

Volume 56



CRM PROCEEDINGS & LECTURE NOTES

Centre de Recherches Mathématiques
Montréal

Analysis and Geometry of Metric Measure Spaces

Lecture Notes of the 50th
Séminaire de Mathématiques
Supérieures (SMS),
Montréal, 2011

Galia Dafni
Robert John McCann
Alina Stancu
Editors



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Foreword

The 50th edition of the Séminaire de Mathématiques Supérieures took place from June 27 to July 8, 2011 on the campus of the Université de Montréal. This university has supported the summer school, which has acquired international renown, for over 50 years, often in collaboration with NATO. The 2011 SMS, on the topic of “Metric measure spaces: geometric and analytic aspects” was the start of a new funding structure in which the school was supported by the Canadian mathematical community, as well as at the North-American level. We are pleased to acknowledge the institutional partners and sponsors who supported this endeavor: the Centre de recherche mathématiques de Montréal (CRM—principal partner), the Fields Institute of Mathematical Sciences, the Pacific Institute for Mathematical Sciences (PIMS), the Mathematical Sciences Research Institute (MSRI), the Institut de sciences mathématiques de Montréal (ISM), the Département de mathématiques et de statistique de l’Université de Montréal (UdeM), Concordia University, and the Canadian Mathematical Society (CMS).

In recent decades, metric-measure spaces have emerged as a fruitful source of mathematical questions in their own right, and as indispensable tools for addressing classical problems in geometry, topology, dynamical systems, and partial differential equations. The summer school was designed to lead young scientists to the research frontier concerning the analysis and geometry of metric-measure spaces, by exposing them to a series of minicourses featuring leading researchers who highlighted both the state-of-the-art and some of the exciting challenges which remain. Attended by over 60 people, mostly graduate students, coming from 13 countries in Africa, Asia, Europe and North America, this 50th edition of the Séminaire de Mathématiques Supérieures was a resounding success.

There were thirteen four-hour minicourses, revealing the breadth and depth of recent work in this area. Versions of eight of these appear in this volume. The notes come under three general themes. The first and perhaps most prevalent was the use of techniques from optimal transportation to define geometric notions on metric measure spaces. In particular the notion of lower curvature bounds, as well as an actual generalization of such a notion to arbitrary metric measure spaces are presented. A shift toward more analytic aspects follows by focussing on the notion of weak gradient and Sobolev spaces of functions in metric measure spaces. As a natural continuation, the final theme consists of functional and geometric inequalities ranging from concentration of measure, isoperimetry and minimizers, to heat kernel estimates on non-compact manifolds, and respectively Sobolev inequalities in the Sierpinsky carpet.

Thus the reader will find in Yann Ollivier's notes a definition of Ricci curvature on discrete metric spaces, based on the idea of comparing the (average transportation) distance between balls to the distance between their centers. Far-reaching consequences of this idea are explored in various geometries. The alternative definition of Lott–Villani and Sturm in path-metric spaces, based on McCann's notion of displacement convexity, and its consequences are also recalled. This is complemented by Robert McCann and Nestor Guillen's introductory notes on optimal transportation, which offer insight into the techniques and applications starting with the classical transportation problem of Monge and Kantorovich and leading to the recent theorems concerning smoothness of optimal maps by Ma, Trudinger, Wang and Loeper. Along the way they are excursions into two-player zero sum games, convex duality and linear programming, fully nonlinear partial differential equations, the economics of optimal pricing, and connections with minimal Lagrangian submanifolds in semi-Riemannian geometry. Minimal submanifolds are the subject of the course of Guy David. Starting with the famous Plateau problem in higher dimensions (not yet solved) as an illustration, David's notes discuss the structure of minimizers to functionals on currents with a given lower-dimensional rectifiable current as boundary. The focus is on lack of smoothness, and minimizers in the sense of Almgren, concluding with the proof of Jean Taylor's theorem which characterizes them. Regularity questions for optimal maps are taken up once again in the lecture notes of Young-Heon Kim. Focusing his attention on the Riemannian setting, Kim outlines the state-of-the art through a series of examples and counterexamples, building up to his recent joint work on Hölder continuity of optimal maps between not-necessarily-smooth distributions of mass.

The second theme of the summer school, roughly coming under the “analytic aspects” of metric spaces, concerned notions of differentiability on metric measure spaces and the corresponding function spaces. The two themes come together in the lecture notes by Luigi Ambrosio, which review and compare various notions of weak gradients and Sobolev spaces in metric measure spaces, such as upper gradients, absolute continuity on lines, and Cheeger's energy, and showed the identification of weak gradients using optimal transportation techniques, without relying on doubling or Poincaré assumptions. Results of Ambrosio with Gigli and Savaré also involve Otto's gradient flow with respect to the Wasserstein distance and the curvature-dimension conditions of Lott–Sturm–Villani.

The final theme could be described as functional and geometric inequalities. The mini-course of Ollivier concludes with results on concentration of measure and joint work with Villani on the Brunn–Minkowski inequality for the discrete hypercube. The notes of Emanuel Milman describe the relations between isoperimetric inequalities, concentration of measure, and functional inequalities such as Poincaré and Sobolev. Based on these, he gives a new proof of a Bobkov's lower bound for the Cheeger constant of a convex domain. For such domains (and more generally, Ricci non-negative metric-measure spaces) the Cheeger constant is proportional to square root of the spectral gap. The inequality in question bounds its reciprocal from above by the product of average distance, and the deviation of this distance, separating an arbitrary point in Euclidean space from the domain.

The relations between various inequalities are also the subject of the course by Thierry Coulhon. Working in the Riemannian context, but this time on a non-compact manifold with volume doubling, Coulhon discusses the implications of

various heat kernel estimates to L^p boundedness of Riesz transforms, an important question which relates back to the second theme since it concerns the compatibility of various definitions of weak derivatives and the corresponding Sobolev spaces. An example was given of a fractal-like manifold where local and global bounds differ. Heat kernel estimates, and in particular this type of different local and global behavior, also played a prominent role in the lecture notes by Martin Barlow, where Gaussian bounds were shown to be equivalent to a parabolic Harnack inequality on the one hand, and doubling and Poincaré on the other. These inequalities (or their failure) were discussed in detail for the Sierpinski carpet.

We hope that this volume succeeds in capturing some of the excitement of the lectures themselves, presenting the reader with a glimpse into this active area of research and its connections with other branches of contemporary mathematics. Just as two of the speakers, Robert McCann and Young-Heon Kim, recounted positive memories of past SMS summer schools they attended as students, it is hoped that the 2011 SMS will remain memorable both for its participants and the readers of this volume, some of whom may even emerge as lecturers and authors in a future edition of the SMS.

Galia Dafni
Robert J. McCann
Alina Stancu

List of Speakers

Luigi Ambrosio (Scuola Normale Superiore di Pisa)

Calculus in metric measure spaces with Ricci curvature bounded from below

Martin T. Barlow (University of British Columbia)

Heat equation on some fractal metric spaces

Thierry Coulhon (Université de Cergy-Pontoise)

Heat kernel estimates, Sobolev type inequalities and Riesz transform on non-compact Riemannian manifolds

Guy David (Université Paris-Sud)

Regularity results for minimal sets

Piotr Hajłasz (University of Pittsburgh)

Sobolev mappings into metric spaces

Vitali Kapovitch (University of Toronto)

Fundamental groups of manifolds with lower Ricci curvature bounds

Young-Heon Kim (University of British Columbia)

Ma-Trudinger-Wang curvature and regularity of optimal transport

Pekka Koskela (University of Jyväskylä)

Quasiconformal mappings and function spaces

Robert J. McCann (University of Toronto)

Optimal transportation

Emanuel Milman (Technion—Israel Institute of Technology)

Isoperimetric, functional and concentration inequalities

Yann Ollivier (CNRS, Université Paris-Sud)

Discrete Ricci curvature with applications

Felix Otto (Max-Planck-Institut für Mathematik in den Naturwissenschaften)

Burger's equation as a gradient flow on two-phase Wasserstein space

Karl-Theodor Sturm (Universität Bonn)

Ricci bounds for metric measure spaces and geometric analysis



Photo: Suzette-Paradis

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This book contains lecture notes from most of the courses presented at the 50th anniversary edition of the Séminaire de Mathématiques Supérieures in Montréal. This 2011 summer school was devoted to the analysis and geometry of metric measure spaces, and featured much interplay between this subject and the emergent topic of optimal transportation. In recent decades, metric measure spaces have emerged as a fruitful source of mathematical questions in their own right, and as indispensable tools for addressing classical problems in geometry, topology, dynamical systems, and partial differential equations. The summer school was designed to lead young scientists to the research frontier concerning the analysis and geometry of metric measure spaces, by exposing them to a series of minicourses featuring leading researchers who highlighted both the state-of-the-art and some of the exciting challenges which remain.

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