MATH 240 Final Exam

December 12, 2025

Mock exam

1. The final exam covers the entire course (see list of topics in separate document for details). The focus will be on material covered after the second midterm (vector calculus).
2. There will be ten problems.
3. No calculators or cheat sheets allowed.

Problem 1

1. Find a scalar equation of the plane containing the points $P(1,0,0)$, $Q(0,2,0)$ and $R(2,2,3)$.
2. Find the area of the parallelogram spanned by the vectors $\vec{a} = \langle 1,1,1 \rangle$ and $b = \langle 1,1,0 \rangle$.
3. Find the volume of the parallellepiped spanned by $\vec{a}, \vec{b}$ and $\vec{c} = \langle 1,3,2 \rangle$.

Problem 1

1. Find the distance of the point $P(4,2,3)$ from the plane $x + y - z = 0$.
2. Find the cosine of the angle between the planes $x + y + z = 1$ and $x - 2y + 2z = 0$.
3. Compute the vector projection $P_{\vec{w}}(\vec{v})$ of $\vec{v} = \langle 1,0,0 \rangle$ onto $\vec{w} = \langle 1,1,2 \rangle$.

Problem 2

1. Parametrize the ellipse $x^2 + \frac{y^2}{4} = 1$ in the form $\vec{r}(t) = \langle a\ \cos t, b \sin t \rangle$ for some a, b.

2. Compute the unit tangent $\vec{T}(t)$ to the ellipse.

3. Compute the curvature $\kappa(t) = \frac{\|\vec{r}''(t) \times \vec{r}'(t)\|}{\|\vec{r}'(t)\|^3}$ of the ellipse and write it as $\kappa(t) = \frac{d}{\left(e+f\ \cos^2(t)\right)^{3/2}}$ for some d, e, f.

4. At which point(s) (x, y) is the curvature greatest?

Problem 2

Consider the helix C parametrized by $\vec{r}(t) = \langle t,\ 2\cos t,\ 2\sin t \rangle$.

1. Compute unit tangent vector function $\vec{T}(t)$.

2. Compute the unit normal vector function $\vec{N}(t)$.

3. Compute the curvature function $\kappa(t)$.

Problem 3

Compute $f(2,3)$ and use linear approximation to compute $f(2.1, 2.8)$ for $f(x,y) = \sqrt{4x^2 + 3y}$.

Problem 3

Verify that the point $P(1,1,1)$ lies on the surface given by $\sqrt{x}+y+z^2 = 3$ and compute the tangent plane at P.

Problem 4

Find the maximum values and minimum values that $f(x, y, z) = x^3 + y^3 + z^3$ takes on the sphere $x^2 + y^2 + z^2 = 1$.

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Find the maximum values and minimum values that $f(x, y, z) = x^4 + y^4 + z^4$ takes on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 4

Find all critical points of the function $f(x, y) = x^4 - 2x^2 + y^2 + 1$ and decide whether they are local maxima, minima, or saddle points.

Problem 5

Compute the double integral $\iint_T x + y\, dA$ where T is the triangle with vertices $P(0, 0)$, $Q(1, 1)$ and $R(1, 2)$.

Problem 5

Compute the double integral $\iint_D x^2 + y^2\, dA$ over the region which arises if the disk $x^2 + y^2 \leq 1$ is removed from the rectangle with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$.

Hint: It may be easier to write the integral over this region as the difference of two integrals.

Problem 6

Compute the volume of the region which is both inside the sphere $x^2 + y^2 + z^2 < 2$ and the cylinder $x^2 + y^2 < 1$.

Hint: Spherical coordinates do not work well here.

Problem 6

Compute the volume and center of mass of region which is both inside the sphere $x^2 + y^2 + z^2 < 1$ and inside the cone $z > \sqrt{x^2 + y^2}$. *By symmetry, you may assume that* $\bar{x} = \bar{y} = 0$.

Problem 7

1. *Is the vector field* $\vec{F}(x, y, z) = \langle yz, xz, xy \rangle$ *conservative? If so, find a potential function.*
2. *Compute the line integral* $\int_C \vec{F} \cdot d\vec{r}$ *where* C *is the straight line segment from* $P(1, 0, 2)$ *to* $Q(4, 5, 1)$.

Problem 7

Compute the line integral $\int_C f\, ds$ *where* C *is the helix segment* $\vec{r}(t) = (2\cos t, 2\sin t, t)$ *with* t *between* 0 *and* 5 *and* $f(x, y, z) = 1 + z$.

Problem 8

Evaluate the surface integral $\iint_S f\, dS$ *where* S *is the ellipsoid* $x^2 + y^2 + \frac{z^2}{4} = 1$ *and*

$$f(x, y, z) = \frac{z^2}{\sqrt{1 + 3x^2 + 3y^2}}.$$

One strategy to solve this is imitating spherical polar coordinates to parametrize the ellipsoid.

Problem 8

Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ *where* $\vec{F}(x, y) = (\sin(x^2) + y)\vec{\imath}$ *and* C *is made up of three straight line segments: From* $(0, 0)$ *to* $(2, 0)$*, from* $(2, 0)$ *to* $(1, 2)$*, and from* $(1, 2)$ *to* $(0, 0)$*.*

Problem 9

An ellipsoid is parametrized by $\vec{r}(\theta, \phi) = \langle \sin\phi\, \cos\theta, \sin\phi\, \sin\theta, 2\cos\phi \rangle$*. A curve* C *on the ellipsoid in physical space corresponds to the curve in parameter space going along straight line segments from* $(\theta_0, \phi_0) = (0, 0)$ *to* $(\theta_1, \phi_1) = (0, \pi/2)$*, from here to* $(\theta_2, \phi_2) = (\pi/2, \pi/2)$*, and finally to* $(\theta_3, \phi_3) = (\pi/2, 0)$*.*

1. *Argue that* C *is in a closed curve in physical space.*
2. *Use Stokes' Theorem to compute the line integral* $\int_C \vec{F} \cdot d\vec{r}$ *where* $\vec{F}(x, y, z) = \langle x + 1, 2y + 3, 4z + 5 \rangle$*.*

Problem 9

Consider the curve C *on the hyperbolic paraboloid* $z = x^2 - y^2$ *which is parametrized by* $0 \leq t < 2\pi$ *using* $\vec{r}(t) = \langle \cos t, \sin t, \cos^2 t - \sin^2 t \rangle = \langle \cos t, \sin t, \cos 2t \rangle$*. Use Stokes' Theorem to compute the line integral* $\int_C \vec{F} \cdot d\vec{r}$ *for* $\vec{F}(x, y, z) = \langle x, x, z \rangle$*.*

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x, y, z) = \langle x, 2y, 3z \rangle$$

where S *is the boundary surface of the hemisphere* $x^2 + y^2 + z^2 < 1$ *and* $z > 0$*, oriented by the exterior unit normal.*

Problem 10

Use the divergence theorem to compute

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \vec{F}(x, y, z) = \langle x + 2,\ 3,\ z + z^2 \rangle$$

where S *is the boundary surface of the tetrahedron with vertices* $(0, 0, 0)$*,* $(1, 0, 0)$*,* $(0, 1, 0)$ *and* $(0, 0, 1)$.