Learning and Individual Differences 61 (2018) 99–108

Contents lists available at ScienceDirect

Learning and Individual Differences

journal homepage: www.elsevier.com/locate/lindif

Towards a mathematically more correct understanding of rational numbers: A longitudinal study with upper elementary school learners

Jo Van Hoof[a,*], Tine Degrande[a], Eva Ceulemans[b], Lieven Verschaffel[a], Wim Van Dooren[a]

[a] Centre for Instructional Psychology and Technology, University of Leuven, Dekenstraat 2, 3000 Leuven, Belgium
[b] Research Group of Quantitative Psychology and Individual Differences, University of Leuven, Tiensestraat 102, 3000 Leuven, Belgium

ARTICLE INFO

Keywords:
Rational numbers
Fraction
Natural number bias
Conceptual change

ABSTRACT

In this study we longitudinally followed 201 upper elementary school learners in the crucial years of acquiring rational number understanding. Using latent transition analysis we investigated their conceptual change from an initial natural number based concept of a rational number towards a mathematically more correct one by characterizing the various intermediate states learners go through. Results showed that learners first develop an understanding of decimal numbers before they have an increased understanding of fractions. We also found that a first step in learners' rational number understanding is an increased understanding of the numerical size of rational numbers. Further, only a limited number of learners fully understand the dense structure of rational numbers at the end of elementary education.

1. Introduction

There is broad agreement in the literature that a good understanding of the rational number domain is highly predictive for the learning of more advanced mathematics (e.g., Siegler, Thompson, & Schneider, 2011). It is therefore worrying that many elementary and secondary school learners and even (prospective) teachers face serious difficulties understanding rational numbers. For instance, Van Hoof, Verschaffel, and Van Dooren (2015) gave the following problem to a representative group of 4th, 6th and 8th graders: "What is half of 1/8?". Only 8% of the 4th, 47% of the 6th, and 63% of the 8th graders could accurately answer this question. Further, a survey of a national representative sample of American algebra teachers showed that a lack of rational number understanding is one of the major sources why learners are not performing well in algebra classes (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Finally, based on their review of 43 studies from all over the world on prospective teachers' rational number understanding, Olanoff, Lo, and Tobias (2014), concluded that most prospective teachers are accurate in performing procedures with rational numbers, but struggle to understand the meanings behind the procedures and the reasons why the procedures work.

An often reported source for the struggle with understanding rational numbers is the natural number bias, i.e., the tendency to (inappropriately) apply properties of natural numbers in rational numbers tasks (e.g., Behr, Lesh, Post, & Silver, 1983; Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Ni & Zhou, 2005; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012, Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015; Vosniadou, 2013).

The literature reports at least three aspects of the natural number bias, relating to size, operations, and density. The first aspect involves the numerical size of numbers. Learners often consider a fraction as two independent numbers, instead of a ratio between the numerator and denominator. This incorrect interpretation of a fraction can lead to the misconception that the numerical value of a fraction increases when the numerator, denominator, or both increase, just like it is the case with natural numbers (e.g., McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015). For example, 1/8 can be judged larger than 1/6, just like 8 is larger than 6. Similarly, in the case of decimal numbers, some learners have been found to wrongly assume that, just like it is the case with natural numbers, longer decimals are larger, while shorter decimals are smaller. For example, these learners judge 0.12 larger than 0.8, just like 12 is larger than 8 (e.g., Meert, Grégoire, & Noël, 2010a, 2010b; Stafylidou & Vosniadou, 2004).

The second aspect concerns the effect of arithmetic operations. After learners did arithmetic with mostly natural numbers only in their first years of schooling, some learners have been found to apply the rules that hold for natural numbers also to rational numbers, also in cases where this is inappropriate. These group of learners assume for example that addition and multiplication will lead to a larger result, while

* Corresponding author.
E-mail addresses: jo.vanhoof@kuleuven.be (J. Van Hoof), tine.degrande@kuleuven.be (T. Degrande), eva.ceulemans@kuleuven.be (E. Ceulemans), lieven.verschaffel@kuleuven.be (L. Verschaffel), wim.vandooren@kuleuven.be (W. Van Dooren).

https://doi.org/10.1016/j.lindif.2017.11.010
Received 14 September 2016; Received in revised form 17 October 2017; Accepted 3 November 2017
1041-6080/

subtraction and division will lead to a smaller result. For example, learners think that 5 ∗ 0.32 will result in an outcome larger than 5 (e.g., Christou, 2015; Van Hoof et al., 2015).

The third aspect is density. Many researchers (e.g., Merenluoto & Lehtinen, 2004; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi et al., 2012, Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof et al., 2015; Vosniadou, 2013) reported a lack of understanding of the dense structure of rational numbers. Contrary to natural numbers that have a discrete structure (each natural number has a successor number; after 5 comes 6, after 6 comes 7, …), rational numbers are densely ordered (between any two rational numbers are always infinitely many other numbers). This difference in structure of both types of numbers leads to frequently found mistakes such as thinking that there are no numbers between two pseudo-successive numbers (e.g., 6.2 and 6.3 or 2/4 and 3/4 (e.g., Merenluoto & Lehtinen, 2004; Vamvakoussi et al., 2011)).

Evidence for this natural number bias has been frequently found in the much higher accuracy levels of learners on congruent rational number tasks (i.e., tasks where natural number reasoning leads to the correct answer, for example: "Which number is the larger one: 0.45 or 0.2?"), compared to their accuracy levels on incongruent tasks (i.e., tasks where natural number reasoning leads to the incorrect solution, for example: "Which number is the larger one: 0.45 or 0.6?").

A lot of research on learners' transition from natural to rational numbers has been described from a conceptual change perspective (but see for example Ni & Zhou, 2005 for alternative views on the origin of the natural number bias). This conceptual change perspective argues that since children encounter natural numbers much more frequently than rational numbers in daily life and in the first years of instruction, they form an idea of what numbers are and how they should behave based on these first experiences with and knowledge of natural numbers. For instance, they think that numbers are discrete, that they "get bigger" with addition or multiplication while subtraction or division makes them "smaller", etc. So, to overcome the natural number bias, a conceptual change revising these initial natural number based understandings is required once rational numbers are introduced in the classroom (e.g., McMullen, 2014; Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof et al., 2015; Vosniadou & Verschaffel, 2004). It should be noted that in the conceptual change literature, there is an ongoing debate on whether learners' initial ideas of concepts are to be characterized as relatively independent fragments (e.g., diSessa, 2013) or as a more or less coherent theory (e.g. Vosniadou, 2013). Nonetheless, in both views conceptual change is considered to be not an all or nothing issue but a gradual and time-consuming process, with many intermediate states between the initial and the correct understanding (e.g., Vosniadou, 2013). Vosniadou (2013) goes further by defining a special class of intermediate states, which she calls "synthetic conceptions". They refer to combinations of elements of the initial idea of number with elements of the new information assimilated in the knowledge structure. An illustration is the synthetic conception of rational numbers as being a collection of unrelated sets of numbers based on their representation (i.e., natural numbers, fractions, and decimal numbers are three unrelated sets of numbers) that are allowed to have different properties. For example, some learners think that there are infinitely many decimal numbers between two decimal numbers, while at the same time they do not accept that there can be infinitely many fractions between two given fractions (Vamvakoussi & Vosniadou, 2010). Therefore, it seems important to investigate in detail how the process of conceptual change occurs from the initial natural number based idea of a rational number towards a mathematically more correct one; which correct insights are gained first, and to characterize the intermediate states that can be found in learners. More specifically, it is essential to investigate whether general patterns in this development can be found. If so, a learner's profile at a certain measurement point can be considered to be predictive for its further development. From an educational perspective, such profiles would be helpful for teachers to provide effective instruction that is adapted to the specific knowledge and needs of each learner (Schneider & Hardy, 2013).

While the natural number bias has already generated substantial research interest in the last decade, empirical evidence on the development of learners' conceptual change in the longer term is scarce. Moreover, studies that try to uncover that development are typically cross-sectional (e.g., Stafylidou & Vosniadou, 2004; Van Hoof et al., 2015). The single exception we are aware of is the recent longitudinal study of McMullen et al. (2015), wherein 263 upper elementary school children have been followed over a one-year time period, including two different school years. The researchers measured children's conceptual understanding of the numerical size and the dense structure of rational numbers. Although the developmental patterns that were found indicated that only a limited number of children showed conceptual change at all in both aspects, it was concluded that a good understanding of the numerical size of rational numbers is a necessary but not a sufficient step for a good understanding of the dense structure of rational numbers. Our study builds on this first longitudinal study, but extends it in several ways.

1.1. The present study

Previous studies stated that many learners need a conceptual change, characterized by several intermediate states and possible synthetic models, in order to come to a good understanding of the rational number domain (see above). However, it remains unclear what these intermediate states consist of and whether there is some consistency in these states across students and across educational systems.

In the present study, we will longitudinally follow the development of rational number understanding of upper elementary school learners in the crucial years of acquiring rational number understanding. The aim of this study is to have a theoretical contribution to the research field by empirically characterizing in detail the intermediate states of learners' conceptual change from an initial natural number based concept of rational numbers towards a mathematically more correct one and by investigating whether these intermediate states have a consistent character across students or not.

Next to the general goal of characterizing the intermediate states, we also want to shed light on three important aspects of learners' rational number development, by extending previous research, and particularly the longitudinal study by McMullen et al. (2015).

First, the results of this study will allow us to take a glance at the question to what extent the development of learners' understanding of rational numbers depends on the kind of rational number instruction that is given (for a broader discussion on cross-national differences in rational number knowledge, see Nguyen, 2015). In Finland, the country where the study of McMullen et al. (2015) took place, rational number instruction is given in intensive periods where the focus in the mathematics class is for a few weeks (mostly) only on rational numbers, whereas in Flanders rational number instruction is spread out throughout several years. By comparing the results of the study of the Finish learners of McMullen et al. (2015) with the results of our study, we have some indications whether the same developmental patterns can be found despite a different curricular approach. Second, keeping in mind that different sorts of misconceptions are found in decimal versus fraction tasks (e.g., Resnick et al., 1989), it is possible that learners' understanding of these two representation types develops differently. This possibility was not yet systematically taken into account in previous research. Our study will allow to address this specifically. Third, while McMullen et al. (2015) concluded that a good understanding of the aspect of size forms a prerequisite to understand the dense structure of rational numbers, the third aspect that we distinguished above, i.e. operations, was not considered in that study. Thus, it remains unclear how learners' understanding of the aspect of operations develops as compared to size and density understanding. Starting from previous research based on the conceptual change perspective (e.g., Vamvakoussi et al., 2012; Van Hoof et al., 2015; Vosniadou, 2013), we expect to find several intermediate states in learners' development. More specifically, concerning the two representation types of rational numbers (decimals and fractions), we conjecture that learners' understanding of both representation types does not necessarily develop simultaneously. It is therefore possible that learners first develop a good understanding of decimal numbers before they develop a good understanding of fractions or vice versa. Next, no predictions could be made about how learners' understanding of the operations aspect develops compared to their understanding of size and density. However, keeping the results of the study of McMullen et al. (2015) in mind, we expected that a first step in learners' conceptual change towards a mathematically more correct concept of a rational number involves a good understanding of the numerical size of rational numbers.

2. Method

2.1. Participants

Participants were recruited from four elementary schools and 11 classrooms in Flanders, Belgium. In total 201 Flemish learners from fourth ($n = 113$) and fifth grade ($n = 88$) participated in this study and 50.2% of the participants were boys. Data were collected according to the ethical guidelines of the university.

2.2. Design

Learners' rational number knowledge was measured three times over the course of two school years, spanning a total time of 15 months: at the beginning (= Time 1, learners were in 4th and 5th grade), and end of Spring 2014 (= Time 2, learners were in 4th and 5th grade) and at the end of Spring 2015 (= Time 3, learners were by that time in 5th and 6th grade). In Flanders, rational number instruction occurs very gradually, spread out over several years. The basic instruction on fractions and decimals typically starts in third grade (unit fractions, very simple fractions and decimals). This already includes instruction on operations with rational numbers during the fourth grade and further instruction spread out until sixth grade. According to the Flemish curriculum, learners should have acquired all knowledge about rational numbers that is measured in our test instrument at the end of the sixth grade. (See Appendix 1 for a sample rational number activity in 4th, 5th, and 6th grade.)

2.3. Tasks

2.3.1. Rational number knowledge test

To measure learners' rational number understanding, we used the Rational Number Knowledge Test (RNKT). This test was already used and validated in previous research investigating the relation between learners' spontaneous focusing on quantitative relations and their rational number understanding (Van Hoof et al., 2016). The test contains 31 incongruent items: 10 size items, 8 operation items, and 13 density items. Because the focus of this study is on learners' ability to overcome the natural number bias, we chose this test instrument that only consists of incongruent items. Table 1 displays examples of items for all three aspects.

No time limit was used, as a time limitation could encourage natural number based reasoning (Vamvakoussi et al., 2012).

2.4. Analysis

Using paired samples t-tests, we analyzed whether significant increases could be found in learners' rational number understanding on all subsets of the test, from Time 1 to Time 3. In a next step, data were analyzed using latent transition analysis (LTA). LTA is a longitudinal data analysis technique designed to detect unknown groups of participants and to model change in group membership over time through transition probabilities (Nylund, 2007). In our study, the groups can be interpreted as developmental states in learners' conceptual change, characterized by a specific answer pattern. As stated by McMullen et al. (2015), LTA can be used to describe the trajectories of learners' conceptual change, since it is able to characterize what type of response patterns learners have in the different developmental states. In our study, a first set of transition probabilities indicates how likely it is that a participant switches from state i at time point 1 to state j at time point 2. A second set of probabilities represents how likely these transitions are when going from time point 2 to time point 3. Hence, taken together, these two sets shed light on the trajectories of learners' conceptual change, and thus on (partial) orderings of the developmental states, unraveling possible developmental paths. While this statistical technique is quite novel in the research field of documenting conceptual change processes, it has already been proven successful in a few studies. For instance, Schneider and Hardy (2013) successfully used this technique to shed light on learners' conceptual development of floating and sinking of objects in liquids. Also McMullen et al. (2015) used it to describe the development of learners' rational number understanding. Our LTA analyses were conducted in the statistical software Mplus version 7.2. We estimated the model parameters using the maximum likelihood estimation with robust standard errors. We restricted the number and nature of the states to be the same over the three measurement points, reducing the number of parameters to be estimated and making it possible to compare the results across measurement points (Schneider & Hardy, 2013). There were no missing data.

Table 1
Examples of both fraction and decimal test items from the Rational Number Knowledge Test per aspect.

Density	Size	Operations
How many numbers are there between 0.74 and 0.75?	Which is the larger number? 0.36 or 0.5	0.36–0.2 = …
How many numbers are there between 0.7 and 0.8? A: zero B: ten C: infinitely many decimals D: infinitely many numbers, decimals as well as fractions	Order the following numbers from small to large 3.682 - 3.2 - 3.84	Is 21: 0.7 bigger or smaller than 21?
What is the smallest possible fraction?	Which is the larger number? 5/8 or 4/3	What is half of 1/8?
How many numbers are there between 3/5 and 4/5?	Order the following numbers from small to large 4/7 2/6 5/10	2/6 + 1/3 = …

3. Results

In the result section, we start with describing the general development of learners' rational number understanding at a more macro level. In a next step, we analyze the specific qualities of this change. We first discuss how we chose the number of developmental states, followed by a description of these different states. Important to note is that the order in which the states are described is based on the accuracy levels on every subtest of the RNKT, but this does not reflect learners' general developmental path of rational number understanding. In a next step, we have a look at the latent transition probabilities to shed light on this general developmental path, since these probabilities show how likely learners switch from one state to another one.

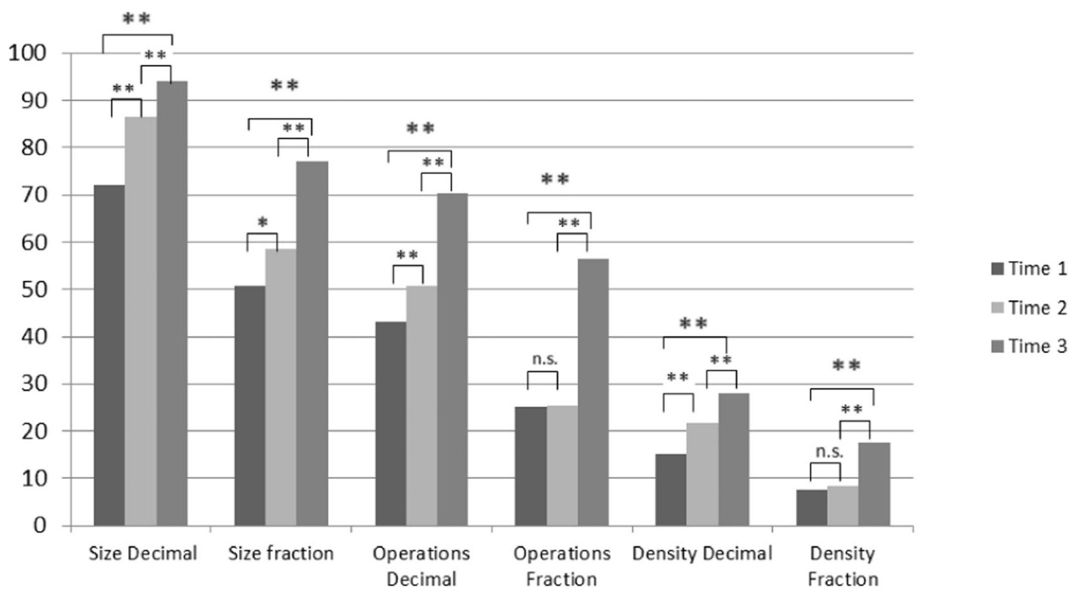


Fig. 1. learners' general development over time on all subsets of the rational number test, with mean accuracy scores (y-axis) on every subtest of the RNKT (x-axis).

3.1. Change at the macro level

Fig. 1 shows learners' accuracy levels on all subsets of the rational number test, going from Time 1 to Time 3. Paired samples t-tests showed that learners' accuracy on all subsets significantly increased going from Time 1 to Time 2 and from Time 2 to Time 3, even after Bonferroni corrections. Only two exceptions were found. Learners' rational number understanding did not significantly increase from Time 1 to Time 2 in the aspect of operations with fractions ($t(200) = -29$, $p = 0.77$) and density of fractions ($t(200) = -81$, $p = 0.42$).

3.2. Choosing the number of states

In the first step of the latent transition analysis, we chose the number of developmental states. We opted to select the six-state solution, based on the following considerations. First, out of the five-, six-, and seven-state solutions, which have the lowest AIC and BIC values, the six-state solution has the highest entropy value, indicating a higher classification certainty (see Table 2). Both the Akaike Information Criterion (AIC; Akaike, 1974) and Bayesian Information Criterion (BIC; Schwarz, 1978) are indicators used to decide which model suits your data best. The model with the lowest AIC-value or BIC-value is the optimal one. These two values indicate an optimal balance between both model fit and model complexity in terms of the number of estimated parameters (= fp). AIC is computed as follows (Akaike, 1974): AIC = $-$ 2logL + 2 × fp. The AIC value implies the assumption that every observation provides new and independent information concerning the underlying model, which can be unrealistic in studies with large sample sizes. For this reason, Schwarz (1978) proposed the BIC criterion, which takes the sample size into account. BIC is computed as follows: BIC = $-$ 2logL + log(I) × fp.

Second, the six-state model is also to be preferred from an interpretation point of view. As shown in Fig. 2 (that displays the mean accuracy scores on all subtests of the RNKT per profile, in the five-, six-, and seven-class solution), the profiles found in the five-class and six-class solution are highly comparable. However, the additional profile in the six-class solution (represented with plus signs) is theoretically more interesting, because it allows to differentiate between learners who strongly tend to think about rational numbers as being discrete (mean accuracy density fractions: 2.80%; mean accuracy on density decimals: 20.35%) and learners who are already much further in their understanding of the dense structure of rational numbers (mean accuracy density fractions: 28.30%; mean accuracy on density decimals: 48.43%). In contrast, the additional profile in the seven-class solution (represented with minus-signs) is theoretically not interesting, because it only makes a distinction between a group of learners with a very good performance on decimal size tasks and a group with an even better performance on the same decimal size tasks (see Fig. 2).

Table 2
Fit measures of the three- to seven-state solution.

Number of states	AIC	BIC	Entropy
3	10,353.22	10,518.38	0.943
4	9800.77	10,028.70	0.978
5	9441.79	9745.70	0.939
6	9279.52	9672.62	0.953
7	9150.68	9646.18	0.938

3.3. Description of the states

The mean accuracy scores on all subtests of the RNKT per state (see Fig. 3) show, first, that learners in the '*Initial*' state are characterized by an initial natural number based understanding of rational numbers. They have a very low accuracy on all subtests, with a maximum subtest score of only 30.68% on decimal operation tasks. Second, learners in the '*Emerging size*' state have low accuracy scores on almost all subtests. Contrary to the 'Initial' state, they already have some understanding of the size of fractions (mean accuracy = 52.23%) and of decimals (mean accuracy = 66.70%). On all other subtests, they score below 50% accuracy. Third, learners in the '*Size decimals*' state are characterized by having a good understanding of the size of decimal numbers, performing almost perfectly on these items. Their mean scores on all other subtests are below 50%. Fourth, learners in the '*Emerging operations*' state have developed a good understanding of the aspect of operations. Moreover, they developed a good understanding of the size of fractions, but still have a natural number based idea of the structure of rational numbers. This is shown by their accuracy scores on decimal density tasks (mean accuracy = 20.38%), but especially on fraction density tasks (mean accuracy = 2.80%). Fifth, learners in the '*Emerging density*' state also have a good understanding of the size and operations aspect, but moreover developed already some understanding of the dense structure of rational numbers (mean accuracy decimal density tasks = 48.43% and mean accuracy fraction density tasks = 28.3%).

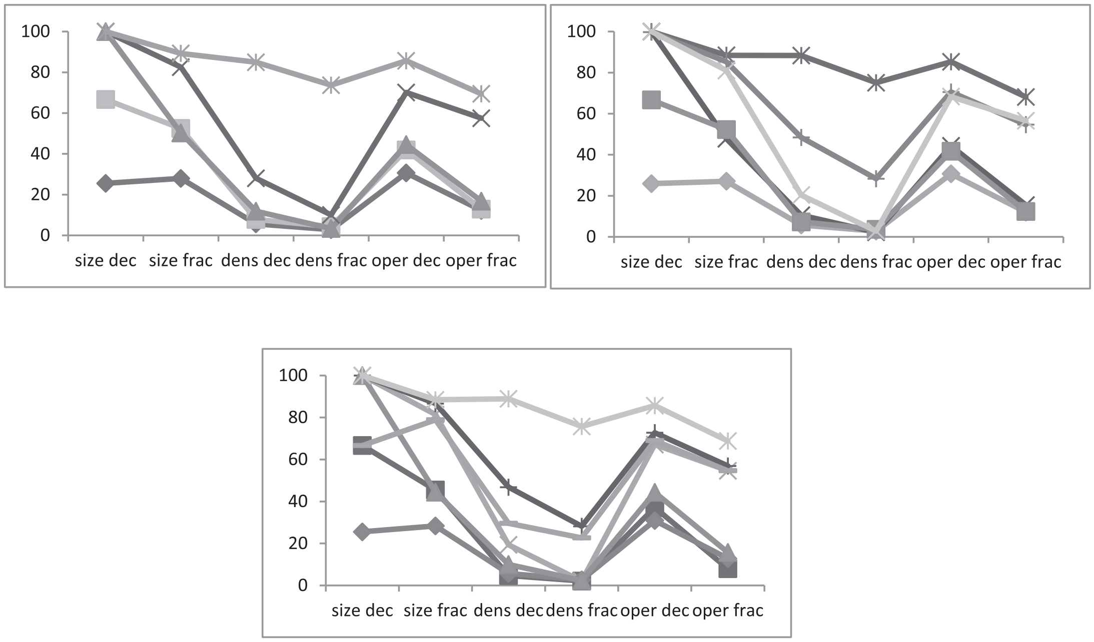


Fig. 2. Profiles in the five-, six-, and seven-class solution, with mean accuracy scores (y-axis) on every subtest of the RNKT (x-axis).

Sixth, learners in the ‘Mathematically more correct’ state show a good understanding on all subtests, with a minimum subtest score of 68.13% on fraction operation tasks.

Interestingly, in every profile (except in the ‘Initial’ profile on size tasks) and in all three aspects of the natural number bias, learners score remarkably higher on the decimal than on the fraction tasks, indicating that understanding the size of decimals, operations with decimals, and decimals' density is easier to achieve than understanding these three aspects for the fraction counterpart. However, the developmental paths are largely parallel. In both cases, most learners first develop a good understanding of the size of rational numbers before they develop an understanding of operations.

As can be seen in Table 3, the number of learners in each profile changed over time. At Time 1, almost half of the learners were in the ‘Initial’ or ‘Emerging size’ state, indicating a low understanding of rational numbers. At Time 2, a clear shift towards a better understanding of rational numbers could already be seen. Especially notable is the doubling of the number of learners in the ‘Size decimals’ group. The growth in rational number understanding was also clearly visible when looking at the sizes of the profiles at Time 3. Remarkably, while half of the learners had an ‘Initial’ or ‘Emerging size’ profile on Time 1, this drops to only 13% of the learners at Time 3. Moreover, almost 62% of the learners have the profile of ‘Emerging operations’ at Time 3, indicating a good understanding of the aspect of operations and size, but still a natural number based understanding of the dense structure of rational numbers. This table also shows that, although there is a difference of one year rational number instruction between the fourth and fifth graders, in both groups there were learners who start in the initial state as well as learners who end in the mathematically more correct state.

3.4. Learners' transition paths

As a second step in our LTA, we checked whether there was a

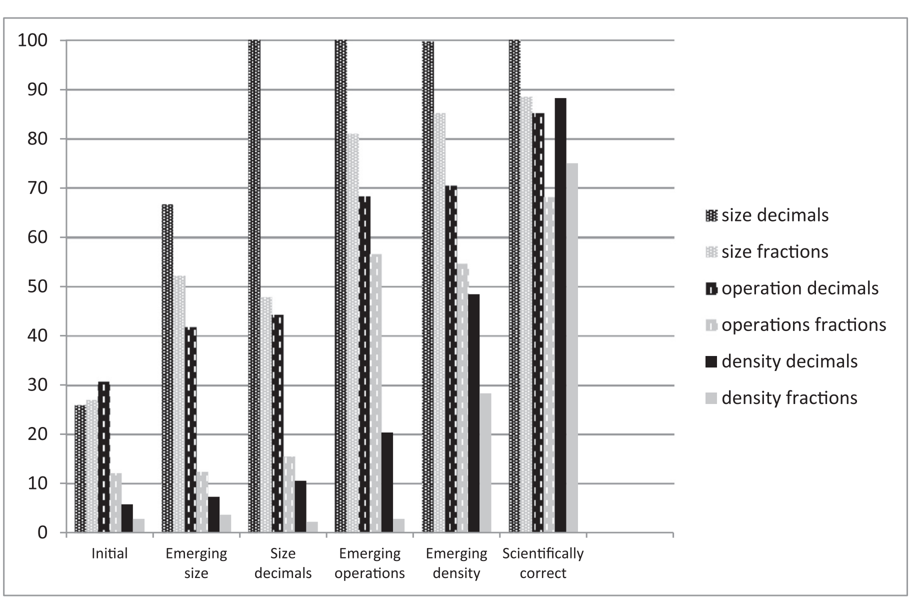


Fig. 3. Accuracy levels (in %) on all aspects of the RNKT per state.

Table 3
Number of learners in each state over time (beginning Spring 2014 (learners were in 4th and 5th grade), end Spring 2014 (learners were in 4th and 5th grade), and end Spring 2015 (learners were in 5th and 6th grade)).

	Begin Spring 2014			End Spring 2014			End Spring 2015		
	Grade 4	Grade 5	Total	Grade 4	Grade 5	Total	Grade 5	Grade 6	Total
Initial	42	4	46	16	2	18	7	2	9
Emerging size	40	12	52	26	13	39	13	4	17
Size decimals	22	21	43	55	23	78	9	3	12
Emerging operations	3	27	30	5	32	37	65	59	124
Emerging density	6	19	25	9	9	18	12	6	18
Mathematically more correct	0	5	5	2	9	11	7	14	21

Table 4
Latent transition probabilities from Time 1 to Time 2 and from Time 2 to Time 3.

T1	T2					
	Initial	Emerging size	Size decimals	Emerging operations	Emerging density	Mathematically more correct
Initial	0.33	0.12	0.37	0.00	0.18	0.00
Emerging size	0.06	0.29	0.53	0.00	0.10	0.02
Size decimals	0.00	0.16	0.70	0.13	0.01	0.00
Emerging operations	0.00	0.00	0.00	0.89	0.04	0.07
Emerging density	0.00	0.12	0.06	0.17	0.45	0.20
Mathematically more correct	0.00	0.00	0.00	0.20	0.20	0.60
T2	**T3**					
	Initial	Emerging size	Size decimals	Emerging operations	Emerging density	Mathematically more correct
Initial	0.33	0.22	0.16	0.28	0.00	0.00
Emerging size	0.05	0.13	0.12	0.70	0.00	0.00
Size decimals	0.01	0.00	0.10	0.72	0.13	0.04
Emerging operations	0.00	0.00	0.00	0.94	0.00	0.06
Emerging density	0.00	0.00	0.00	0.10	0.37	0.53
Mathematically more correct	0.00	0.00	0.00	0.29	0.00	0.71

general trend going from the initial natural number based idea of a rational number (‘Initial’ state) to the mathematically more correct one (‘Mathematically more correct’ state). Therefore we had a look at the Latent Transition Probabilities (LTP) (see Table 4), that show the probabilities of learners transitioning from one state to another one through the different time periods. Overall, the states stayed more stable from Time 1 to Time 2 compared to the stability over Time 2 to Time 3. This is not surprising given that there was less time between Time 1 and 2 than between Time 2 and 3. Further, the ‘Emerging operations’ state stands out as being the most stable state. Learners who are in this group at Time 1 have 89% chance of staying in this group at Time 2. In the same line, learners who have the ‘Emerging operations’ state at Time 2 have 94% chance of having the same state at Time 3. This suggests that once learners at the end of elementary education have developed a good understanding of the operations and size with rational numbers, they most often do not develop further and hence do not yet have a good understanding of the dense structure of rational numbers. If we take a look at the highest latent transition probabilities, as they indicate the transitions that occur most, we see that from Time 1 to Time 2 learners from both the ‘Initial’ state and the ‘Emerging size’ state at Time 1 have a high chance of ending up in the ‘Size decimals’ at Time 2. This suggests that learners with an initial natural number based understanding of rational numbers at Time 1 first have an increased understanding of the size of decimal rational numbers. In the transition from Time 2 to Time 3, learners from both the ‘Emerging size’ and the ‘Size decimals’ state have a very high chance of ending up in the ‘Emerging operations’ state. This shows that learners who have an initial natural number based understanding of rational numbers, except for the size of decimal numbers, are very likely to develop an increased understanding of operations with rational numbers (both decimals and fractions) and the size of fractions in a next step, while they still have an initial natural number based understanding of the dense structure of rational numbers.

While a large group of learners who have a good understanding of operations first go through the early states of a good understanding of size, no such developmental path is found in the transition probabilities in the group of learners with (good) understanding of density. Very few learners of these qualitatively different group go through previous states. This suggests that the two states ‘Emerging density’ and ‘Mathematically more correct’ describe qualitatively different learners who understand density as opposed to the rest of the learners who do not see the dense structure of rational numbers. As shown in Table 4, at least 10% of the learners in the state of ‘emerging density’ and ‘mathematically correct’ fall back into the state of ‘emerging operations’ state between time points, indicating that this understanding of density is not perfectly stable.

Moreover, while there are 216 possible transition paths in our six profile-solution (6ˆ3), only 53 transition paths were actually found. If we only focus on those transition paths taken by at least five learners (out of 201, thus 2.5% of the sample), there are only 11 remaining transition paths that describe the evolution of more than 62% of the learners in our study.

4. Conclusion and discussion

We longitudinally followed a large group of elementary school learners in the crucial years of their rational number development. Using latent transition analysis we documented their conceptual change from an initial natural number based concept of a rational number towards a mathematically more correct concept of a rational number. The latent transition analysis revealed six developmental states that allowed to differentiate between learners with an initial natural number based, some, or a good understanding of the three aspects of the natural number bias.

Our results add to the existing theory by characterizing the several different intermediate states going from learners' initial natural number based concept of rational numbers towards a mathematically more correct one. Some of these intermediate states showed the co-existence of inconsistent parts of rational number understanding, c.f. synthetic models (see introduction). An example of such a synthetic model can be found in the “decimal size” profile, which represents a large group of learners at the end of elementary education (39% of the learners had this profile at time 2). In the “decimal size” profile, learners have a very good understanding of the size of decimal numbers, while they still have a quite naïve understanding of the size of fractions.

The finding that there can be six different profiles distinguished in learners' rational number understanding shows that although learners received similar rational number instruction, individual differences could be found at every time point in learners' conceptual understanding of fractions and decimal numbers. It should be noted however that although we found several rational number understanding profiles and differences in learners' learning trajectories, we also found that the number of rational number profiles ($n = 6$) and transition paths ($n = 56$, of which only 11 were somewhat frequent) was much smaller than the number of participants in this study ($n = 201$). This indicates that learners' conceptual change from an initial to a more correct concept of rational numbers is constrained along certain patterns, and general developmental paths can be described. Results showed that learners first develop a good understanding of decimal numbers before they have a good understanding of fractions. We further found that if learners have an increased understanding of rational numbers, they first develop a good understanding of the size of rational numbers, especially in the case of decimal numbers (‘Size decimals’ state). Once learners have a good understanding of the size of rational numbers, they can also develop a good understanding of operations with rational numbers (‘Emerging operations’ state). However, in our sample, most learners do not reach a higher state than ‘Emerging operations’, as is for example shown in the finding that 23 learners stayed in the ‘Emerging operations’ state across all time points. Moreover, 19% of all the learners in our sample stay in the same state over the 15 month period comprised in our study. This is in big contrast with the findings of McMullen et al. (2015) who found that 82% of the learners stayed in the same state across all time points over a 12 month period. Moreover, the transition paths teach us that learners rarely fall back in a ‘lower’ state. At last, a group of qualitatively different learners was found who, in contrast to the rest of the learners, understand the dense structure of rational numbers.

This study extends previous research in three ways. First, as stated above, by comparing the results of the study of the Finnish learners of McMullen et al. (2015) with the results of the present study, we have some indications whether the same developmental patterns can be found despite a different curricular approach. Both similarities and differences between the two studies were present. As in the study of McMullen et al. (2015), we found that the understanding of the size of rational numbers forms a first necessary step in learners' rational number understanding. This finding is consistent with the integrated theory of numerical development (Siegler et al., 2011), which states that understanding the numerical sizes of fractions forms a crucial step in the understanding of fractions. We also found that the aspect of density elicits the most difficulties with learners. However, contrary to the Finnish learners, the Flemish learners in this study more frequently developed a better understanding of rational numbers. In total only 19% of all the learners in our sample stayed in the same state (while this was 82% in the Finnish study). Further, more learners developed some or a good understanding of the dense structure of rational numbers, while this group of learners was much smaller in the study of McMullen et al. (2015). It should be noted, however, that the stability observed by McMullen et al. (2015) can partly be explained by the fact that, due to the design of their study, they were unable to differentiate between states that differ by number type (i.e., fractions or decimals) or operation.

Second, because previous research indicated different type of mistakes and misconceptions between decimal and fraction tasks (e.g., Resnick et al., 1989), we hypothesized that learners' understanding of both representation types might develop differently. Results showed that this was not the case. While for each aspect of the natural number bias learners first develop a good understanding with decimal numbers before they develop a good understanding with fractions, the developmental paths are largely parallel. In both cases, most learners first develop a good understanding of the size of rational numbers before they develop an understanding of operations.

Third, until now, it remained unclear how learners' understanding of the operations aspect develops compared to their size and density understanding. Based on the trends that we observed, we can characterize the development from the initial natural number based to the mathematically more correct idea of rational number as follows: First, learners develop a good understanding of the size of decimal numbers, followed by a good understanding of the size of fractions. Once learners have a good understanding of the size of rational numbers, they develop an understanding of operations with rational numbers (first decimals, then fractions). A qualitatively different group of learners also develop their understanding of the dense structure of rational numbers (first with decimals, then with fractions), without necessarily going through the profiles of good understanding of size and operations. Important to note is that this is in contrast with previous studies (e.g., McMullen et al., 2015; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015) that concluded that learners with a good understanding of the dense structure of rational numbers first need to have a good understanding of both the size and operations with rational numbers. Moreover, we found that a good understanding of the dense structure of rational numbers is not an all or nothing issue, since both the ‘Emerging density’ and ‘Mathematically more correct’ state of understanding both have been found to be unstable states.

We also have some suggestions for future research. First, future research investigating learners' conceptual change regarding rational numbers would benefit from following learners for a longer time period than one year. From our data it is clear that most of the learners do not have a mathematically correct understanding of the rational number system by the end of elementary education. Especially the aspect of density is still difficult for many learners. Therefore, it is valuable to follow learners also in their first years of secondary education to be able to investigate how learners' understanding of the density aspects develops. In addition, more cross-cultural comparative studies are necessary in order to investigate the extent to which the development of learners' concept of rational number has certain universal features, or whether a different curriculum elicits a qualitatively different development. Moreover, since we found in this study that learners' understanding of decimals and fractions do not develop simultaneously, further research should pay particular attention to the distinction between rational number tasks with decimals compared to fractions.

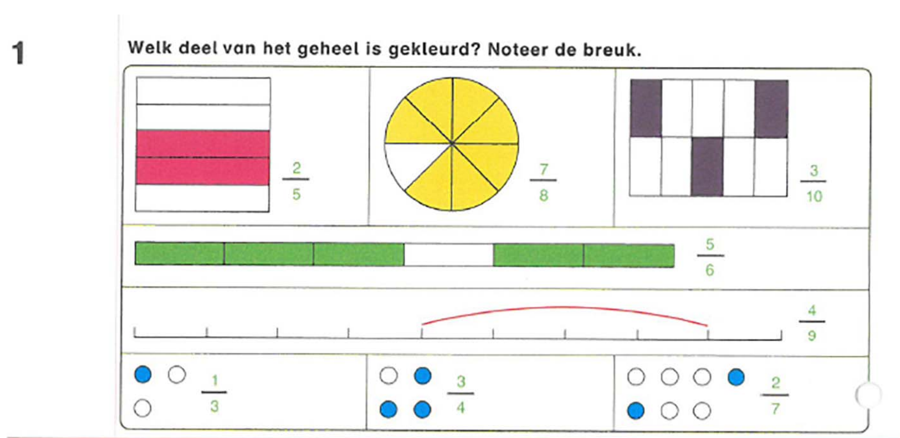
We continue with an important educational implication. From the theoretical background, we know that the process of conceptual change is gradual, time-consuming, and far from easy. Still, while instruction aimed at conceptual change in mathematics needs a lot of effort, research has shown that it can be successful under appropriate conditions (Vamvakoussi, Vosniadou, & Dooren, 2013). Vosniadou, Ioannides, Dimitrakopoulou, and Papademetriou (2001) highlight several recommendations to design learning environments aimed at conceptual change. First, they emphasize that curriculum designers should focus on a deep exploration and understanding of a few concepts instead of superficially covering a great amount of material. This suggestion to focus rather than to be comprehensive may make the endeavor also more feasible in a classroom setting, given limited instruction time. In this light, another implication of the present study is that it can be seen in line with the learning trajectory research. Indeed, as stated by Clemens and Sarama (2004), a clear instructional sequence is an important aspect of learning trajectories. The results of the present study show that a first step in learners' rational number understanding is a good understanding of the size of rational numbers (first decimals and later fractions). Therefore, we would suggest that instruction in the beginning explicitly focuses on the numerical size of rational numbers before introducing the more advanced content, such as operations with rational numbers. Second, Vosniadou et al. (2001) suggest that teachers should explicitly take into account learners' prior knowledge and, in the same line, distinguish new information that is consistent or inconsistent with this prior knowledge, facilitate learners' metaconceptual awareness, motivate learners to change their beliefs and presuppositions, produce cognitive conflict, and use models and external representations to clarify certain scientific explanations. However, current instructional practices, at least in Flanders, do not seem to meet these suggestions. We refer to the research of Debou and Verschetze (2012), who found that the most commonly used textbooks in the Flemish classroom pay almost no explicit attention to the (conceptual) differences between natural and rational numbers. On the contrary, they tend to give explicit attention to the continuity with natural numbers in their instruction about rational numbers by emphasizing the similarities between both types of numbers. For example, multiplication is taught through the model of repeated addition, and that the common way to teach division is through equal sharing. As stated by Vamvakoussi (2015), it is important to note that the natural number bias should not only be associated with its adverse effect of learners' prior knowledge on their further learning. Using natural number knowledge acts as a facilitator too, namely in contexts that are compatible (congruent) with natural number knowledge. However, since natural number properties cannot always be applied in tasks with rational numbers, there is a need for a stronger awareness of the possible negative consequences of introducing rational numbers without an explicit attention for both the similarities and differences with natural numbers (see for example Greer, 1994). This might help to avoid later conceptual difficulties with these operations in learners. Put differently, it is not because we found in this study that learners develop in their understanding of rational number concepts from fourth until sixth grade, that this is the only moment concepts develop. Also the years before rational number instruction have an influence on learners' rational number understanding (such as teaching multiplication as repeated addition).

In sum, this study showed that while learners' conceptual change from an initial to a mathematically more correct concept of rational numbers does not have one unique character, learners' conceptual change processes are constrained and general developmental paths can be described. This study moreover indicated that an important first step in these general developmental paths is a good understanding of the size of rational numbers.

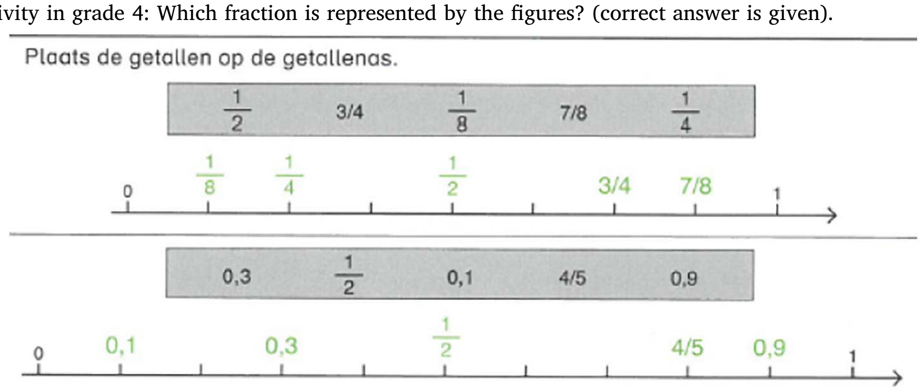
Acknowledgments

Jo Van Hoof holds a post-doctoral fellowship of the Research Foundation–Flanders (FWO).

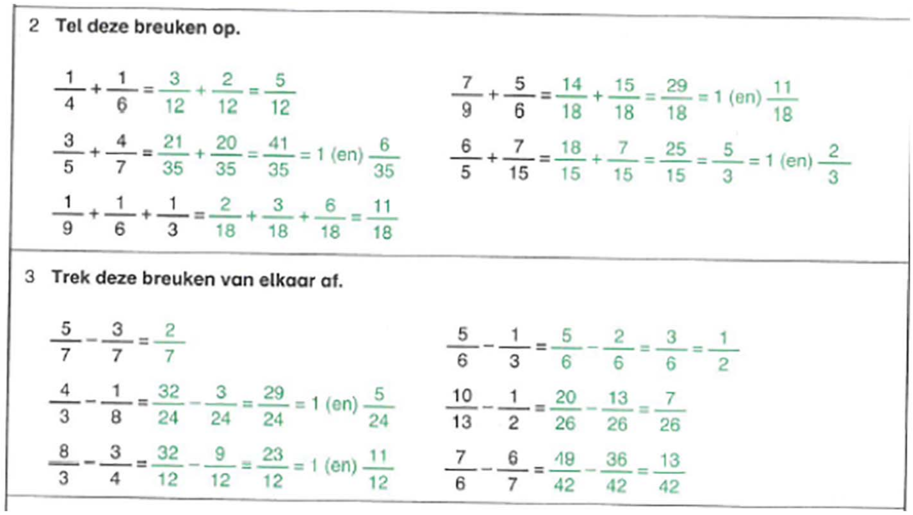
Appendix A. Appendix 1



Rational number activity in grade 4: Which fraction is represented by the figures? (correct answer is given).



Rational number activity in grade 5: Fill in the given numbers on the number line (correct answer is given).



2 **Tel deze breuken op.**

$\frac{1}{4}+\frac{1}{6}=\frac{3}{12}+\frac{2}{12}=\frac{5}{12}$

$\frac{3}{5}+\frac{4}{7}=\frac{21}{35}+\frac{20}{35}=\frac{41}{35}=1\text{ (en) }\frac{6}{35}$

$\frac{1}{9}+\frac{1}{6}+\frac{1}{3}=\frac{2}{18}+\frac{3}{18}+\frac{6}{18}=\frac{11}{18}$

$\frac{7}{9}+\frac{5}{6}=\frac{14}{18}+\frac{15}{18}=\frac{29}{18}=1\text{ (en) }\frac{11}{18}$

$\frac{6}{5}+\frac{7}{15}=\frac{18}{15}+\frac{7}{15}=\frac{25}{15}=\frac{5}{3}=1\text{ (en) }\frac{2}{3}$

3 **Trek deze breuken van elkaar af.**

$\frac{5}{7}-\frac{3}{7}=\frac{2}{7}$

$\frac{4}{3}-\frac{1}{8}=\frac{32}{24}-\frac{3}{24}=\frac{29}{24}=1\text{ (en) }\frac{5}{24}$

$\frac{8}{3}-\frac{3}{4}=\frac{32}{12}-\frac{9}{12}=\frac{23}{12}=1\text{ (en) }\frac{11}{12}$

$\frac{5}{6}-\frac{1}{3}=\frac{5}{6}-\frac{2}{6}=\frac{3}{6}=\frac{1}{2}$

$\frac{10}{13}-\frac{1}{2}=\frac{20}{26}-\frac{13}{26}=\frac{7}{26}$

$\frac{7}{6}-\frac{6}{7}=\frac{49}{42}-\frac{36}{42}=\frac{13}{42}$

Rational number activity in grade 6: Add/subtract these fractions (correct answer is given).

References

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control, AC-19*, 716–723. http://dx.doi.org/10.1109/TAC.1974.1100705.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh, & M. Landau (Eds.). *Acquisition of mathematics concepts and processes* (pp. 91–125). New York, NY: Academic Press.

Christou, K. P. (2015). Natural number bias in operations with missing numbers. *ZDM Mathematics Education, 47*, 747–758. http://dx.doi.org/10.1007/s11858-015-0675-6.

Clemens, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Debou, E., & Verschetze, L. (2012). *De overgang van natuurlijke naar rationale getallen. Een curriculumanalyse van drie Vlaamse wiskundemethoden [The transition from natural to rational numbers. An analysis of three Flemish methods of mathematics]*. (Unpublished master's thesis)Belgium: Katholieke Universiteit Leuven, Faculteit Psychologie en Pedagogische Wetenschappen.

diSessa, A. A. (2013). A bird's eye view of "pieces" vs. "coherence" controversy. In S. Vosniadou (Ed.). *Handbook of conceptual change research* (pp. 31–48). (2nd ed.). New York, NY: Routledge.

Gómez, D., Jiménez, A., Bobadilla, R., Reyes, C., & Dartnell, P. (2014). Exploring fraction comparison in school children. *Enero, 10*, 1–10.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel, & J. Confrey (Eds.). *The development of multiplicative reasoning in the learning of mathematics* (pp. 61–85). New York, NY: State University Press.

Hoffer, T. B., Venkataraman, L., Hedberg, E. C., & Shagle, S. (2007). *Final report on the national survey of algebra teachers for the national math panel.* Chicago, IL: National Opinion Research Center at the University of Chicago.

McMullen, J. (2014). Spontaneous Focusing on Quantitative Relations and the Development of Rational Number Conceptual Knowledge ((Doctoral dissertation). Retrieved from) https://www.doria.fi/bitstream/handle/10024/94538/AnnalesB380.pdf?sequence=2.

McMullen, J., Laakkonen, E., Hannula-Sormunen, M., & Lehtinen, E. (2015). Modeling the developmental trajectories of rational number concept(s). *Learning and Instruction, 37*, 14–20. http://dx.doi.org/10.1016/j.learninstruc.2013.12.004.

Meert, G., Grégoire, J., & Noël, M.-P. (2010a). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology, 107*, 244–259. http://dx.doi.org/10.1016/j.jecp.2010.04.008.

Meert, G., Grégoire, J., & Noël, M.-P. (2010b). Comparing 5/7 and 2/9: Adults can do it by accessing the magnitude of the whole fractions. *Acta Psychologica, 135*, 284–292. http://dx.doi.org/10.1016/j.actpsy.2010.07.014.

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the processes of change. *Learning and Instruction, 14*, 519–534. http://dx.doi.org/10.1016/j.learninstruc.2004.06.016.

Nguyen, H. T. T. (2015). L'apprentissage des nombres naturals et rationnels. *Etude comparative entre le Vietnam et la Belgique*. ((Doctoral dissertation). Retrieved from) https://www.google.be/url?sa=t&rct=j&q=&esrc=s&source=web&cd=1&ved=0ahUKEwimgJjR7uDNAhWLLMAKHRHbDK8QFgggMAA&url=http%3A%2F%2Fdial.academielouvain.be%2Fdownloader%2Fdownloader.php%3Fpid%3Dboreal%3A160838%26datastream%3DPDF_01&usg=AFQjCNHW4XqegA2gG9z2PoDryxwA1EGrQA&sig2=JgoZsHtwr2wP_yYd_33TZw&cad=rja.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*, 27–52. http://dx.doi.org/10.1207/s15326985ep4001_3.

Nylund, K. L. (2007). Latent transition analysis: Modeling extensions and an application to peer victimization ([Doctoral dissertation]. Retrieved from) http://www.statmodel.com/download/Nylund%20dissertation%20Updated1.pdf.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction, 28*, 64–72. http://dx.doi.org/10.1016/j.learninstruc.2013.05.003.

Olanoff, D., Lo, J.-J., & Tobias, J. M. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast, 11*(2), 267–310.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20*, 8–27. http://dx.doi.org/10.2307/749095.

Schneider, M., & Hardy, I. (2013). Profiles of inconsistent knowledge in children's pathways of conceptual change. *Developmental Psychology, 49*, 1639–1649. http://dx.doi.org/10.1037/a0030976.

Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of Statistics, 6*, 461–464. http://dx.doi.org/10.1214/aos/1176344136.

Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*, 273–296. http://dx.doi.org/10.1016/j.cogpsych.2011.03.001.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14*, 503–518. http://dx.doi.org/10.1016/j.learninstruc.2004.06.015.

Vamvakoussi, X. (2015). The development of rational number knowledge: Old topic, new insights. *Learning and Instruction, 37*, 50–55. http://dx.doi.org/10.1016/j.learninstruc.2015.01.002.

Vamvakoussi, X., Christou, K. P., Mertens, L., & Van Dooren, W. (2011). What fills the gap between discrete and dense? Greek and Flemish students' understanding of density. *Learning and Instruction, 21*, 676–685. http://dx.doi.org/10.1016/j.learninstruc.2011.03.005.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior, 31*, 344–355. http://dx.doi.org/10.1016/j.jmathb.2012.02.001.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*, 453–467. http://dx.doi.org/10.1016/j.learninstruc.2004.06.013.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*, 181–209. http://dx.doi.org/10.1080/07370001003676603.

Vamvakoussi, X., Vosniadou, S., & Van Dooren, W. (2013). The framework theory approach applied to mathematics learning. In S. Vosniadou (Ed.). *International handbook of research on conceptual change* (pp. 305–321). (2nd ed.). New York, NY: Routledge.

Van Hoof, J., Degrande, T., McMullen, J., Hannula-Sormunen, M., Lehtinen, E., Verschaffel, L., & Van Dooren, W. (2016). The relation between learners' spontaneous focusing on quantitative relations and their rational number knowledge. *Studia Psychologica, 58*, 156–170.

Van Hoof, J., Janssen, R., Verschaffel, L., & Van Dooren, W. (2015). Inhibiting natural knowledge in fourth graders: towards a comprehensive test instrument. *ZDM, 47*, 849–857. http://dx.doi.org/10.1007/s11858-014-0650-7.

Van Hoof, J., Vandewalle, J., Verschaffel, L., & Van Dooren, W. (2015). In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction, 37*, 30–38. http://dx.doi.org/10.1016/j.learninstruc.2014.03.004.

Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural

number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics, 90*, 39–56. http://dx.doi.org/10.1007/s10649-015-9613-3.
Vosniadou, S. (Ed.). (2013). *International handbook of research on conceptual change*(2nd ed.). New York, NY: Routledge.
Vosniadou, S., Ioannides, C., Dimitrakopoulou, A., & Papademetriou, E. (2001). Designing learning environments to promote conceptual change in science. *Learning and Instruction, 11*(4–5), 381–419.
Vosniadou, S., & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning and Instruction, 14*, 445–451. http://dx.doi.org/10.1016/j.learninstruc.2004.06.014.