axiom of choice to pick a representative for each class. The two constructions are the same in spirit, differing only in where we choose to be general and where to pick a specific value.
Just like above we chose f, such that $f(0) = 1$ and thereby $f(n) = n + 1$ for $n \in \mathbb{N}$[1]. So we have that

$$f(x + n) = f(x) + n + 1$$

So if define f on the interval $(0, 1)$, we will then indirectly also define f on all of $\mathbb{R}$. Let's now define our equivalence classes. We say x and y are equivalent, iff they differ by an integer amount. So $x \sim y \Leftrightarrow x - y \in \mathbb{N}$.
Let Z be the set of all the representatives for the equivalence classes. We can actually even more specifically pick Z to be the intervall $(0, 1)$. Writting $x = z_\alpha + n$ and $y = z_\beta + m$, we should have that

$$f(z_\alpha + f(z_\beta)) + n + m + 2 = f(z_\alpha) + f(z_\beta) + n + m + 2 \Leftrightarrow f(z_\alpha + f(z_\beta)) = f(z_\alpha) + f(z_\beta).$$

So if we can fulfill this equation for all $z_\alpha, z_\beta \in Z$, then the function f will be in $\mathcal{F}$. The easiest way to make this happen, is to use the fact that we have $f(x + n) = f(x) + f(n)$ anyway already. So define $f(z_\alpha) =: n_\alpha \in \mathbb{Z}$ for $z_\alpha \in Z$. One can check that the function f defined like this (independant of our choices for n_α) is indeed a function in $\mathcal{F}$.
We now to want to find "good" choices for the n_α, such that for a given $q \notin T$, we never have $f(x) = qx$. Let x be such a possible candidate. Note that $x \notin \mathbb{N}$, by $q \notin T$. We have

$$f(z_\xi + m) = q(z_\xi + m) \Leftrightarrow m - qm = qz_\xi - n_\xi - 1 \Leftrightarrow m = \frac{1 - z_\xi + n_x i}{q - 1}.$$

So the fraction on the right should be an integer. Well let's just choose n_ξ such that it isn't. Since $\frac{1}{q-1}$ isn't an integer this is indeed possible. So for each $q \notin T$, we can construct a function showing $q \notin S$, implying $S \subset T$, finishing the proof.

[1]As a matter of fact on can show that all functions in $\mathcal{F}$ can be 'streched' to be equal to these values on the integers.

12. For a positive integer m, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$. Let n be a positive integer and let $\mathcal{S}$ be a non-empty collection of subsets of $[n]$. A function $f\colon [n] \to [n+1]$ is called *kawaii* if there exists $A \in \mathcal{S}$ such that for all $B \in \mathcal{S}$ with $A \neq B$ we have

$$\sum_{a \in A} f(a) > \sum_{b \in B} f(b).$$

Prove that there are always at least n^n *kawaii* functions, irrespective of $\mathcal{S}$.

Solution: We will prove there is an injection from functions $f : [n] \to [n]$ and kawaii functions.

Let f be an arbitrary function in the latter set. Let S be any set in $\mathcal{S}$ whose sum of images is **larger or equal** to all other sums of images over sets in $\mathcal{S}$. Now, add 1 to the image of every element of S. You now obtain a function that is forcibly kawaii, irrespective of whether f was kawaii. Furthermore, this function is clearly invertible from the image back to the domain and therefore an injection.

Note that several equivalent approaches are possible, for example considering functions whose images are $2, \ldots, n+1$ and then taking away 1 from the images of elements not in a maximal set.

Marking Scheme: The following points are additive:

- 1P: Trying to find an injection between functions $[n] \to [n]$ or $[n] \to \{2, \ldots, n+1\}$ and kawaii functions.
- 1P: The idea of modifying the images of some set whose sum is maximal **or** its complementary.
- 4P: Providing the injection.
- 1P: Justifying that this mapping is an injection.

If contestants provide the flawed injection where they do not modify functions that are already kawaii, they are penalised **2 points**.