AMA1D01C

The History of Ancient Chinese and World Mathematics

Lecture Notes Set #11

A Brief Introduction to Ancient Japanese Mathematics

簡介　日本數學史

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References :

- **David Eugene SMITH and Yoshio MIKAMI 三上義夫, *A History of Japanese Mathemstics*, 1914.**
- **FUKAGAWA Hidetoshi 深川英俊 and Tony ROTHMAN, *Sacred Mathematics Japanese Temple Geometry*, 2008.**
- **MURATA Tamotsu 村田全, *Indigenous Japanese Mathematics, Wasan*, Journal of Japanese Trade and Industry, 2001.**
- **Eberhard KNOBLOCH • Hikosaburo KOMATSU 小松彦三郎 • Dun LIU 劉鈍, *Seki, Founder of Modern Mathematics in Japan. A Commemoration on His Tercentenary*, 2013.**

Many scholars have written extensively on the topic, just to name a few:

- 川原秀城 **(translated)**
- 洪萬生
- 城地茂
- 李文林
- 徐澤林
- 黃俊瑋

Web resources: 京都大學
Kyoto University Rare Materials Digital Archive
Mathematics in Classics (Dep.Math.)
https://rmda.kulib.kyoto-u.ac.jp/en/collection/math?status=1&page=3

Web resources : 和算の館
http://www.wasan.jp/

According to Smith and Mikami, the development of Japanese mathematics could be divided into 6 periods:
(Note: other scholars divided the timeline in different ways)

- Earliest Period: Before 552
- Second Period: 552 — 1600
- Third Period: 1600 — 1675
- Fourth Period: 1675 — 1775
- Fifth Period: 1775 — 1868
- Sixth Period: After 1868

Other scholars divide the timeline differently, e.g.

- 遠藤利貞 (1843-1915)《大日本數學史》
- 藤原松三郎 (1881-1946)《明治前日本數學史》
- 李文林, 徐澤林《和算選粹》
- 黃俊瑋《江戶時期和算發展之分期》2013.

Chronological list of events:

- **Awakening Period: 1st — 7th centuries**
    - **Mathematics from Ancient China indirectly by way of Korea**
    - **Asuka period 飛鳥時代 592 — 710**
    - **Empress Suiko 推古天皇 (554-628) and Prince Shōtoku 聖德太子(572-621)**
    - **Hōryū-ji 法隆寺** or **Hōryū Gakumonji 法隆学問寺 was founded in 607**
    - **First Official Contacts with China 遣隋使 / 遣唐使，645 大化革新.**
    - **In 701, Emperor Monbu 文武天皇 establish a university system, nine Chinese mathematics books were recognised**
        - **Chou-pei (周髀), Sun-tsu (孫子), Liu-chang (六章), San-kai Chung-cha (三開重差), Wu-tsao (五曹), Hai-tao (海島), Chiu-szu (九司) , Chiu-Chang (九章), Chui-She (綴術).**
- **Nara Period 奈良時代 : 710 — 794.《養老令·學令》718,《養老令·令義解》733.**
- **Heian Period 平安時代 : 794 — 1185**
- **Kamakura Period 鎌倉時代 : 1185 — 1333**
- **Samurai Governments' Period: 1192 — 1573 / Civil War Period: 1467 — 1602**
    - **In the late 16th century (1592–1598), Toyotomi Hideyoshi 豐臣秀吉 (1537–1598) invaded the Korean 朝鮮 peninsula and brought back to Japan the Korean's version of the Chinese mathematics books《算學啟蒙》,《楊輝算法》.**
    - **The Chinese versions of 《算學啟蒙》(1299) and《楊輝算法》(1275) were already lost in China at the time [兩書在當時大明時期已經亡佚.]**
- **Edo Period 江戶時代 or Tokugawa period (德川時代) : 1603 — 1867**

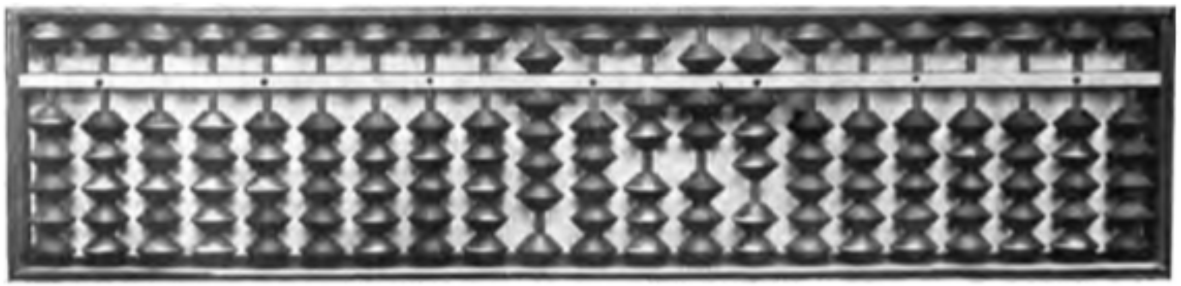
Computational Tools:

北中國數學(算木(籌)數學)

算木, さん, ぎ sangi / computing rods

南中國數學(珠算數學)

算盤, そろばん, soroban (Japanese abacus)



Thanks Professor Xiaojun Chen
for her comments earlier,
and subsequently,
this page was added.

二五珠算盤

一四珠算盤

go-dama　五玉
ichi-dama 一玉

early from China　二五珠算盤

around 1850　一五珠算盤

1893 (明治26年)　**西川秀三郎『數學報知』第七七號**

一四珠算盤 (天一地四)

big argument

大正　昭和　時代的 四珠論

鹽野直道　大正13年 (1924)

接任命為教育部圖書監修官

base 16, for certain things, was only adopted in China
(for example, 1斤 = 16兩),
but not in Japan, thus, no real need to keep 5 beads below.

1781 乳井貢 Mitsugi Nyui, and others, proposed to remove one bead

乳井貢『初學算法』(安永十年 (1781))：
『...常用的算盤；上二珠，下五珠，其上二珠有何用處，而且其下五珠的最下面一珠，又有何用處，當然主二珠中之一珠是無用處，因此算盤只有上一珠，下四珠即可...。』

Just before entering into The Edo Period (和算 wasan)

The book *Suanfa Tongzong* 算法統宗 was written by
Cheng Dawei (1533 — 1606, 明代) (程大位) in 1592.
A concise version of the book
Suanfa Zuanyao (A compendium of calculating methods) 演算法纂要
was also written by Cheng in 1598.

Mōri Kambei / Mōri Shigeyoshi 毛利勘兵衛 / 毛利重能 translated the book *Suanfa Zuanyao* into Japanese, and was also writing and teaching about the use of soroban.

The story has it that Mori was sent by Toyotomi Hideyoshi 豊臣秀吉 to China (明代) to bring the mathematics/computational knowledge back to Japan (before Toyotomi invaded Korea).

Smith and Mikami argued that, Mori may not have been to China,
and doubtful about the account of Mori's journey mentioned in
Sampo Tamatebako 和洋普通算法玉手箱 written by Fukuda Riken 福田理軒 in 1879.

Edo period 江戸時代 or Tokugawa period (徳川時代) 1603 - 1867

(和算 wasan)

- Mōri Kambei / Mōri Shigeyoshi 毛利勘兵衛 / 毛利重能
- Yoshida Mitsuyoshi 吉田光由 (1598 — 1672)
- Imamura Tomoaki 今村知商
- Muramatsu Shigekiyo 村松茂清 (1608 – 1695)
- Araki Murahide 荒木村英（1640—1718）
- Seki Takakazu 關孝和 (1642 – 1708)
- Takebe Kataaki 建部賢明 (1661 – 1716)
- Takebe Katahiro 建部賢弘 (1664 –1739)
- Matsunaga Yoshisuke 松永良弼 (1690 — ?)
- Kurushima Yoshihiro 久留島義太 / 久留島喜内 (? — 1757)
- Arima Yoriyuki 有馬賴徸（1714 – 1783）
- Fujita Sadasuke 藤田貞資 (藤田定資) (1734 – 1807)
- Ajima Naonobu 安島直円 (1732 – 1798)
- Aida Yasuaki 會田安明（1747 – 1817）
- Shin'ichi Kawabe 川邊信一
- Fujita Kagen 藤田嘉言 (1765 – 1821)
- Wada Yenzō Nei 和田寧 (1787 – 1840)

Modern Period (洋算 yōsan)

- Hōjō Tokiyuki 北条時敬 (1858 – 1929)
- Teiji Takagi Teiji Takagi 高木貞治 (1875 – 1960)
- Mikami Yoshio 三上義夫（1875 – 1950）

- 1587年豐臣秀吉下令驅逐傳教士，實行禁教。
- 1597年「二十六聖人殉教」事件，6 名傳教士、3 名日本傳教士與 17 名日本信徒在長崎被處以死刑。

Sakoku 鎖國

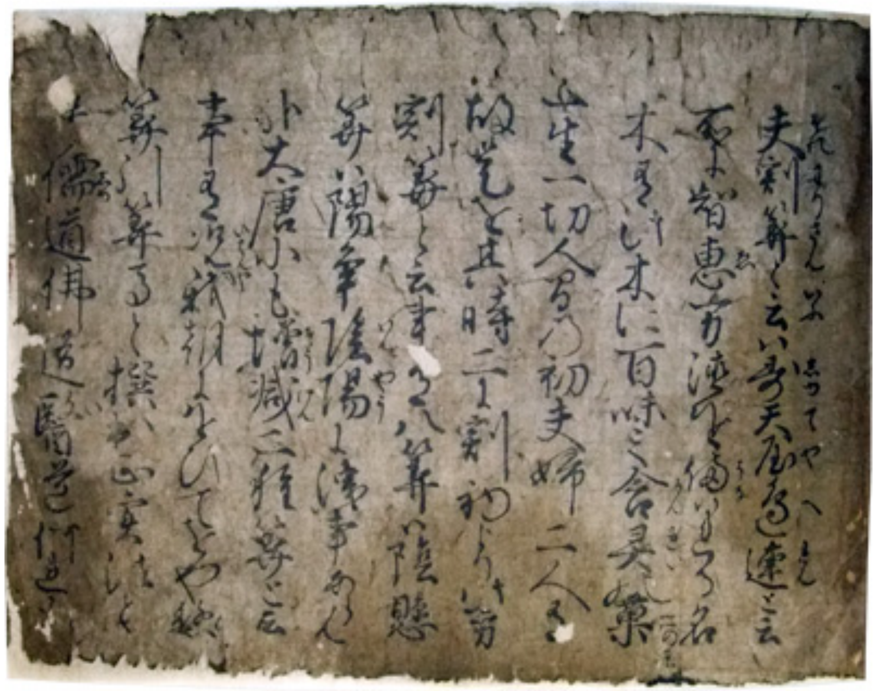
- 1633年第1次鎖國令。禁止奉書船以外船隻渡航。奉書制度即出海船隻除朱印狀外，還必須有幕府老中簽發的文書特許與“南蠻”進行貿易的船隻。 當時只有特權商人才能得到該種許可文書。
- 此外，禁止滯留外國5年以上的日本人回國。

(from the Dutch) Rangaku 蘭學 around 1720

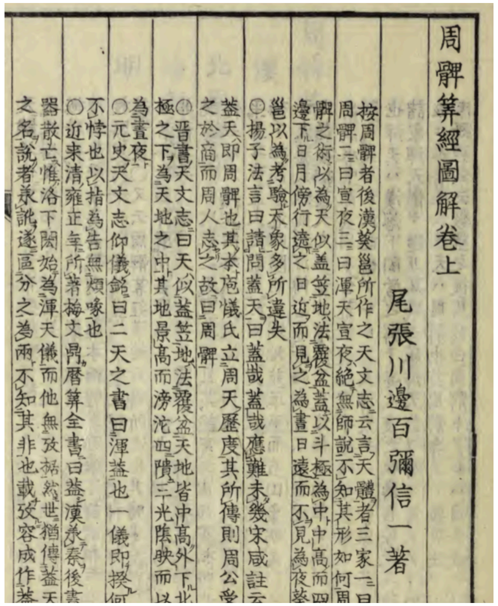
1808年發生在日本的英國船襲擊事件(菲頓號事件 HMS Phaeton 1782)

1854年 黑船事件 James Biddle 培理率領九艘軍艦抵達幕府被迫與美國締結《神奈川條約》

吉田光由《塵劫記》1627

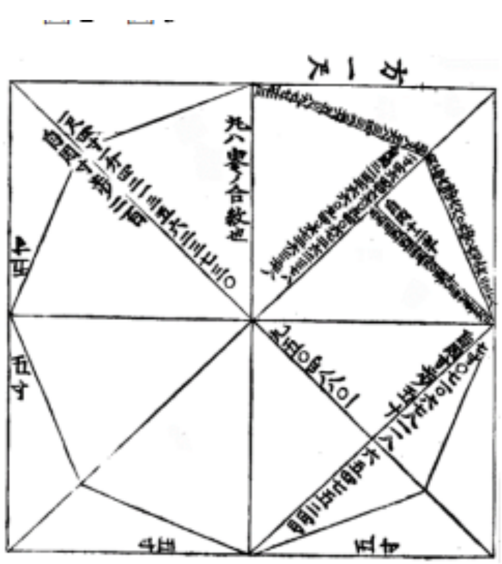


毛利重能《割算書》1622

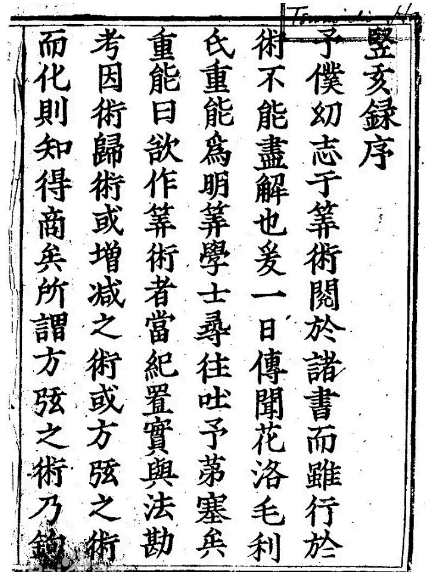


周髀算經圖解卷上　尾張川邊百彌信一著

川邊信一《周髀算經圖解》1786



村松茂清《算俎》1663



竪亥録序

予僕幼志于筭術閲於諸書而雖行於
術不能盡解也爰一日傳聞花洛毛利
氏重能爲明筭學士尋往叱予茅塞矣
重能曰欲作筭術者當紀置實與法勘
考因術歸術或增減之術或方弦之術
而化則知得商矣所謂方弦之術乃鈎

今村知商《竪亥録》1639

Seki Takakazu 關孝和
or
關新助孝和
(1642 – 1708)
算聖

Pictures from the reference:
Eberhard Knobloch • Hikosaburo Komatsu • Dun Liu, ***Seki, Founder of Modern Mathematics in Japan. A Commemoration on His Tercentenary*****, 2013.**

Mathematical Methods for Exploring Subtle Points [發微算法 Hatsubi Sampo]

Compendium of Mathematics [括要算法 Katsuyo Sampo]

Complete Book of Mathematics [大成算經 Taisei Sankei]

傍書法, 天元演段術, 歸原整法, 點竄法 (Let x be …. algebra)
圓理 (arc-length, area of 2D regions, volume of 3D objects, mathematical analysis)

發微算法

古今算法記一十五問之荅術

平圓解空門一問

今有平圓内如圖平圓空三箇外餘寸平積百二十步只云從中圓徑寸而小圓徑寸者短五寸問大中小圓徑幾何

荅曰依左術得小圓徑

術曰立天元一爲小圓徑加入云數爲中圓徑自之得數寄甲位 列小圓徑自之得數倍之加入甲位以圓周率乘之得數寄乙位 列外餘積四之以圓徑率乘之得數加入乙位爲因圓周率大圓徑冪寄丙位 列小圓徑以甲位相乘亦以圓周率相乘得數寄丁位 列中圓徑四之得内減小圓徑餘以丙位乘之得内減丁位餘自乘之爲因中圓徑四箇與小圓徑二箇和冪因中圓徑冪因圓周率冪大圓徑冪寄左 列併中圓徑四箇與小圓徑二箇得數自之以甲位相乘亦以丙位相乘亦以圓周率相乘得數與寄左相消得開方式五乘方飜法開之得小圓徑仍推前術得大中圓徑各合問

平立重積門四問五問之荅術

今有大立方小平方各一只云平方積爲實開立方之

發微算法 Hatsubi Sampo

Three circles with radius r_1, r_2 and r_3 are inscribed within a circle as shown,

and each one is tangent to the other two, and to the original circle with radius r_0 .

The total area of the space inside the circumscribing circle and outside the three circles is 120 square units 平方步.

The diameters of the two smaller circles are equal, so $r_2 = r_3$,

and each is 5 units 寸 less than the diameter of the larger one.

Find the diameters of the circles.

1步 = 60寸

1平方步 = 3600平方寸

Since the two smaller circles are of the same size, they both have radius r_2.

Note that 1步$=$6尺, and 1尺$=$10寸. Hence, 1步$=$60寸.
Therefore, 1平方步 $=$3600 平方寸.

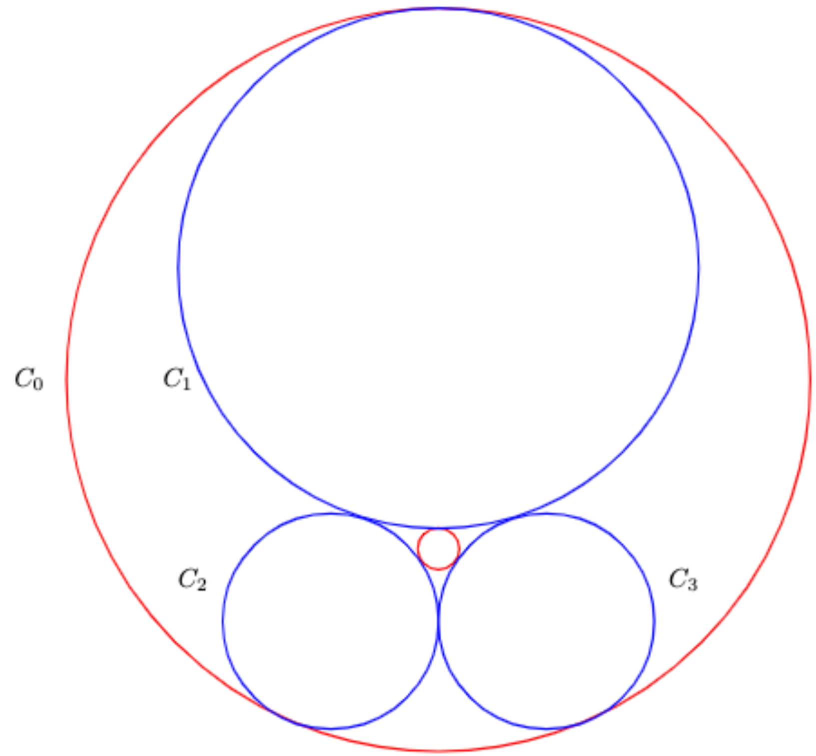
Thus, the space inside the circumscribing circle and outside the three circles can be written as $\pi r_0^2 - \pi r_1^2 - \pi r_2^2 - \pi r_2^2 = 120 \times 3600$ in 平方寸 units, or

$$r_0^2 - r_1^2 - r_2^2 - r_2^2 \quad = \quad \frac{120 \times 3600}{\pi}.$$

Moreover, the diameters of each of the two smaller circles is 5 units 寸 less than the diameter of the larger one. Thus,

$$2r_1 \quad = \quad 2r_2 + 5$$

in 寸 units.



Kissing Circles Theorem

Apollonius circle packing Problem, (Descartes-Soddy Circle Theorem) :

- Apollonius of Perga (240 BC –190 BC)
- Rene Descartes (1596–1650)
- Princess Elizabeth of Bohemia (1618–1680)
- Seki Takakazu 關孝和 (1642-1708)
- Jakob Steiner (1796—1863) Steiner's Chain Porism
- Philip Beecroft (1818–1862) duel circles, published in *Lady's and Gentleman's diary* in 1842
- Frederick Soddy (1877–1956) (Nobel laureate for Chemistry in 1921)

Let r_1, r_2 and r_3 be radius of blue circles C_1, C_2, and C_3, that they touched each other tangentially. There are two red circles (the big outer circle, and the small inner circle). The theorem says that, r_0 (radius of red circles) satisfy the following

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\pm\frac{1}{r_0}\right)^2 \quad = \quad 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_0^2}\right).$$

(if r_0 is the radius of the outer circle, the choice of the $\pm$ sign should be $-$)

For Students who know how to use CoCalc to numerically solve for this <u>Kissing Circles</u> problem stated in 發微算法 Hatsubi Sampo 平圓解空門一問 (Decrates-Soddy Circle Theorem)

Kernel: SageMath 9.5

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/kissingCircles.ipynb
https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/kissingCircles.pdf

In [1]:
```
# CoCalc code to numerically solve for this Kissing Circles problem
# stated in 發微算法 Hatsubi Sampo 平圓解空門一問
var('r0 r1 r2')
```
Out[1]: (r0, r1, r2)

In [2]:
```
eq1=(r0)^2-(r1)^2-(r2)^2==RR(120*3600/pi)
```

In [3]:
```
eq2=2*r1==2*r2+5
```

In [4]:
```
eq3=(1/r1+1/r2+1/r2-1/r0)^2==2*(1/(r1)^2+1/(r2)^2+1/(r2)^2+1/(r0)^2)
```

In [5]:
```
solns=solve([eq1,eq2,eq3],r0,r1,r2)
```

In [6]:
```
# how many sets of solutions
len(solns)
```
Out[6]: 6

In [7]:
```
# which set is real solutions for r0>=0, r1>=0 and r2>=0.
soln=tuple(i for i in range(len(solns)) if (solns[i][0].rhs()>=0)and(solns[i]
[1].rhs()>=0)and(solns[i][2].rhs()>=0))[0]
```

In [8]:
```
# the only real solution set for r0>=0, r1>=0 and r2>=0.
show(solns[soln])
```
Out[8]:

$$[r_0 = 492.1816443594646, r_1 = 230.0837696335078, r_2 = 227.5837563451777]$$

A page of Digression:

Just a few more words on Kissing Circles:

Coxeter, H. S. M. gave a version of the proof in The Presidential address delivered to The Canadian Mathematical Congress at

The joint meeting with the Mathematical Association of America in Toronto in August 28, 1967.

https://www.cambridge.org/core/services/aop-cambridge-core/content/view/A16EF894E64D18EB7DCD16799E668536/S0008439500056162a.pdf/problem_of_apollonius1.pdf

There are other beautiful web pages online about the proof, e.g.
https://euler.genepeer.com/from-herons-formula-to-descartes-circle-theorem

Descartes-Soddy can also be written in quadratic form:

Let $k_0 = \frac{1}{r_0}$, $k_1 = \frac{1}{r_1}$, $k_2 = \frac{1}{r_2}$, $k_3 = \frac{1}{r_3}$. These are known as curvatures.
Then, Descartes-Soddy

$$\left(\frac{1}{r_0}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2 - 2\left(\frac{1}{r_0^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right) \quad = \quad 0$$

can be expressed as

$$\begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{bmatrix} \quad = \quad 0.$$

$\lambda = 2$ (multiplicity 1)

$\lambda = -2$ (multiplicity 3)

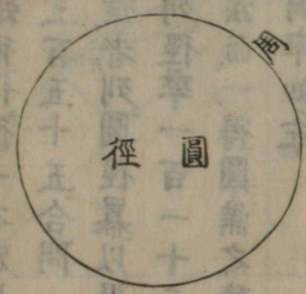
括要算法巻貞

関氏孝和先生遺編

荒木村英撿閲

大高由昌 校訂

圓周之圖



求圓周率術

第一

假如有圓滿徑一尺則

問圓周率若干

答曰 徑一百一十三

周三百五十五

依環矩術得徑一之定周而以零約術得徑一百一十三周三百五十五合問

求積者列圓徑冪以周率三百五十五相乘得數爲實列徑率一百一十三四之得四百五十二爲法實如法而一得圓滿之積而已

求周率如左

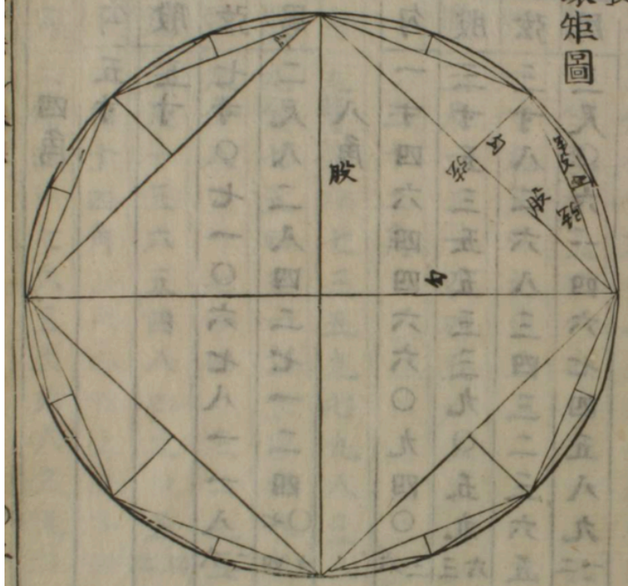
圓率解

第一

徑一尺圓內如圖容四角次容八角次容一十六角次容三十二角次第如此至一十三萬一千零七十二角各以勾股術求弦以角數相乘之各得截周 各所得勾股弦及周

數列于後

環矩圖



括要算法 Katsuyo Sampo

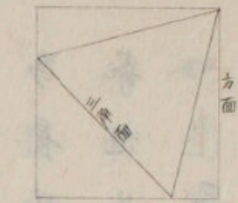
$$\pi \approx \frac{355}{113}$$ 密率

1 尺 = 10 寸

1 寸 = 10 分

1 平方尺 = 10000 平方分

一倍也



假如有方內容三角只云方面一尺問三
角面

答曰三角面一尺〇三分五厘二七弱毛六

術曰立天元一為三角面〇自之得數三之為
四段三角中徑冪〇〇寄九列方面自之得
數八之為四段方斜冪〇〇內減三角面冪与寄九
餘〇〇〇自之為因寄〇〇〇〇〇〇〇再寄列三
九數四段三角面冪角面自之以
寄九相乘得〇〇〇〇〇就分四之与再寄相消
得開〇〇〇〇〇〇〇三乘方開之得三角面也
方式

此術得式諸級悉有四因之過也
如是等著得答數則雖無差或損開出之功或有
諸數之繁冗也

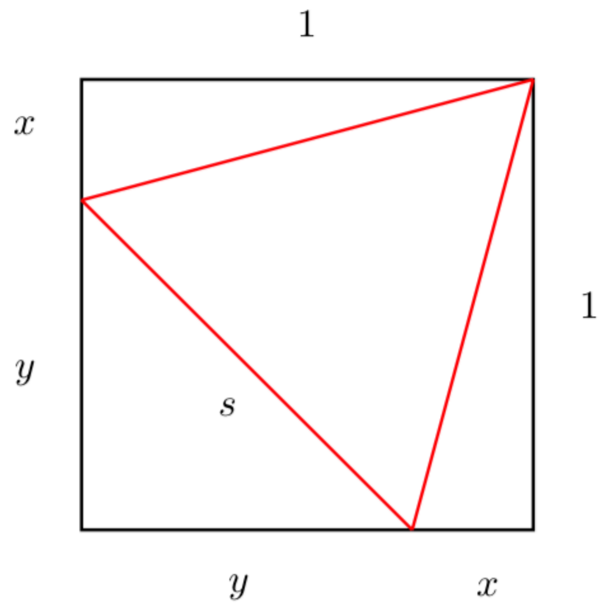
滞 問三

假如有絹八尺布一丈四尺四寸共價一十九錢
只云每一錢絹不及布八寸問絹布價

答曰絹價一十錢 布價九錢

術曰列共價一十九錢以絹八尺相乘得一百五十二錢為実列
布一丈四尺四寸加入不及八寸共得一丈五尺二寸為法実如法
而一得絹價以減共價餘即布價也
此術不察題意而求之故雖合于此問據佗數
則悉有差也

《大成算經》by 關孝和 Seki Takakazu , 建部賢明 Takebe Kataakira, and 建部賢弘 Takebe Katahiro
大成算經 Taisei Sankei, published in 1710 (after Seki Takakazu's death in 1708).

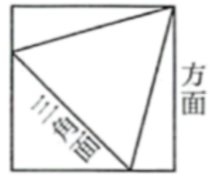


A problem listed in 大成算經 Taisei Sankei, Chapter 16 (page 35).

Suppose the side of the square is 1, and inscribed in it an equilateral triangle with one vertex on the corner of the square. Let x, and y be defined as in the diagram, and s be the side of the triangle.

Thus, we have 3 equations 3 unknowns:

$$\begin{aligned} x^2 + 1^2 &= s^2 \\ y^2 + y^2 &= s^2 \\ x + y &= 1. \end{aligned}$$



假如有方内容三角只云方面一尺問三角面

答曰三角面一尺○三分五釐七毫六二弱

Again, for those who know how to use CoCalc to numerically solve for this problem

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/TaiseiSankeiChapter16Page35.ipynb

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/TaiseiSankeiChapter16Page35.pdf

Kernel: SageMath 9.5

In [1]:
```
# CoCalc code to numerically solve for this problem listed in
# 大成算經 Taisei Sankei, Chapter 16, page 35
var('x y s')
```
Out[1]: (x, y, s)

In [2]:
```
eq1=x^2+1==s^2
```

In [3]:
```
eq2=2*y^2==s^2
```

In [4]:
```
eq3=x+y==1
```

In [5]:
```
solns=solve([eq1,eq2,eq3],x,y,s)
```

In [6]:
```
# how many sets of solutions
len(solns)
```
Out[6]: 4

In [7]:
```
# which set is real solutions for x>=0, y>=0 and s>=0.
soln=tuple(i for i in range(len(solns)) if (solns[i][0].rhs()>=0)and(solns[i]
[1].rhs()>=0)and(solns[i][2].rhs()>=0))[0]
```

In [8]:
```
# the only real solution set for x>=0, y>=0 and s>=0.
show(solns[soln])
```
Out[8]:

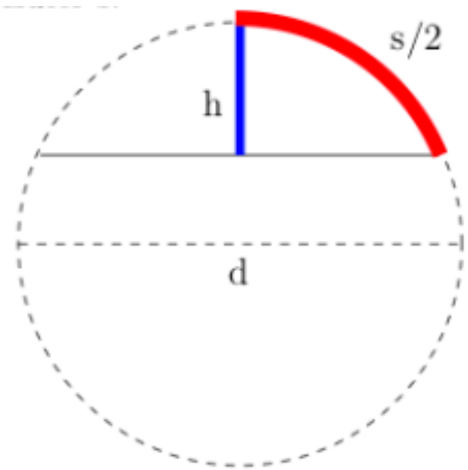
$$\left[x = -\sqrt{3} + 2, y = \sqrt{3} - 1, s = 2\sqrt{-\sqrt{3} + 2}\right]$$

In [9]:
```
# the numerical value of s is given by
show(RR(solns[soln][2].rhs()))
```
Out[9]:

$$1.03527618041008$$

In 1722, Takebe Katahiro 建部賢弘 (1664 - 1739 AD) (a student of Seki Takakazu 關孝和) has published his famous work Tetsujutsu Sankei 綴術算經.

[Takebe Katahiro was the mathematician commissioned by General Tokugawa Yoshimune 德川吉宗 to cartograph the map 《日本總繪圖》 from 1719 to 1723, one of the significant symbol of the Kyoho Reforms 享保改革].



In Chapter 12 of Tetsujutsu Sankei, the square of half the arc with sagitta h in a circle of diameter d was considered, that is

$$f(h) = \left(\frac{s}{2}\right)^2 = \left(d\ \sin^{-1}\left(\sqrt{\frac{h}{d}}\right)\right)^2$$

Takebe was calling this quantity the definite half back arc squared 定半背冪. A set of procedures was then presented in the book to give arbitrarily close numerical approximations to the value (square of arc-sine). Essentially, it was a power series. Rewriting the result using modern notation, we have

$$f(h) = dh\left\{1 + \sum_{n=1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!}\left(\frac{h}{d}\right)^n\right\}$$

Takebe's result was 15 years earlier than that of Euler's.

矢徑相乘為泛半背冪。矢自乘，三除之，為一差;

置一差，矢乘徑除，又八乘一十五除之，為二差;

置二差，矢乘徑除，又九乘一十四除之，為三差;

置三差，矢乘徑除，又三十二乘四十五除之，為四差;

置四差，矢乘徑除，又二十五乘三十三除之，為五差;

置五差，矢乘徑除，又七十二乘九十一除之， 為六差

七差以上准之。

以逐差各加泛半背冪，為定半背冪。

一差	二差	三差	四差	五差	六差	七差	八差	九差	十差
段奇	段偶	段奇	段偶	段奇	段偶	段奇	段偶	段奇	段偶
乘法一	乘法八	乘法九	乘法三十二	乘法二十五	乘法七十二	乘法四十九	乘法一百二十八	乘法八十一	乘法二百
矢數一自乘	矢數二自乘倍	矢數三自乘	矢數四自乘倍	矢數五自乘	矢數六自乘倍	矢數七自乘	矢數八自乘倍	矢數九自乘	矢數十自乘倍
除法三	除法一十五	除法一十四	除法四十五	除法三十三	除法九十一	除法六十	除法一百五十三	除法九十五	除法二百三十一
八矢數三 右矢數一 相乘	左矢數五 右矢數三 相乘	左矢數七 右矢數二 相乘	左矢數九 右矢數五 相乘	左矢數十一 右矢數三 相乘	左矢數十三 右矢數七 相乘	左矢數十五 右矢數四 相乘	左矢數十七 右矢數九 相乘	左矢數十九 右矢數五 相乘	左矢數廿一 右矢數十一 相乘

definite half back arc squared 定半背冪 = 泛半背冪 + 一差 + 二差 + 三差 + 四差 +......

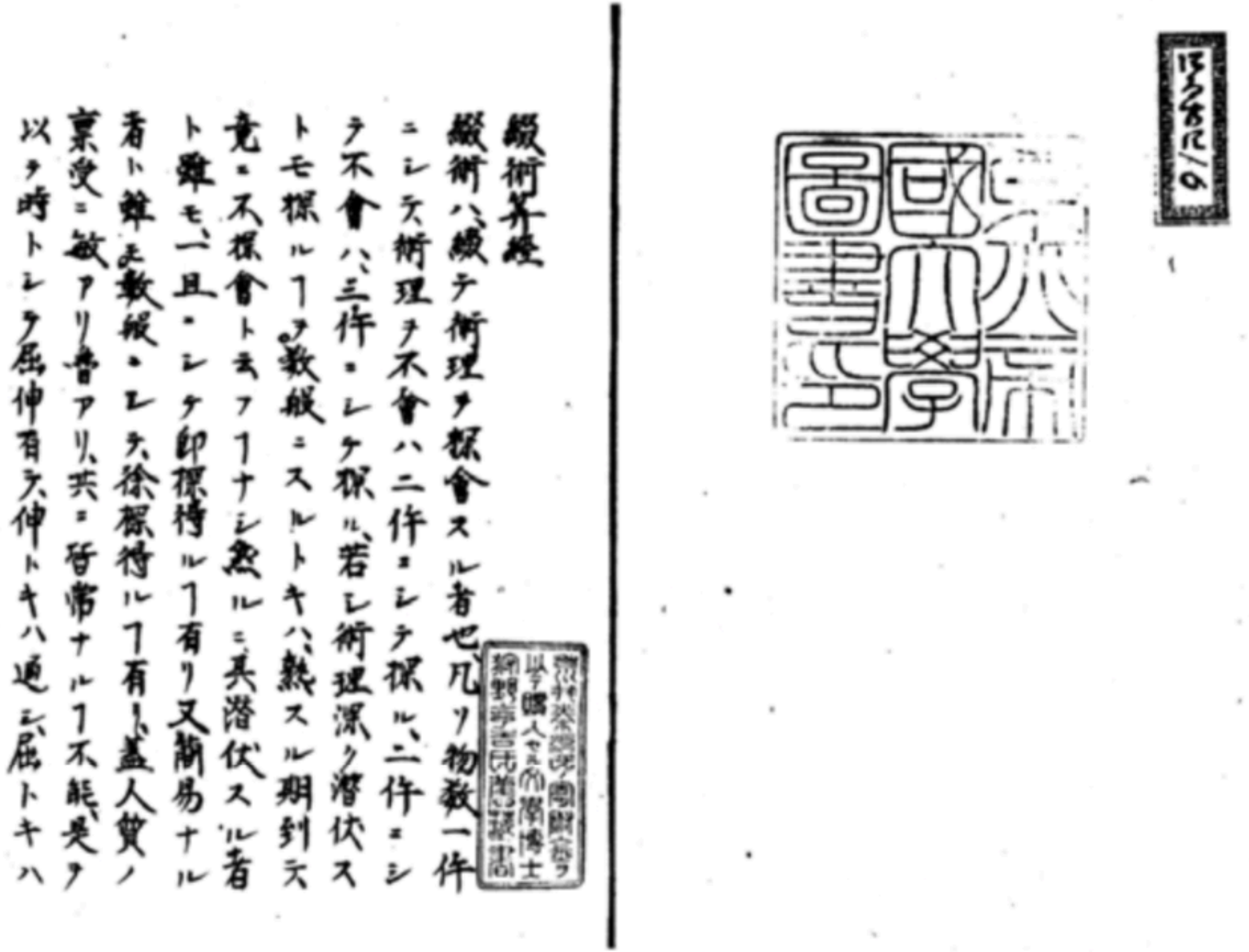
泛半背冪 = hd

一差 = $\dfrac{h^2}{3} = hd\dfrac{1}{3}\left(\dfrac{h}{d}\right)$

二差 = $\dfrac{h^2}{3}\,\dfrac{h}{d}\,\dfrac{8}{15} = hd\dfrac{8}{45}\left(\dfrac{h}{d}\right)^2$

三差 $= \dfrac{8}{45}\,\dfrac{h^3}{d}\,\dfrac{h}{d}\,\dfrac{9}{14} = hd\dfrac{4}{35}\left(\dfrac{h}{d}\right)^3$

四差 $= \dfrac{4}{35}\,\dfrac{h^4}{d^2}\,\dfrac{h}{d}\,\dfrac{32}{45} = hd\dfrac{128}{1575}\left(\dfrac{h}{d}\right)^4$



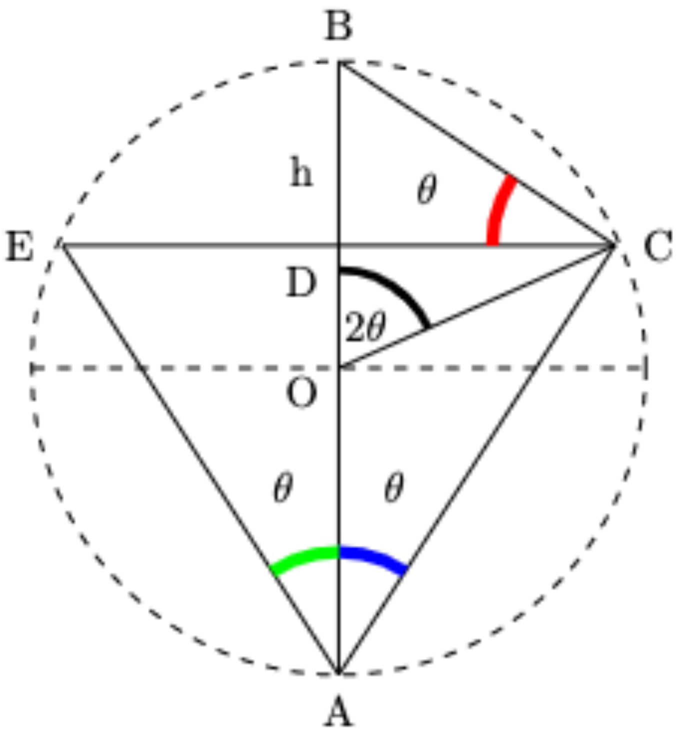
綴術算經

綴術ハ綴テ術理ヲ探會スル者也凡ソ物數一件ニシテ術理ヲ不會ハ二件ニシテ探ル二件ニシテ不會ハ三件ニシテ探ル若シ術理深ク潜伏ストモ探ルヿヲ數綴ニスルトキハ熟スル期到テ竟ニ不探會トテフヿナシ然ルニ其潜伏スル者ト雖モ一旦ニシテ卽探得ルヿ有リ又簡易ナル者ト雖モ數綴ニシテ徐探得ルヿ有ト盖人質ノ稟受ニ敏アリ昏アリ共ニ昏常ナルヿ不能矣ヲ以テ時トシテ屈伸有テ伸トキハ通シ屈トキハ

《綴術算經》

(version of the Tohoku University Library / 狩野本版本)

Note: To see why the definite half back arc squared 定半背冪 has the arcsine squared form:



Let $\angle BCE = \theta$.
By Proposition 20 of Euclid Elements (also known as Inscribed Angle Theorem), $\angle BCE = \angle BAE = \theta$.
By symmetry, $\angle BAC = \angle BAE = \theta$.

Also, by Proposition 20 of Euclid Elements, $\angle BOC = 2\angle BAC = 2\theta$.

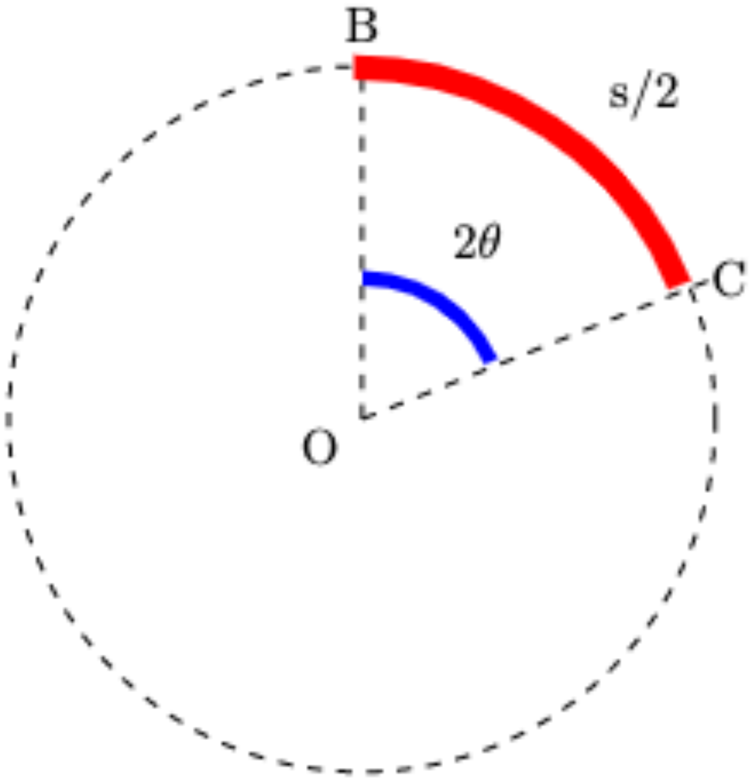
Since AB is the diameter passing through the center O, thus, by Thales's theorem $\angle ACB$ is a right angle.

Consider the right-angled triangle $\triangle ABC$ where C is the right angle, thus, we have $\sin(\theta) = \dfrac{BC}{AB} = \dfrac{BC}{d}$. Or, $BC = d\sin(\theta)$.

Consider the right-angled triangle $\triangle BCD$ where D is the right angle, thus, we have $\sin(\theta) = \dfrac{BD}{BC} = \dfrac{h}{BC}$. Or, $BC = \dfrac{h}{\sin(\theta)}$.

Combining the two results, we have $BC = d\sin(\theta) = \dfrac{h}{\sin(\theta)}$. Therefore, $\sin^2(\theta) = \dfrac{h}{d}$.

We can rewrite it as, $\theta = \sin^{-1}\left(\sqrt{\dfrac{h}{d}}\right)$.



The ratio between the length of the red arc $\frac{s}{2}$ to the circumference is $\frac{s/2}{d\pi}$, and the ratio between the angle of the sector to the whole revolution is $\frac{2\theta}{2\pi}$. The two ratios are the same, therefore, $\frac{s}{2} = d\theta$.

Thus, the definite half-back arc squared is given by $\left(\frac{s}{2}\right)^2 = \left(d\sin^{-1}\left(\sqrt{\frac{h}{d}}\right)\right)^2$.

By rearranging, we have

$$\frac{\text{定半背冪}}{\text{泛半背冪}} = \frac{1}{hd}\left(\frac{s}{2}\right)^2 = \frac{d^2}{hd}\left(\sin^{-1}\left(\sqrt{\frac{h}{d}}\right)\right)^2 = \frac{\left(\sin^{-1}\left(\sqrt{\frac{h}{d}}\right)\right)^2}{\left(\frac{h}{d}\right)}$$

Let $x = \frac{h}{d}$, we thus define $g(x)$ to be

$$g(x) = \frac{\left(\sin^{-1}(\sqrt{x})\right)^2}{x}.$$

Again, you might like to check with CoCalc on the Maclaurin series

In [1]:
```
g(x)=arcsin(sqrt(x))^2/x
show(g)
```
Out[1]:

$$x \mapsto \frac{\arcsin\left(\sqrt{x}\right)^2}{x}$$

In [2]:
```
show(limit(diff(g,x,1),x=0)/factorial(1))
```
Out[2]:

$$x \mapsto \frac{1}{3}$$

In [3]:
```
show(limit(diff(g,x,2),x=0)/factorial(2))
```
Out[3]:

$$x \mapsto \frac{8}{45}$$

In [4]:
```
show(limit(diff(g,x,3),x=0)/factorial(3))
```
Out[4]:

$$x \mapsto \frac{4}{35}$$

In [5]:
```
show(limit(diff(g,x,4),x=0)/factorial(4))
```
Out[5]:

$$x \mapsto \frac{128}{1575}$$

In [6]:
```
show(limit(diff(g,x,5),x=0)/factorial(5))
```
Out[6]:

$$x \mapsto \frac{128}{2079}$$

In [7]:
```
#alternatively, we can find the maclaurin / taylor series in one go
show(taylor(g(x),x,0,8))
```
Out[7]:

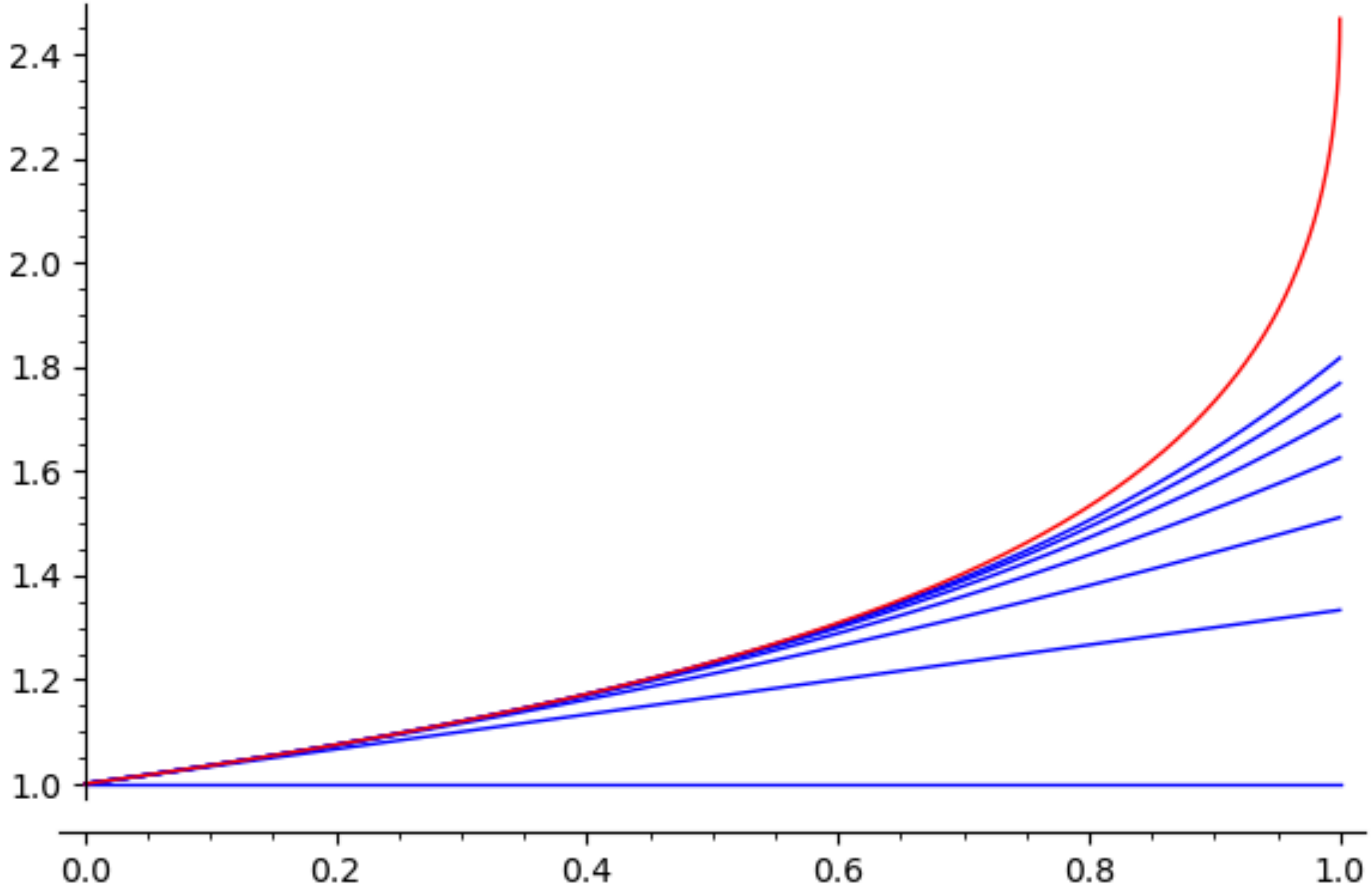
$$\frac{32768}{984555}x^8 + \frac{256}{6435}x^7 + \frac{1024}{21021}x^6 + \frac{128}{2079}x^5 + \frac{128}{1575}x^4 + \frac{4}{35}x^3 + \frac{8}{45}x^2 + \frac{1}{3}x + 1$$

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/TakebeCoCalc.ipynb

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/TakebeCoCalc.pdf

In [8]:
```
p0=plot(limit(g(x),x=0),x,0,1)
p1=plot(taylor(g(x),x,0,1),x,0,1)
p2=plot(taylor(g(x),x,0,2),x,0,1)
p3=plot(taylor(g(x),x,0,3),x,0,1)
p4=plot(taylor(g(x),x,0,4),x,0,1)
p5=plot(taylor(g(x),x,0,5),x,0,1)
p6=plot(taylor(g(x),x,0,6),x,0,1)
pg=plot(g(x),x,0,1,rgbcolor='red')
p0+p1+p2+p3+p4+p5+p6+pg
```

Out[8]:



Kurushima Yoshihiro 久留島義太 / 久留島喜内 (? — 1757)

《久氏遺書》中的〈算學粹沙〉

$$\begin{vmatrix} 東 & 春 & 青 & 木 \\ 南 & 夏 & 赤 & 火 \\ 西 & 秋 & 白 & 金 \\ 北 & 冬 & 黑 & 水 \end{vmatrix}$$

4 by 4 determinant

Laplace expansion
cofactor expansion

expansion along the first two rows

$\binom{4}{2} = 6$ terms

東夏內去南春㊀　東赤內去南青㊁　東火內去南木㊂　春赤內去夏青㊃
夏木內去南火㊄　青火內去赤木㊅　西冬內去北秋㊅　西黑內去北白㊄
西水內去北金㊃　秋黑內去冬白㊂　冬金內去秋水㊁　白水內去黑金㊀

一一相乘，二二相乘，三三相乘，四四相乘，五五相乘，六六相乘，相併分正負起本術。

$$\begin{vmatrix} 東 & 春 & 青 & 木 \\ 南 & 夏 & 赤 & 火 \\ 西 & 秋 & 白 & 金 \\ 北 & 冬 & 黑 & 水 \end{vmatrix} = \begin{vmatrix} 東 & 春 \\ 南 & 夏 \end{vmatrix} \cdot \begin{vmatrix} 白 & 金 \\ 黑 & 水 \end{vmatrix} - \begin{vmatrix} 東 & 青 \\ 南 & 赤 \end{vmatrix} \cdot \begin{vmatrix} 秋 & 金 \\ 冬 & 水 \end{vmatrix} + \begin{vmatrix} 東 & 木 \\ 南 & 火 \end{vmatrix} \cdot \begin{vmatrix} 秋 & 白 \\ 冬 & 黑 \end{vmatrix}$$

$$+ \begin{vmatrix} 春 & 青 \\ 夏 & 赤 \end{vmatrix} \cdot \begin{vmatrix} 西 & 金 \\ 北 & 水 \end{vmatrix} - \begin{vmatrix} 春 & 木 \\ 夏 & 火 \end{vmatrix} \cdot \begin{vmatrix} 西 & 白 \\ 北 & 黑 \end{vmatrix} + \begin{vmatrix} 青 & 木 \\ 赤 & 火 \end{vmatrix} \cdot \begin{vmatrix} 西 & 秋 \\ 北 & 冬 \end{vmatrix}$$

Again, for those who might like to check with CoCalc

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/KurushimaYoshihiro.ipynb

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/KurushimaYoshihiro.pdf

Kernel: SageMath 9.5

In [1]:
```
var('a11 a12 a13 a14 a21 a22 a23 a24 a31 a32 a33 a34 a41 a42 a43
a44')
A=matrix([[a11,a12,a13,a14],[a21,a22,a23,a24],[a31,a32,a33,a34],
[a41,a42,a43,a44]])
show(A)
```
Out[1]:
$$\left(\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right)$$

In [2]:
```
Asub1122=matrix([[a11,a12],[a21,a22]])
show(Asub1122)
```
Out[2]:
$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)$$

In [3]:
```
Asub3344=matrix([[a33,a34],[a43,a44]])
show(Asub3344)
```
Out[3]:
$$\left(\begin{array}{cc} a_{33} & a_{34} \\ a_{43} & a_{44} \end{array}\right)$$

In [4]:
```
Asub1123=matrix([[a11,a13],[a21,a23]])
show(Asub1123)
```
Out[4]:
$$\left(\begin{array}{cc} a_{11} & a_{13} \\ a_{21} & a_{23} \end{array}\right)$$

In [5]:
```
Asub3244=matrix([[a32,a34],[a42,a44]])
show(Asub3244)
```
Out[5]:
$$\left(\begin{array}{cc} a_{32} & a_{34} \\ a_{42} & a_{44} \end{array}\right)$$

In [6]:
```
Asub1124=matrix([[a11,a14],[a21,a24]])
show(Asub1124)
```
Out[6]:
$$\left(\begin{array}{cc} a_{11} & a_{14} \\ a_{21} & a_{24} \end{array}\right)$$

In [7]:
```
Asub3243=matrix([[a32,a33],[a42,a43]])
show(Asub3243)
```
Out[7]:
$$\left(\begin{array}{cc} a_{32} & a_{33} \\ a_{42} & a_{43} \end{array}\right)$$

In [8]:
```
Asub1223=matrix([[a12,a13],[a22,a23]])
show(Asub1223)
```
Out[8]:
$$\left(\begin{array}{cc} a_{12} & a_{13} \\ a_{22} & a_{23} \end{array}\right)$$

In [9]:
```
Asub3144=matrix([[a31,a34],[a41,a44]])
show(Asub3144)
```
Out[9]:
$$\left(\begin{array}{cc} a_{31} & a_{34} \\ a_{41} & a_{44} \end{array}\right)$$

In [10]:
```
Asub1224=matrix([[a12,a14],[a22,a24]])
show(Asub1224)
```
Out[10]:
$$\left(\begin{array}{cc} a_{12} & a_{14} \\ a_{22} & a_{24} \end{array}\right)$$

In [11]:
```
Asub3143=matrix([[a31,a33],[a41,a43]])
show(Asub3143)
```
Out[11]:
$$\left(\begin{array}{cc} a_{31} & a_{33} \\ a_{41} & a_{43} \end{array}\right)$$

In [12]:
```
Asub1324=matrix([[a13,a14],[a23,a24]])
show(Asub1324)
```
Out[12]:
$$\left(\begin{array}{cc} a_{13} & a_{14} \\ a_{23} & a_{24} \end{array}\right)$$

In [13]:
```
Asub3142=matrix([[a31,a32],[a41,a42]])
show(Asub3142)
```
Out[13]:
$$\left(\begin{array}{cc} a_{31} & a_{32} \\ a_{41} & a_{42} \end{array}\right)$$

In [14]:
```
allKY=expand(det(Asub1122)*det(Asub3344)-det(Asub1123)*det(Asub3244)+det(Asub1124)*det(Asub3243)+det(Asub1223)*det(Asub3144)-det(Asub1224)*det(Asub3143)+det(Asub1324)*det(Asub3142))
```
In [15]:
```
bool(allKY==det(A))
```
Out[15]: True

和算
藝道化

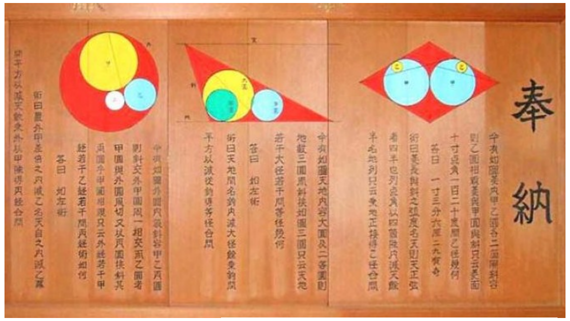
江戸時代的和算流派

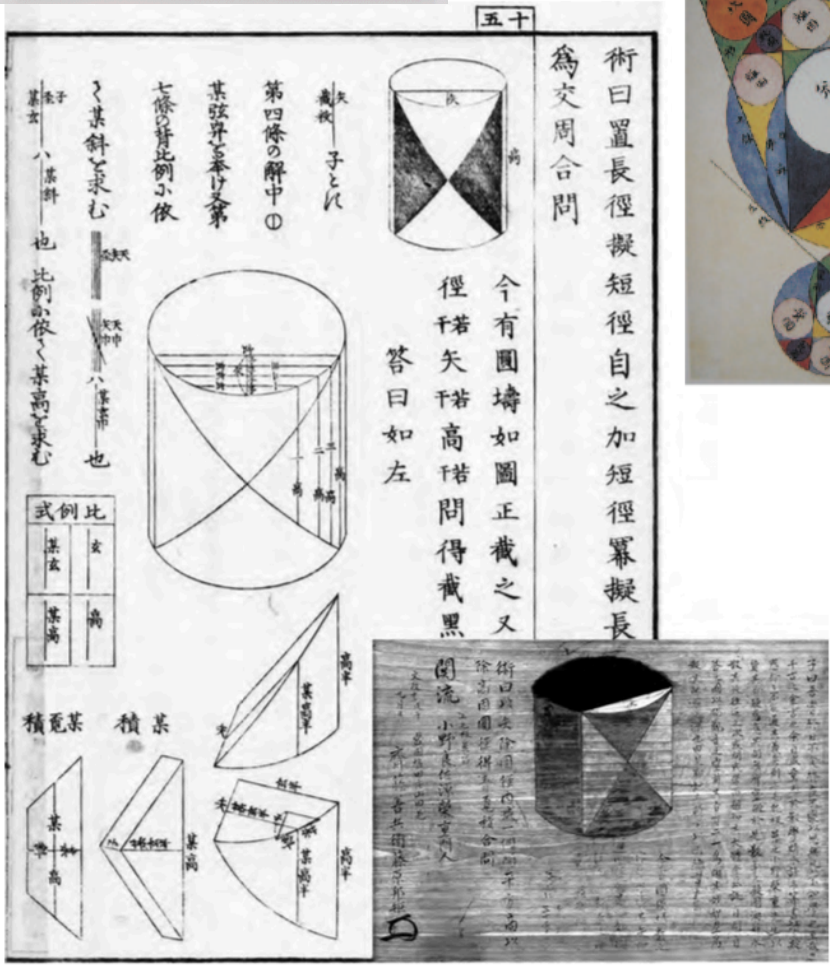
- 關流
- 最上流 (さいじょうりゅう **Saijōryū** / 會田安明)
- 宅間流 (たくまりゅう / 宅間能清)
- 中西流
- 三池流
- 宮城流
- 大島流
- 麻田流
- 真元流
- 至誠贊化流

Sangaku

算額　さんがく

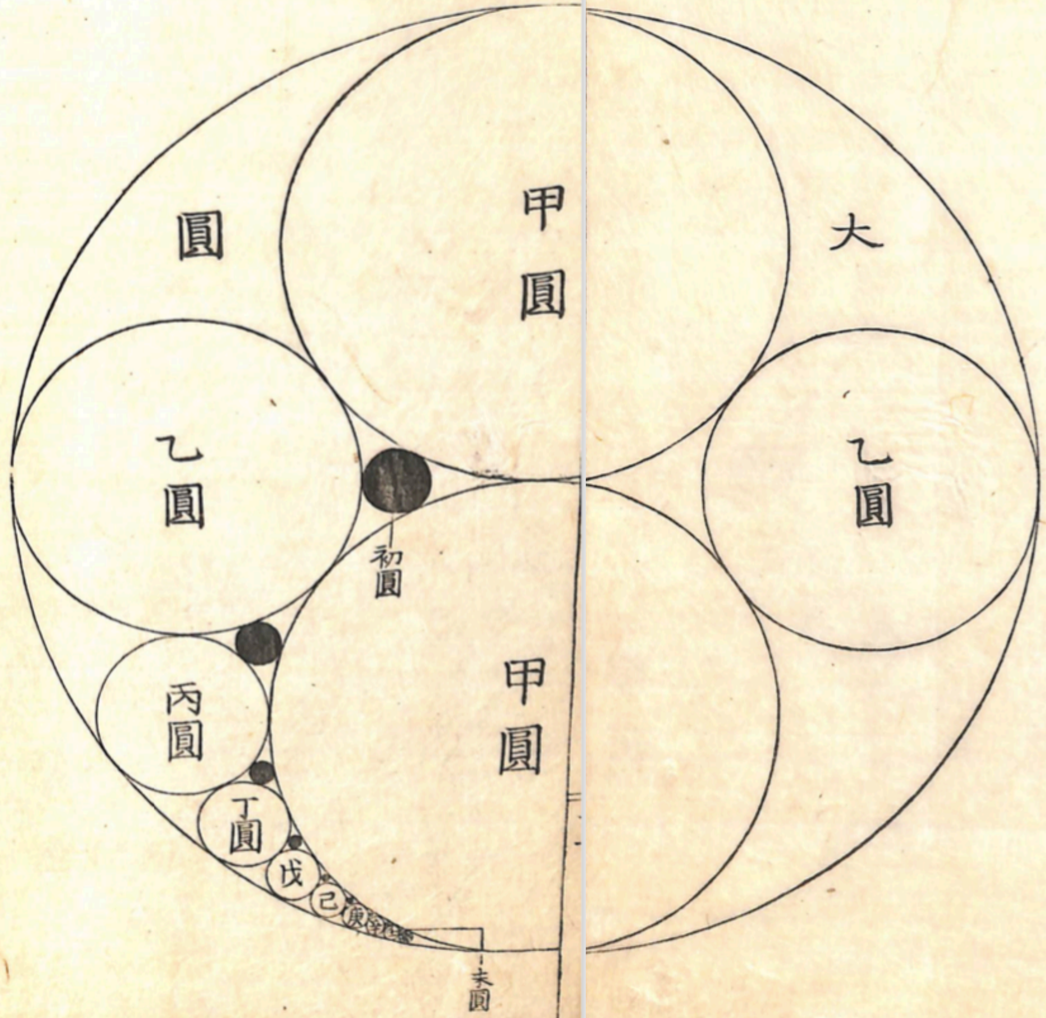
Japanese geometrical problems or theorems on wooden tablets which were placed as offerings at Shinto shrines or Buddhist temples during the Edo perio





京都清水寺　1847　福田理軒　奉納

所懸于東都本所柳島妙見堂者一事



今有如圖大圓中容累圓與挾圓以下至于九圓者假爲末圓而圖之大圓徑九十七寸五分末圓徑一分不知其挾圓之總計也問從初圓至末圓之總計幾何

答曰一十六箇

術曰置大圓徑以末圓徑除之內減二十四箇餘平方開之加一箇折半之得圓數合問

關流藤田貞資門人

龜井隱岐守家士

堀田仁助泉尹

天明八年戊申三月

problem originally proposed by
堀田仁助 Hotta Nisuke (1745-1829) in 1788

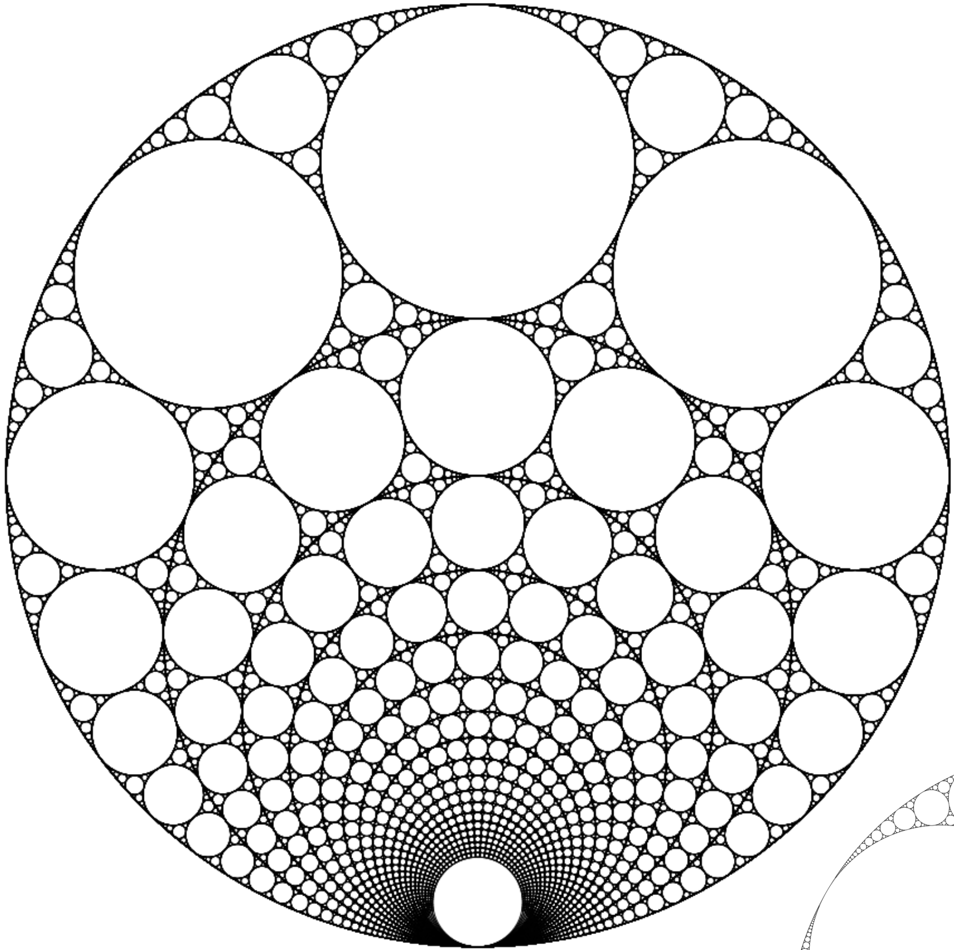
Fujita Kagen 藤田嘉言

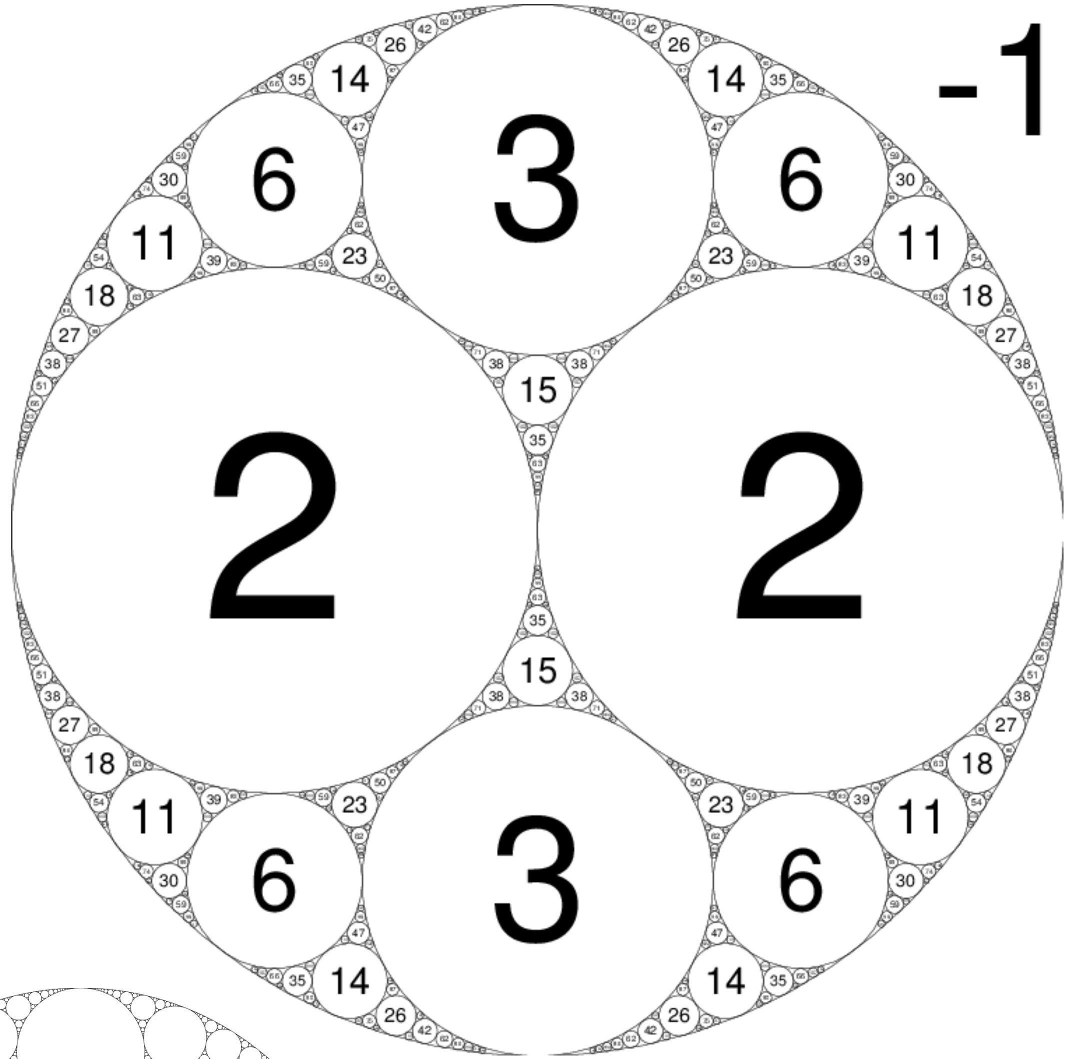
神壁算法, Shinpeki sanpō, 1796

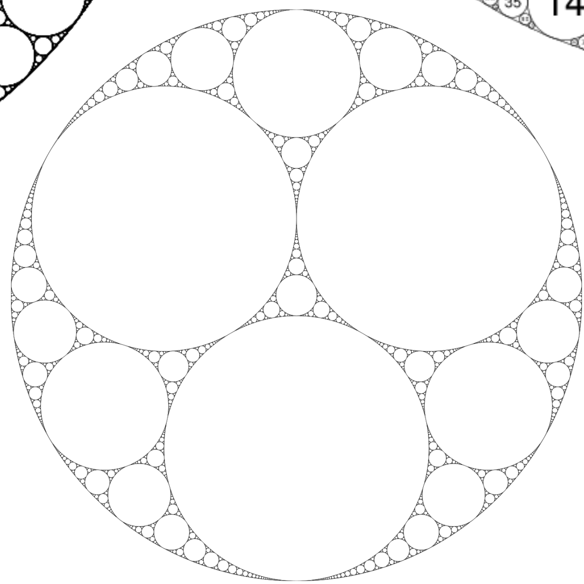
Mathematical Problems Suspended Before the temple

(Decrates-Soddy Circle Theorem)

Apollonian gasket **Leibniz packing**







Again, for those who might like to use CoCalc to find Apollonius Circles

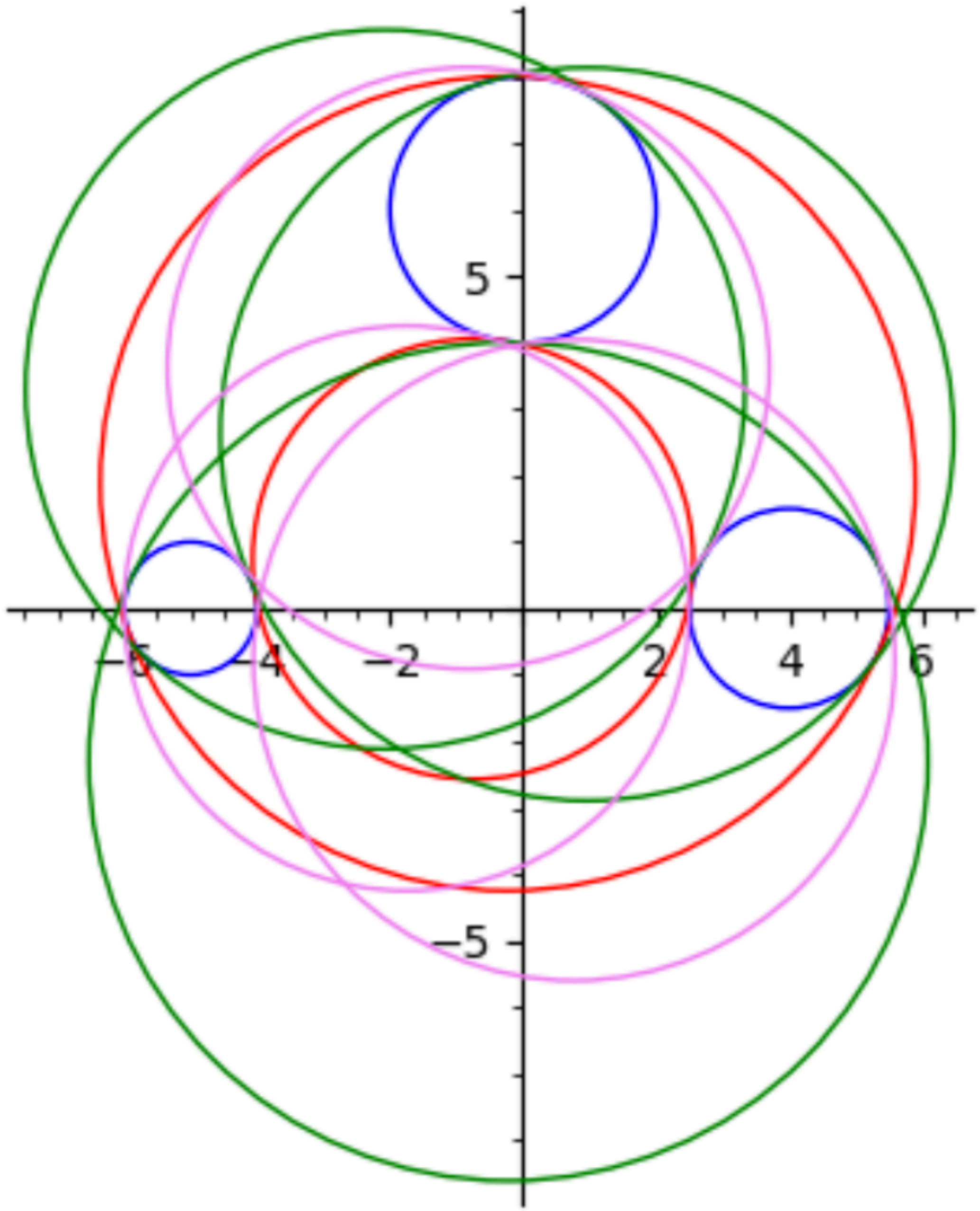
Kernel: SageMath 9.5

In [1]:

```
from collections import namedtuple
import math
Circle = namedtuple('Circle', 'x, y, r')
def solveApollonius(c1, c2, c3, s1, s2, s3):
    x1, y1, r1 = c1
    x2, y2, r2 = c2
    x3, y3, r3 = c3
    v11 = 2*x2 - 2*x1
    v12 = 2*y2 - 2*y1
    v13 = x1*x1 - x2*x2 + y1*y1 - y2*y2 - r1*r1 + r2*r2
    v14 = 2*s2*r2 - 2*s1*r1
    v21 = 2*x3 - 2*x2
    v22 = 2*y3 - 2*y2
    v23 = x2*x2 - x3*x3 + y2*y2 - y3*y3 - r2*r2 + r3*r3
    v24 = 2*s3*r3 - 2*s2*r2
    w12 = v12/v11
    w13 = v13/v11
    w14 = v14/v11
    w22 = v22/v21-w12
    w23 = v23/v21-w13
    w24 = v24/v21-w14
    P = -w23/w22
    Q = w24/w22
    M = -w12*P-w13
    N = w14 - w12*Q
    a = N*N + Q*Q - 1
    b = 2*M*N - 2*N*x1 + 2*P*Q - 2*Q*y1 + 2*s1*r1
    c = x1*x1 + M*M - 2*M*x1 + P*P + y1*y1 - 2*P*y1 - r1*r1
    # Find a root of a quadratic equation. This requires the circle
centers not to be e.g. colinear
    D = b*b-4*a*c
    rs = (-b-math.sqrt(D))/(2*a)
    xs = M+N*rs
    ys = P+Q*rs
    return Circle(xs, ys, rs)
```

https://rosettacode.org/wiki/Problem_of_Apollonius#Python https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/ApolloniusCircles.ipynb

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/ApolloniusCircles.pdf

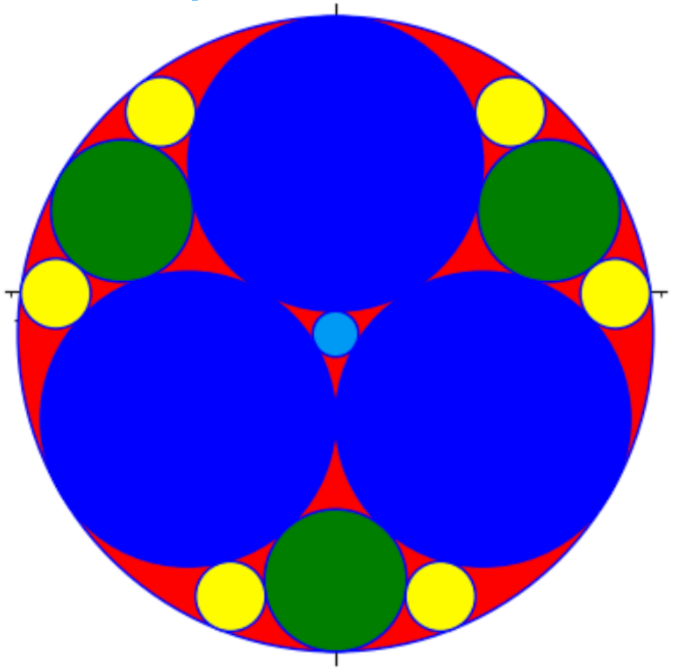


3 initial circles (blue)

in this case, generate 8 Apollonius Circles

Or, you might like to use CoCalc to try to get a plot of Apollonius Gasket

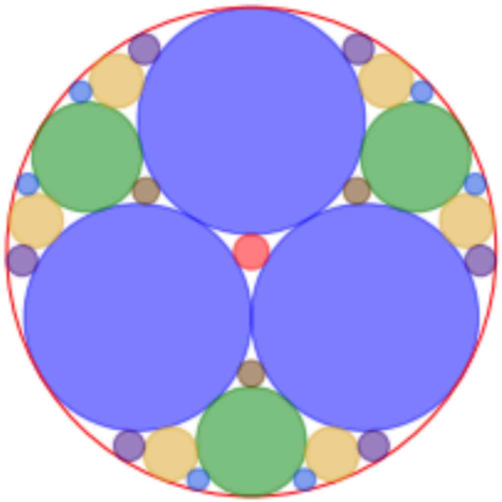
https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/ApolloniusGasket.ipynb
https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/ApolloniusGasket.pdf



I just stopped at 14 circles. You might like to carry on.

Or, you might like to use Google Colab / python (jupyter notebook)

https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/apolloniusColab.ipynb
https://www.polyu.edu.hk/ama/profile/hwlee/AMA1D01C/apolloniusColab.pdf

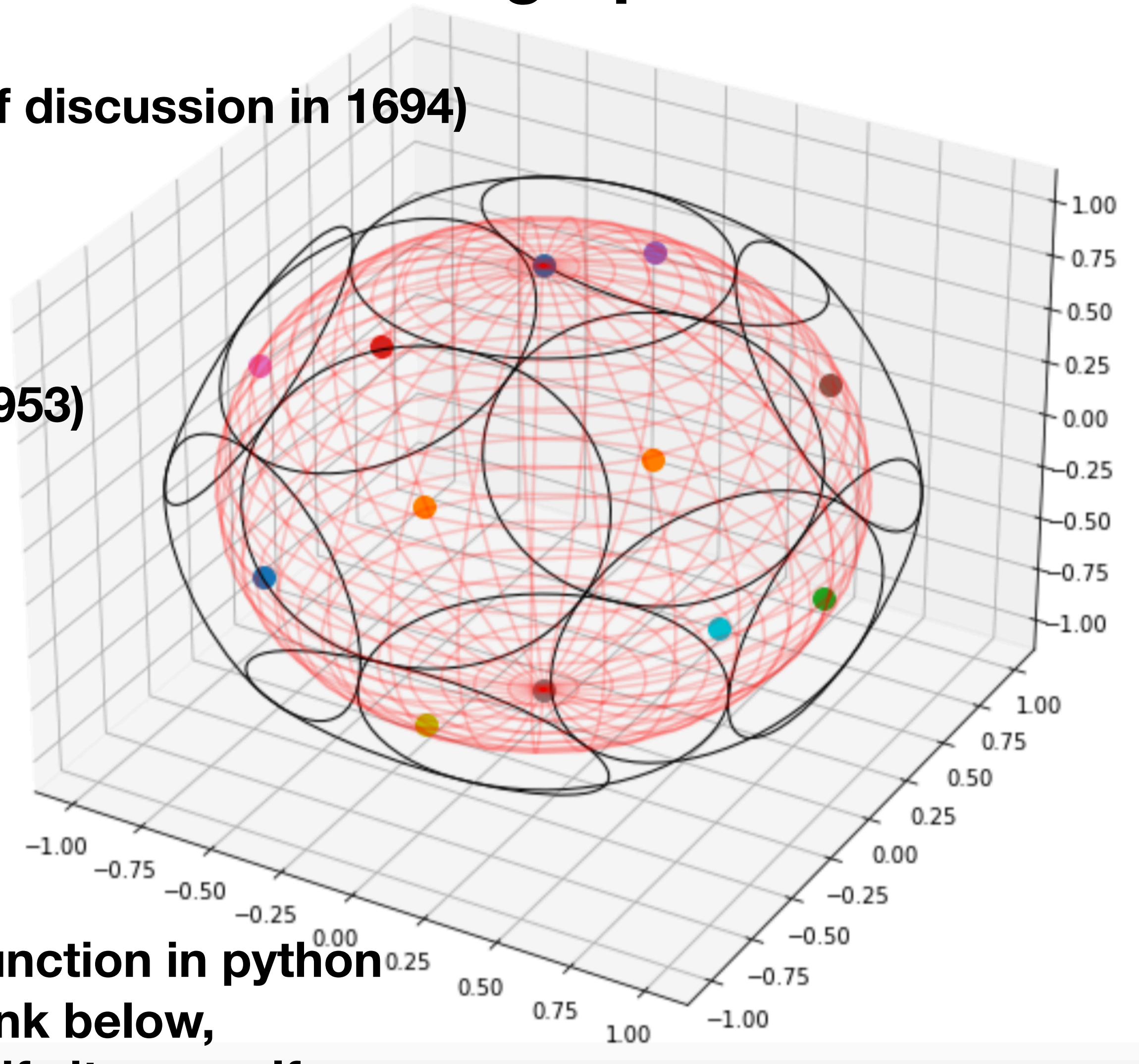


I just stopped at 29 circles. You might like to carry on.

Another page of Digression:

Just a few more words on Kissing Spheres: The Impossibility of the 13th Kissing Sphere

- Newton and Gregory (a record of discussion in 1694)
- Conway and Sloane (1993)
- Bender (1874)
- Hoppe (1874)
- Günther (1875)
- Schütte and van der Waerden (1953)
- Leech (1956).



I use GEKKO and the 3D plot function in python
The code can be found in the link below,
you are welcome to try it / modify it yourself

https://www.polyu.edu.hk/ama/profile/hwlee/GEKKO-MISER/TwelvePointsOnSphere.pdf