Number 41 *SUMMER 2026*

R M M

ROMANIAN MATHEMATICAL MAGAZINE

SOLUTIONS

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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PROBLEMS FOR JUNIORS

JP.601 If $a,b,c>0$ and $\lambda\geq 0$ then:

$$(\lambda+1)\sum\frac{a^2}{b+\lambda c}+2\sqrt{3(ab+bc+ca)}\geq 3(a+b+c)$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We use pqr Method. We denote $p=a+b+c; q=ab+bc+ca, r=abc$.

Due to homogeneity we can assume $ab+bc+ca=3$.

We obtain

$$\sum\frac{a^2}{b+\lambda c}=\sum\frac{a^3}{a(b+\lambda c)}\overset{Holder}{\geq}\frac{(\sum a)^3}{3\sum a(b+\lambda c)}=\frac{p^3}{3\cdot(\lambda+1)\sum bc}=\frac{p^3}{3(\lambda+1)q}=$$
$$=\frac{p^3}{3(\lambda+1)\cdot 3}=\frac{p^3}{9(\lambda+1)}.$$

It suffices to prove that:

$$(\lambda+1)\cdot\frac{p^3}{9(\lambda+1)}+2\sqrt{3}\cdot 3\geq 3p\Leftrightarrow\frac{p^3}{9}+6\geq 3p\Leftrightarrow p^3-27p+54\geq 0\Leftrightarrow$$
$$\Leftrightarrow(p-3)^2(p+6)\geq 0.$$

Equality holds if and only if $a=b=c$.

Note: For $\lambda=1$ we obtain the propose problem by Nguyen Viet Hung in Pure Inequalities 3/2022: If $a,b,c>0$ then

$$\sum\frac{a^2}{b+c}+\sqrt{3(ab+bc+ca)}\geq\frac{3}{2}(a+b+c)$$

Nguyen Viet Hung – Vietnam

Solution 2 by Soumava Chakraborty=Kolkata-India

$$(\lambda+1)\sum_{cyc}\frac{a^2}{b+\lambda c}+2\sqrt{3\sum_{cyc}ab}=(\lambda+1)\sum_{cyc}\frac{a^3}{ab+\lambda ca}+2\sqrt{3\sum_{cyc}ab}$$

$$\stackrel{Holder}{\geq}(\lambda+1).\frac{\left(\sum_{cyc}a\right)^3}{3(\lambda+1)\sum_{cyc}ab}+2\sqrt{3\sum_{cyc}ab}=\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}ab}+\sqrt{3\sum_{cyc}ab}+\sqrt{3\sum_{cyc}ab}$$

$$\stackrel{AM-GM}{\geq}3.\sqrt[3]{\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}ab}\sqrt{3\sum_{cyc}ab}.\sqrt{3\sum_{cyc}ab}}=3\sum_{cyc}a$$

$$\therefore(\lambda+1)\sum_{cyc}\frac{a^2}{b+\lambda c}+2\sqrt{3(ab+bc+ca)}\geq 3(a+b+c)\ \forall\, a,b,c>0\ and\ \lambda\geq 0,$$

$$''='' \text{ iff } a=b=c\ (QED)$$

JP.602 If $a,b,c>0$ and $\lambda\geq 0$ then:

$$2\sum\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum bc\geq 2\sum a^2$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using means inequality, we have:

$$\frac{2a^3}{b+\lambda c}+\frac{a(b+\lambda c)}{2}\geq 2\sqrt{\frac{2a^3}{b+\lambda c}\cdot\frac{a(b+\lambda c)}{2}}=2a^2,$$

with equality for $\frac{2a^3}{b+\lambda c}=\frac{a(b+\lambda c)}{2}\Leftrightarrow 2a=b+\lambda c.$

Writing the other analogs inequalities and summing we obtain:

$$2\sum\frac{a^3}{b+\lambda c}+\sum\frac{a(b+\lambda c)}{2}\geq 2\sum a^2\Leftrightarrow 2\sum\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum bc\geq 2\sum a^2$$

Equality holds if and only if $a=b=c$ and $\lambda=1$.

Solution 2 by Omkumar Rao-India

$$\sum_{cyc}\frac{a^3}{b+\lambda c}=\sum_{cyc}\frac{a^4}{ab+\lambda ac}$$

$$\textit{Applying Bergstrom's inequality}:$$

$$\sum_{cyc}\frac{a^4}{ab+\lambda ac}\geq\frac{(\sum a^2)^2}{\lambda\sum ab+\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}\geq 2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum ab\geq 2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}+\frac{\lambda+1}{2}\sum ab$$

$$\textit{Applying AM}-\textit{GM Inequality on RHS}:$$

$$2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}+\frac{\lambda+1}{2}\sum ab\geq 2\sqrt{\frac{2(\sum a^2)^2}{(\lambda+1)\sum ab}\cdot\frac{\lambda+1}{2}\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum ab\geq 2\sum a^2$$

$$Therefore, LHS\geq 2\sum a^2=RHS$$

JP.603 If $a,b,c>0$ then:

$$2\left(a^a+b^b+c^c\right)\geq a^b+a^c+b^a+b^c+c^a+c^b$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Lemma: If $a,b>0$ then:

$$a^a+b^b\geq a^b+b^a\quad (1)$$

Proof:

WLOG: $a\geq b\Rightarrow \ln a\geq \ln b$

$(a,b);(\ln a,\ln b)$ **are similarly ordered. By rearrangement's inequality:**

$$a\ln a+b\ln b\geq a\ln b+b\ln a$$

Let be $f:\mathbb{R}\to\mathbb{R}; f(x)=e^x,\ f'(x)=e^x; f''(x)=e^x>0\Rightarrow f$ **convexe**

By Karamata's inequality:

$$e^{a\ln a}+e^{b\ln b}\geq e^{a\ln b}+e^{b\ln a}$$

$$a^a+b^b\geq a^b+b^a$$

Back to the problem. By (1):

$$a^a+b^b\geq a^b+b^a$$

$$b^b+c^c\geq b^c+c^b\quad (2)$$

$$c^c+a^a\geq a^c+c^a\quad (3)$$

By adding (1); (2); (3):

$$2\left(a^a+b^b+c^c\right)\geq a^b+a^c+b^a+b^c+b^a+c^b+c^a$$

Equality holds for $a=b=c$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall\, x>0, x^x \overset{?}{\geq} x \Leftrightarrow x.\ln x \overset{?}{\geq} \ln x \Leftrightarrow (\ln x)(x-1) \overset{?}{\geq} 0 \to \text{true}\ \forall\, x \in (0,1)$$

and also true $\forall\, x \geq 1 \because \ln x \geq 0$ and $x-1 \geq 0 \therefore x^x \geq x\ \forall\, x>0 \to (1)$

We prove : $a^a + b^b \geq a^b + b^a\ \forall\, a,b>0$ and if $a=b=1$, then : LHS $-$ RHS $=0$ and if exactly one variable $=1$ and WLOG we may assume $a=1$, then remains to prove : $b^b \overset{?}{\geq} b \to$ true via (1); we now dive into the following cases :

$\boxed{\text{Case 1}}$ $a \geq b > 1$ or $b > 1 > a > 0$ and then : $a^a + b^b \overset{?}{\geq} a^b + b^a$

$$\Leftrightarrow b^{a-b}.\left(\left(\frac{a}{b}\right)^a - 1\right) \boxed{\overset{?}{\geq}_{(*)}} \left(\frac{a}{b}\right)^b - 1;\ \text{Now, } (a-b) \text{ and } (\ln a - \ln b) \text{ have same sign}$$

$$\therefore (a-b)\left(\ln\frac{a}{b}\right) \geq 0 \Rightarrow a\left(\ln\frac{a}{b}\right) \geq b\left(\ln\frac{a}{b}\right) \Rightarrow \left(\frac{a}{b}\right)^a \geq \left(\frac{a}{b}\right)^b$$

$$(\because \ln \text{ is an } \uparrow \text{ function}) \Rightarrow b^{a-b}.\left(\left(\frac{a}{b}\right)^a - 1\right) \geq b^{a-b}.\left(\left(\frac{a}{b}\right)^b - 1\right) \overset{?}{\geq} \left(\frac{a}{b}\right)^b - 1$$

$$\Leftrightarrow \left(\left(\frac{a}{b}\right)^b - 1\right)\left(b^{a-b} - 1\right) \boxed{\overset{?}{\geq}_{(**)}} 0$$

$$\text{For } a \geq b > 1, \text{we have} : b\left(\ln\frac{a}{b}\right) \geq 0 \text{ and } (a-b)(\ln b) \geq 0$$

$$\Rightarrow \left(\left(\frac{a}{b}\right)^b - 1\right), \left(b^{a-b} - 1\right) \geq 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\text{For } b > 1 > a > 0, \text{we have} : b\left(\ln\frac{a}{b}\right) \leq 0 \text{ and } (a-b)(\ln b) \leq 0$$

$$\Rightarrow \left(\left(\frac{a}{b}\right)^b - 1\right), \left(b^{a-b} - 1\right) \leq 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2}}$ $a > 1 > b > 0$ or $b \geq a > 1$ and then : $a^a + b^b \overset{?}{\geq} a^b + b^a$

$$\Leftrightarrow a^{a-b}.\left(1 - \left(\frac{b}{a}\right)^a\right) \boxed{\overset{?}{\geq}_{(\bullet)}} 1 - \left(\frac{b}{a}\right)^b;\ \text{Now, } (b-a) \text{ and } (\ln b - \ln a) \text{ have same sign}$$

$$\therefore (b-a)\left(\ln\frac{b}{a}\right) \geq 0 \Rightarrow b\left(\ln\frac{b}{a}\right) \geq a\left(\ln\frac{b}{a}\right) \Rightarrow 1 - \left(\frac{b}{a}\right)^a \geq 1 - \left(\frac{b}{a}\right)^b \Rightarrow$$

$$a^{a-b}\left(1 - \left(\frac{b}{a}\right)^a\right) \geq a^{a-b}.\left(1 - \left(\frac{b}{a}\right)^b\right) \overset{?}{\geq} 1 - \left(\frac{b}{a}\right)^b \Leftrightarrow \left(1 - \left(\frac{b}{a}\right)^b\right)\left(a^{a-b} - 1\right) \boxed{\overset{?}{\geq}_{(\bullet\bullet)}} 0$$

$$\text{For } a > 1 > b > 0, \text{we have} : 0 \geq b\left(\ln\frac{b}{a}\right) \text{ and } (a-b)(\ln a) \geq 0$$

$$\Rightarrow \left(1 - \left(\frac{b}{a}\right)^b\right), \left(a^{a-b} - 1\right) \geq 0 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\text{For } b \geq a > 1, \text{we have} : b\left(\ln\frac{b}{a}\right) \geq 0 \text{ and } (a-b)(\ln a) \leq 0$$

$$\Rightarrow\left(1-\left(\frac{b}{a}\right)^{b}\right).\left(a^{a-b}-1\right)\le 0\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}$$

$\boxed{\text{Case 3}}$ $1>a\ge b>0$ and then : $a^a+b^b\overset{?}{\ge}a^b+b^a\Leftrightarrow a^a-a^b\overset{?}{\ge}b^a-b^b$

Let $f(x)=x^a-x^b$ and so, we require to prove : $f(a)\overset{?}{\ge}f(b)$ and we have $f'(x)=ax^{a-1}-bx^{b-1}=x^{b-1}\left(ax^{a-b}-b\right)\to(2)$

Now, $(b-1)\ln x>0$ ($\because 1>a\ge x\ge b>0\Rightarrow(b-1),\ln x<0$) $\Rightarrow x^{b-1}>1>0$ and hence (2) $\Rightarrow$ in order to prove : $f'(x)\ge 0$, it suffices to prove :

$$ax^{a-b}-b\overset{?}{\ge}0\text{ and }(a-b)(\ln x-\ln b)\ge 0\therefore x^{a-b}\ge b^{a-b}\text{ and }\because a>0$$

$$\therefore\text{ it suffices to prove : }ab^{a-b}\overset{?}{\ge}b\Leftrightarrow a\ \boxed{\underset{(i)}{\overset{?}{\ge}}}\ b^{1-a+b}$$

$$\text{Let }a=b^m\text{ and then : }m-1=\frac{\ln a}{\ln b}-1=\frac{\ln a-\ln b}{\ln b}\le 0\ (\because a\ge b\text{ and }b<1)$$

$$\Rightarrow m\le 1\to(3)\text{ and now, (i)}\Leftrightarrow b^m\overset{?}{\ge}b^{1-b^m+b}\Leftrightarrow m\ln b\overset{?}{\ge}(1-b^m+b)\ln b$$

$$\Leftrightarrow m\overset{?}{\le}1-b^m+b\ (\because b<1\Rightarrow\ln b<0)\Leftrightarrow b^m\ \boxed{\underset{(ii)}{\overset{?}{\le}}}\ 1+b-m$$

Indeed via (3) and Bernoulli, $b^m\le 1+(b-1)m\overset{m\le 1}{\le}1-m+b\Rightarrow$ (ii) $\Rightarrow$ (i) is true $\Rightarrow f'(x)\ge 0\Rightarrow f(x)\uparrow$ on $[b,a]\Rightarrow f(a)\ge f(b)\Rightarrow a^a+b^b\ge a^b+b^a$

$\boxed{\text{Case 4}}$ $1>b\ge a>0$ and then : Let $F(x)=x^b-x^a$ & so, we require to prove : $F(b)\overset{?}{\ge}F(a)$ and we have $F'(x)=bx^{b-1}-ax^{a-1}=x^{a-1}\left(bx^{b-a}-a\right)\to(4)$

Now, $x^{a-1}>0\therefore(4)\Rightarrow$ to prove : $F'(x)\ge 0$, suffices to prove : $bx^{b-a}\overset{?}{\ge}a$ and $(b-a)(\ln x-\ln a)\ge 0\therefore x^{b-a}\ge a^{b-a}\therefore$ it suffices to prove : $ba^{b-a}\overset{?}{\ge}a$

$$\Leftrightarrow b\ \boxed{\underset{(m)}{\overset{?}{\ge}}}\ a^{1-b+a};\text{ Let }b=a^n;\text{ then : }n-1=\frac{\ln b-\ln a}{\ln a}\le 0\to(5)\text{ and now, (m)}\Leftrightarrow$$

$$a^n\overset{?}{\ge}a^{1-a^n+a}\Leftrightarrow n\ln a\overset{?}{\ge}(1-a^n+a)\ln a\Leftrightarrow n\overset{?}{\le}1-a^n+a\Leftrightarrow a^n\ \boxed{\underset{(n)}{\overset{?}{\le}}}\ 1+a-n$$

Bernoulli, (5) $\Rightarrow a^n\le 1+(a-1)n\overset{n\le 1}{\le}1-n+a\Rightarrow$ (n) is true $\Rightarrow F(x)\uparrow$ on $[a,b]$ $\Rightarrow F(b)\ge F(a)\therefore$ with all cases, $a^a+b^b\ge a^b+b^a$ & analogs $\therefore 2\left(a^a+b^b+c^c\right)\ge a^b+a^c+b^a+b^c+c^a+c^b\ \forall\, a,b,c>0$, " = " iff $a=b=c$ (QED)

JP.604 If $a,b>0;n\in\mathbb{N}^*$ then:

$$\frac{a^{2n}}{b^{2n}}+a^{2n}b^{4n}+\frac{1}{a^{4n}b^{2n}}\ge\frac{b}{a}+a^2b+\frac{1}{ab^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

For $n=1$ we will prove that:

$$\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\geq\frac{b}{a}+a^2b+\frac{1}{ab^2}\qquad (1)$$

Denote:

$$x=\frac{a}{b};y=ab^2;z=\frac{1}{a^2b}$$

Let's observe that $xyz=1$. Inequality (1) can be written:

$$x^2+y^2+z^2\geq\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\quad x^2+y^2+z^2\geq\frac{xy+yz+zx}{xyz}$$

$$x^2+y^2+z^2\geq xy+yz+zx$$

$$2x^2+2y^2+2z^2\geq 2xy+2yz+2zx$$

$$x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\geq 0$$

$$(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

Back to the problem:

$$\frac{a^{2n}}{b^{2n}}+a^{2n}b^{4n}+\frac{1}{a^{4n}b^{2n}}=\frac{\left(\frac{a^2}{b^2}\right)^n}{1^{n-1}}+\frac{(a^2b^4)^n}{2^{n-1}}+\frac{\left(\frac{1}{a^4b^2}\right)^n}{1^{n-1}}\overset{RADON}{\geq}$$

$$\geq\frac{\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)^n}{(1+1+1)^{n-1}}=\frac{\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)^{n-1}}{3^{n-1}}\cdot\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)\geq$$

$$\overset{AM-GM}{\geq}\frac{\left(3\sqrt[3]{\frac{a^2}{b^2}\cdot a^2b^4\cdot\frac{1}{a^4b^2}}\right)^{n-1}}{3^{n-1}}\cdot\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)\geq$$

$$\overset{(1)}{\geq}\frac{3^{n-1}}{3^{n-1}}\cdot\left(\frac{b}{a}+a^2b+\frac{1}{ab^2}\right)=\frac{b}{a}+a^2b+\frac{1}{ab^2}$$

Equality holds for $a=b=1$.

Solution 2 by Tapas Das-India

$$Let\ x=\frac{a^{2n}}{b^{2n}},y=a^{2n}b^{4n},z=\frac{1}{a^{4n}b^{2n}}\ then\ \frac{b}{a}=x^{\frac{1}{2n}},a^2b=y^{\frac{1}{2n}},\frac{1}{ab^2}=z^{\frac{1}{2n}},xyz=1$$

$$so\ we\ need\ to\ show,x+y+z\geq x^r+y^r+z^r,where\ r=\frac{1}{2n}<1$$

$$\frac{x+y+z}{3}\overset{AM-GM}{\geq}\sqrt[3]{xyz}\overset{xyz=1}{=}1 \text{ and } \frac{x^r+y^r+z^r}{3}\leq\left(\frac{x+y+z}{3}\right)^r \text{ as } r\in(0,1)$$

$$\text{so clearly}, \frac{x^r+y^r+z^r}{3}\leq\left(\frac{x+y+z}{3}\right)^r\leq\frac{x+y+z}{3}\left(\text{as } \frac{x+y+z}{3}\geq 1 \ \&\ r\in(0,1)\right)$$

$$\text{so,}\quad x+y+z\geq x^r+y^r+z^r$$

Equality holds for x=y=z=1 or a=b=1

Solution 3 by Marin Chirciu-Romania

Using substitutions $(x,y,z)=\left(\frac{a}{b},ab^2,\frac{1}{a^2b}\right)$, $x,y,z>0$, $xyz=1$ **the inequality can be written:**

$$x^{2n}+y^{2n}+z^{2n}\geq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Leftrightarrow x^{2n}+y^{2n}+z^{2n}\geq xy+yz+zx \text{ see:}$$

$$x^{2n}+y^{2n}+z^{2n}\overset{SOS}{\geq}x^ny^n+y^nz^n+z^nx^n\overset{Lema}{\geq}xy+yz+zx.$$

Lemma.

If $A,B,C>0$, $ABC=1$ **and** $n\in\mathbf{N}^*$ **then**

$$A^n+B^n+C^n\geq A+B+C.$$

Proof:

For $n=1$ **we obtain:** $A+B+C=A+B+C$.

For $n\geq 2$:

$$\sum A^n\overset{Holder}{\geq}\frac{\left(\sum A\right)^n}{3^{n-1}}=\sum A\left(\frac{\sum A}{3}\right)^{n-1}\overset{AM-GM}{\geq}\sum A\left(\sqrt[3]{ABC}\right)^{n-1}=\sum A.$$

Lemma is used for $(A,B,C)=(xy,yz,zx)$, $xyz=1\Rightarrow ABC=1$.

Equality holds for $x=y=z\Leftrightarrow\frac{a}{b}=ab^2=\frac{1}{a^2b}\Leftrightarrow a=b=1$.

JP.605 Find $x,y\in\mathbb{R}$ such that:

$$4^{x+y}+\sqrt{2026^x+2026^y-2}=2^{x+y+1}-1.$$

Proposed by Adrian Gobej – Romania

Solution 1 by proposer

The relationship in the statement is written $\sqrt{2026^x+2026^y-2}+(2^{x+y}-1)^2=0\Rightarrow$

$$\Rightarrow\begin{cases}\sqrt{2026^x+2026^y-2}=0(R_1)\\ \text{and}\\ 2^{x+y}-1=0\Rightarrow x=-y\end{cases}$$

From $x=-y \Rightarrow$ relationship (R_1) becomes $2026^x+2026^{-x}=2 \Leftrightarrow$

$\Leftrightarrow 2026^x+\frac{1}{2026^x}=2 \Leftrightarrow (2026^x-1)^2=0 \Leftrightarrow 2026^x=1 \Leftrightarrow x=0$ and so $x=y=0$.

Solution 2 by Omkumar Rao-India

$4^{x+y}-2^{x+y+1}+1 \quad +\sqrt{2026^x+2026^y-2}$=0

$\underbrace{(2^{x+y}-1)^2}_{\geq 0}+\underbrace{\sqrt{2026^x+2026^y-2}}_{\geq 0}$=0

$(2^{x+y}-1)^2$=0$\Rightarrow 2^{x+y}$=2^0 ⟹y=-x

$\sqrt{2026^x+2026^y-2}$=0$:\ (y=-x)$

$2026^x+\frac{1}{2026^x}$=2. By AM-GM Inequality, for m>0: m+$\frac{1}{m}\geq 2$

Hence: 2026^x=1⟹x=0 and since x+y=0, y=0. Therefore,the unique solution is (x,y)=$(0,0)$.

JP.606 Solve the following equation:

$$2016^{2-x^2}+2016^{2017x^2-4\sqrt{x}-2013}=2017$$

Proposed by Adrian Gobej – Romania

Solution by proposer

Conditions $x\in[0;+\infty)$. LHS can be written:

$$E(x)=2016\cdot 2016^{1-x^2}+2016^{2017x^2-4\sqrt{x}-2013}=$$

$$=\underbrace{2016^{1-x^2}+2016^{1-x^2}+\cdots+2016^{1-x^2}}_{2016\ terms}+\underbrace{2016^{2017x^2-4\sqrt{x}-2013}}_{1\ term}\overset{AM\geq GM}{\geq}$$

$$\overset{AM\geq GM}{\geq} 2017\sqrt[2017]{\underbrace{2016^{1-x^2}\cdot 2016^{1-x^2}\cdot\ldots\cdot 2016^{1-x^2}}_{2016\ terms}\cdot 2016^{2017x^2-4\sqrt{x}-2013}}=$$

$$=2017\sqrt[2017]{2016^{2016-2016x^2}\cdot 2016^{2017x^2-4\sqrt{x}-2013}}=$$

$$=2017\sqrt[2017]{2016^{2016-2016x^2+2017x^2-4\sqrt{x}-2013}}=$$

$$=2017\sqrt[2017]{2016^{x^2-4\sqrt{x}+3}}\ (R_1)$$

As $x^2-4\sqrt{x}+3 \underset{not\ \sqrt{x}=t}{=} t^4-4t+3=(t-1)^2(t^2+2t-3)=$ (returning to the notation)$=\left(\sqrt{x}-1\right)^2\left(x+2\sqrt{x}-3\right)$ we obtain using (R_1):

$$\left.\begin{array}{r} E(x) \ge 2017\sqrt[2017]{2016^{(\sqrt{x}-1)^2(x+2\sqrt{x}+3)}} \\ \left.\begin{array}{r} As\ (\sqrt{x}-1)^2 \ge 0 \\ x+2\sqrt{x}+3 = (\sqrt{x}+1)^2+2 \ge 2 \end{array}\right\} \Rightarrow (\sqrt{x}-1)^2(x+2\sqrt{x}+3) \ge 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow E(x) \ge 2017\sqrt[2017]{2016^0} = 2017.$$

As in the previous inequalities we have equality if and only if $x = 1 \Rightarrow x = 1$ is a unique solution of the given equation.

JP.607 Solve for reals:

$$\sqrt[4]{8x-7} + \sqrt[4]{3-2x^4} = 2$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Existence conditions: $8x-7 \ge 0$ and $3-2x^4 \ge 0$.

With means inequality we obtain:

1. $(8x-7)+1+1+1 \ge 4\sqrt[4]{(8x-7)\cdot 1\cdot 1\cdot 1} = 4\sqrt[4]{8x-7}$, with equal for $8x-7 = 1 \Leftrightarrow x = 1$.

2. $(3-2x^4)+1+1+1 \ge 4\sqrt[4]{(3-2x^4)\cdot 1\cdot 1\cdot 1} = 4\sqrt[4]{3-2x^4}$, with equal for $3-2x^4 = 1 \Leftrightarrow x = 1$.

It follows:

$$2 = \sqrt[4]{8x-7} + \sqrt[4]{3-2x^4} \le \frac{(8x-7)+1+1+1}{4} + \frac{(3-2x^4)+1+1+1}{4} =$$

$$= \frac{8x-2x^4+2}{4} \Rightarrow 2 \le \frac{8x-2x^4+2}{4} \Leftrightarrow x^4-4x+3 \le 0 \Leftrightarrow$$

$$\Leftrightarrow (x-1)^2(x^2+2x+3) \le 0 \Leftrightarrow x = 1.$$

We deduce that $x = 1$ is the unique solution of the equation.

Solution 2 by Jose Luis Diaz – Barrero – Spain

For the fourth roots to be defined over the real numbers, the radicands must be non-negative:

1. $8x-7\ge 0\Rightarrow x\ge\frac{7}{8}$

2. $3-2x^4\ge 0\Rightarrow x^4\le\frac{3}{2}\Rightarrow x\le\sqrt[4]{\frac{3}{2}}$

Thus, any real solution x must lie within the interval:

$$x\in\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$$

By inspection, we test $x=1$, which lies within our valid domain:

$$\sqrt[4]{8(1)-7}+\sqrt[4]{3-2(1)^4}=\sqrt[4]{1}+\sqrt[4]{1}=1+1=2$$

Thus, $x=1$ is a valid real solution.

Let us define the function $f(x)=\sqrt[4]{8x-7}+\sqrt[4]{3-2x^4}$ on the interval $\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$.

To determine its behavior, we compute its first derivative $f'(x)$:

$$f'(x)=\frac{1}{4}(8x-7)^{-\frac{3}{4}}\cdot 8+\frac{1}{4}(3-2x^4)^{-\frac{3}{4}}\cdot(-8x^3)$$

$$f'(x)=\frac{2}{(8x-7)^{\frac{3}{4}}}-\frac{2x^3}{(3-2x^4)^{\frac{3}{4}}}$$

Setting $f'(x)=0$ to find the critical points:

$$\frac{2}{(8x-7)^{\frac{3}{4}}}=\frac{2x^3}{(3-2x^4)^{\frac{3}{4}}}\Rightarrow\frac{1}{(8x-7)^{\frac{3}{4}}}=\frac{x^3}{(3-2x^4)^{\frac{3}{4}}}$$

Raising both sides to the power of $\frac{4}{3}$ yields:

$$\frac{1}{8x-7}=\frac{x^4}{3-2x^4}\Leftrightarrow 8x^5-5x^4-3=0$$

Since we know $x=1$ satisfies this equation, we can factor out $(x-1)$ using polynomial long division or synthetic division:

$$(x-1)(8x^4+3x^3+3x^2+3x+3)=0$$

For any x in our domain $\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$, the value of x is strictly positive. Consequently, the polynomial factor $(8x^4+3x^3+3x^2+3x+3)$ is always strictly greater than zero and has no real roots. Therefore, $x = 1$ is the unique critical point of $f(x)$ in this domain. We can check the sign of $f'(x)$ around $x = 1$:

- For $\frac{7}{8} \le x < 1$, we have $f'(x) > 0$, meaning $f(x)$ is strictly increasing.
- For $1 < x \le \sqrt[4]{1.5}$, we have $f'(x) < 0$, meaning $f(x)$ is strictly decreasing.

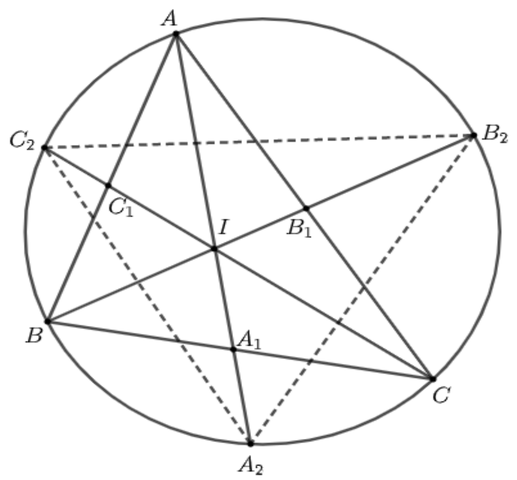
Since $f(x)$ strictly increases to a peak at $x = 1$ and strictly decreases thereafter, $x = 1$, is a strict global maximum. The maximum value achieved by the function is exactly 2. Thus, the equation $f(x) = 2$ has exactly one real solution $x = 1$.

JP.608 Let be the triangle ABC, AA_1, BB_1, CC_1 internal bisectors and A_2, B_2, C_2 contact points to the bisectors with the circumcircle of the triangle. Prove that: $A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \ge Rs$ where p represent the semiperimeter and R the circumradii of triangle ABC.

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

We have $\rho(A_1) = AA_1 \cdot A_1A_2 = BA_1 \cdot A_1C \Rightarrow A_1A_2 = \frac{BA_1 \cdot A_1C}{A_1A_2}$ **(1)**



From Bisector Theorem, we get: $BA_1 = \frac{ac}{b+c}$ and $CA_1 = \frac{ab}{b+c}$ **(2)**

But $AA_1 = \frac{2bc}{b+c} \cos\frac{A}{2}$ **(3)**

From (1), (2) and (3), it follows that: $A_1A_2 = \frac{a^2}{2(b+c)\cos\frac{A}{2}}$ (4)

From Law of sines: $\frac{B_2C_2}{\sin\left(\frac{B+C}{2}\right)} = 2R \Rightarrow B_2C_2 = 2R \cdot \cos\frac{A}{2}$ (5)

From (4) and (5) we get: $AA_2 \cdot B_2C_2 = \frac{a^2R}{b+c}$

Therefore, by adding we obtain:

$$\sum_{cyc} A_1A_2 \cdot B_2C_2 = R\sum_{cyc}\frac{a^2}{b+c} \overset{Bergstrom}{\geq} R \cdot \frac{(a+b+c)^2}{2(a+b+c)} = R \cdot \frac{a+b+c}{2} = Rs.$$

JP.609 In acute triangle ABC, A', B', C' are symmetric points of the points A, B, C to the sides BC, AC, and AB respectively. Prove that:

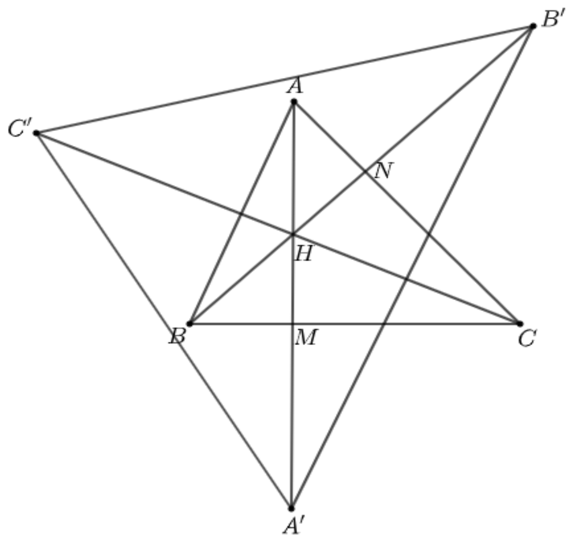
$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 4\left(\frac{r}{R}\right)^2 + 8 \cdot \frac{r}{R} - 1$$

where $\sigma[ABC]$ represent area of ΔABC.

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

Let be $\{H\} = AA' \cap BB' \cap CC'$. From $h_a = 2R\sin B \sin C$ and $HM = 2R\cos B\cos C$,



then $HA' = 2R\cos(B-C)$ and analogs. Hence, we get:

$$\sigma[A'B'C'] = \frac{1}{2}(HA' \cdot HB' \cdot \sin C + HB' \cdot HC' \cdot \sin A + HC' \cdot HA' \cdot \sin B) =$$

$$= 2R^2 \sum_{cyc} \cos(B-C)\cos(C-A)\sin C =$$

$$= 2R^2 \sin A \sin B \sin C \sum_{cyc} \frac{\cos(B-C)\cos(C-A)}{\sin A \sin B} \quad (1)$$

But $\sigma[ABC] = 2R^2 \sin A \sin B \sin C$ and

$$\sum_{cyc} \frac{\cos(B-C)\cos(C-A)}{\sin A \sin B} = 3 + 8\cos A \cos B \cos C \quad (2)$$

From (1) and (2), it follows: $\frac{\sigma[A'B'C']}{\sigma[ABC]} = 3 + 8\cos A \cos B \cos C \quad (3)$

But $\cos A \cos B \cos C = \frac{s^2-(2R+r)^2}{4R^2}$ (4)

From (3) and (4), it follows:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 3 + \frac{2(s^2-4R^2-4Rr-r^2)}{R^2} \quad (5)$$

Using Walker's inequality $s^2 \geq 2R^2 + 8Rr + 3r^2$ to obtain:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} \geq 3 - \frac{4R^2 + 8Rr + 4r^2}{R^2} = 4\left(\frac{r}{R}\right)^2 + 8 \cdot \frac{r}{R} - 1$$

JP.610 Solve for real numbers:

$$\log_{2\sqrt{8+2\sqrt{15}}}(x^2+x+2) = \log_{\sqrt{4+\sqrt{15}}}(x^2+x+1)$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

Let be $\log_{2\sqrt{8+2\sqrt{15}}}(x^2+x+2) = \log_{\sqrt{4+\sqrt{15}}}(x^2+x+1) = y \Rightarrow$

$$\begin{cases} x^2+x+1 = \left(\sqrt{4+\sqrt{15}}\right)^y \\ x^2+x+2 = \left(2\sqrt{8+2\sqrt{15}}\right)^y \end{cases} \Leftrightarrow \left(4+\sqrt{15}\right)^y + 1 = \left(2\sqrt{2} \cdot \sqrt{4+\sqrt{15}}\right)^y.$$

We observe that $y = 2$ is a solution, $\left(4+\sqrt{15}\right)^2 + 1 = \left(2\sqrt{2} \cdot \sqrt{4+\sqrt{15}}\right)^y \Leftrightarrow$

$32 + 8\sqrt{15} = 32 + 8\sqrt{15}$ true. Let us prove that $y = 2$ is the unique solution:

$$1+\left(\frac{1}{4+\sqrt{15}}\right)^y=\left(\frac{2\sqrt{2}}{\sqrt{4+\sqrt{15}}}\right)^y$$

Let be the function $f(y)=1+\left(\frac{1}{4+\sqrt{15}}\right)^y$ strictly decreases because $\frac{1}{4+\sqrt{15}}<1$,

and $g(y)=\left(\frac{2\sqrt{2}}{\sqrt{4+\sqrt{15}}}\right)^y$ strictly increases function because $\frac{2\sqrt{2}}{\sqrt{4+\sqrt{15}}}>1$.

Hence, $y=2$ is the unique solution i.e. $x^2+x+1=4+\sqrt{15}\Leftrightarrow x^2+x-3-\sqrt{15}=0$

with $x_{1,2}=\frac{-1\pm\sqrt{3+5\sqrt{15}}}{2}$

JP.611 Find the angle between the real plans:

$$P_1: 2x+y+3z-1=0$$

$$P_2: 3x-2y+z+1=0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

The normal vectors of P_1, P_2 are:

$$\overrightarrow{u_1}(2,1,3)=2\vec{\imath}+\vec{\jmath}+3\vec{k}$$

$$\overrightarrow{u_2}(3,-2,1)=3\vec{\imath}-2\vec{\jmath}+\vec{k}$$

$$\overrightarrow{u_1}\cdot\overrightarrow{u_2}=2\cdot3+1\cdot(-2)+3\cdot1=6-2+3=7$$

$$|\overrightarrow{u_1}|=\sqrt{2^2+1^2+3^2}=\sqrt{14}$$

$$|\overrightarrow{u_2}|=\sqrt{3^2+(-2)^2+1}=\sqrt{14}$$

$$\cos(\sphericalangle(\overrightarrow{u_1},\overrightarrow{u_2}))=\frac{\overrightarrow{u_1}\cdot\overrightarrow{u_2}}{|\overrightarrow{u_1}|\cdot|\overrightarrow{u_2}|}=\frac{7}{\sqrt{14}\cdot\sqrt{14}}=\frac{7}{14}=\frac{1}{2}$$

$$\sphericalangle(\overrightarrow{u_1},\overrightarrow{u_2})=\sphericalangle(P_1,P_2)=\frac{\pi}{3}$$

Solution 2 by José Luis Díaz – Barrero – Spain

The acute angle θ between two planes is equal to the angle between their normal vectors, which is given by the formula:

$$\cos\theta=\frac{|\overrightarrow{n_1}\cdot\overrightarrow{n_2}|}{\left||\overrightarrow{n_1}|\right|\left||\overrightarrow{n_2}|\right|}$$

The normal vector coefficients correspond directly to the coefficients of x, y and z in each plane equation:

$$\overrightarrow{n_1} = (2, 1, 3)$$
$$\overrightarrow{n_2} = (3, -2, 1)$$
$$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = (2)(3) + (1)(-2) + (3)(1) = 6 - 2 + 3 = 7$$
$$||\overrightarrow{n_1}|| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$
$$||\overrightarrow{n_2}|| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

Substituting the values back into the cosine formula:

$$\cos\theta = \frac{|7|}{\sqrt{14} \cdot \sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

Taking the inverse cosine ($\arccos$):

$$\theta = \arccos\left(\frac{1}{2}\right) = 60^\circ \quad \text{(or } \frac{\pi}{3} \text{ radians)}$$

Thus, the angle between the two planes is 60°.

JP.612 Find the angle between the line d and the real plan P.

$$d: \begin{cases} x = 2 + 3t \\ y = 3 - 2t \,; t \in \mathbb{R}; P: 2x + y + z - 5 = 0 \\ z = 1 + 5t \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$d: \begin{cases} x - 2 = 3t \\ y - 3 = -2t \\ z - 1 = 5t \end{cases} \Rightarrow \begin{cases} \dfrac{x-2}{3} = t \\ \dfrac{y-3}{-2} = t \,; t \in \mathbb{R} \\ \dfrac{z-1}{5} = t \end{cases}$$

$$d: \frac{x-2}{3} = \frac{y-3}{-2} = \frac{z-1}{5}$$

The directory vector of d is: $\vec{v}(3, -2, 5) = 3\vec{i} - 2\vec{j} + 5\vec{k}$

The normal vector of P is: $\vec{u}(2, 1, 1) = 2\vec{i} + \vec{j} + \vec{k}$

$$\vec{u} \cdot \vec{v} = 3 \cdot 2 - 2 \cdot 1 + 5 \cdot 1 = 6 - 2 + 5 = 9$$

$$|\vec{u}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$|\vec{v}| = \sqrt{3^2 + (-2)^2 + 5^2} = \sqrt{9 + 5 + 25} = \sqrt{38}$$

$$\cos(\sphericalangle(\vec{u}, \vec{v})) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{9}{\sqrt{6} \cdot \sqrt{38}} = \frac{9}{2\sqrt{3 \cdot 19}} = \frac{9\sqrt{57}}{2 \cdot 3 \cdot 19}$$

$$\cos(\sphericalangle(\vec{u}, \vec{v})) = \frac{3\sqrt{57}}{38}$$

$$\mu(\sphericalangle(\vec{u}, \vec{v})) = \arccos\left(\frac{3\sqrt{57}}{38}\right)$$

$$\sphericalangle(d, P) = \frac{\pi}{2} - \arccos\left(\frac{3\sqrt{57}}{38}\right) = arcsin\left(\frac{3\sqrt{57}}{38}\right)$$

Solution 2 by José Luis Díaz – Barrero – Spain

To find the angle θ between a line and a plane, we use the relationship between the direction vector of the line $\vec{u}$ and the normal vector of the plane $\vec{n}$. Because the normal vector is perpendicular to the plane, the angle α between $\vec{u}$ and $\vec{n}$ is complementary to the angle θ we want to find $\theta = 90^\circ - \alpha$. Therefore, we use the sine formula:

$$\sin(\theta) = \frac{|\vec{u} \cdot \vec{n}|}{||\vec{u}|| ||\vec{n}||}$$

- Direction vector of the line $\vec{u}$: Look at the coefficients of the parameter t in the line's parametric equations, to obtain: $\vec{u} = (3, -2, 5)$
- Normal vector of the plane $\vec{n}$: Look at the coefficients of x, y, and z in the plane's equation, to obtain: $\vec{n} = (2, 1, 1)$

Then,

$$\vec{u} \cdot \vec{n} = (3)(2) + (-2)(1) + (5)(1) = 9,$$

and $|\vec{u} \cdot \vec{n}| = 9$.

On the other hand,

- Magnitude of $\vec{u}$:

$$||\vec{u}|| = \sqrt{3^2 + (-2)^2 + 5^2} = \sqrt{9 + 4 + 25} = \sqrt{38}$$

- **Magnitude of $\vec{n}$:**

$$||\vec{n}|| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

Therefore,

$$\sin(\theta) = \frac{9}{\sqrt{38}\cdot\sqrt{6}} = \frac{9}{\sqrt{228}} = \frac{9}{2\sqrt{57}},$$

and the exact angle is given by

$$\theta = \arcsin\left(\frac{9}{2\sqrt{57}}\right) = arcsin\left(\frac{9\sqrt{57}}{2\cdot 3\cdot 19}\right) = arcsin\left(\frac{3\sqrt{57}}{38}\right)$$

Solution 3 by Omkumar Rao-India

$$\textit{The angle between a line and a plane is determined by the angle between the line's direction vector and the plane's normal sector. For line d, the direction vector is } \vec{d} = (3,-2,5),$$

$$\textit{and for plane P, the normal vector is } \vec{n} = (2,1,1).$$

$$\textit{The angle } \theta \textit{ between the line and the plane is given by:}$$

$$\sin\theta = \frac{|\vec{d}.\vec{n}|}{|\vec{d}|.|\vec{n}|}, \qquad \vec{d}.\vec{n} = 9$$

$$|\vec{d}| = \sqrt{38}, \qquad |\vec{n}| = \sqrt{6}$$

$$\sin\theta = \frac{3\sqrt{57}}{38}$$

$$\textit{The angle between the line d and the plane P is } \theta = \sin^{-1}\left(\frac{3\sqrt{57}}{38}\right)$$

JP.613 Find the angle between the lines:

$$d_1: \begin{cases} x = 3 - 2t \\ y = 1 + t \\ z = -1 + 3t \end{cases}; d_2: \begin{cases} x = 2 + 4t \\ y = -1 + 2t \\ z = 4 - t \end{cases}; t \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$d_1:\begin{cases}\frac{x-3}{-2}=t\\ \frac{y-1}{1}=t\\ \frac{z+1}{3}=t\end{cases};d_2:\begin{cases}\frac{x-2}{4}=t\\ \frac{y+1}{2}=t\\ \frac{z-4}{-1}=t\end{cases};t\in\mathbb{R}$$

$$d_1:\frac{x-3}{-2}=\frac{y-1}{1}=\frac{z+1}{3};d_2:\frac{x-2}{4}=\frac{y+1}{2}=\frac{z-4}{-1}$$

The directions vectors of d_1,d_2:

$$\overrightarrow{u_1}(-2,1,3)=-2\vec{\imath}+\vec{\jmath}+3\vec{k}$$

$$\overrightarrow{u_2}(4,2,-1)=4\vec{\imath}+2\vec{\jmath}-\vec{k}$$

$$\overrightarrow{u_1}\cdot\overrightarrow{u_2}=-2\cdot 4+1\cdot 2+3\cdot(-1)=-8+2-3=-9$$

$$|\overrightarrow{u_1}|=\sqrt{(-2)^2+1^2+3^2}=\sqrt{4+1+9}=\sqrt{14}$$

$$|\overrightarrow{u_2}|=\sqrt{4^2+2^2+(-1)^2}=\sqrt{16+4+1}=\sqrt{21}$$

$$\cos\left(\sphericalangle(\overrightarrow{u_1},\overrightarrow{u_2})\right)=\frac{\overrightarrow{u_1}\cdot\overrightarrow{u_2}}{|\overrightarrow{u_1}|\cdot|\overrightarrow{u_2}|}=\frac{-9}{\sqrt{14}\cdot\sqrt{21}}=\frac{-9}{7\sqrt{6}}=\frac{-9\sqrt{6}}{7\cdot 6}=\frac{-3\sqrt{6}}{14}$$

$$\mu\left(\sphericalangle(\overrightarrow{u_1},\overrightarrow{u_2})\right)=\arccos\left(\frac{-3\sqrt{6}}{14}\right)=\pi-\arccos\left(\frac{3\sqrt{6}}{14}\right)$$

Solution 2 by José Luis Díaz – Barrero – Spain

To find the acute angle θ between the two lines d_1 and d_2, we need to find the angle between their respective direction vectors. These vectors are formed by the coefficients of the parameter t, as it is well-known.

For the first line d_1, the direction vector is $\overrightarrow{v_1}=(-2,1,3)$.

For the second line d_2, the direction vector is $\overrightarrow{v_2}=(4,2,-1)$.

The acute angle θ between two lines with direction vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ is given by:

$$\cos(\theta)=\frac{|\overrightarrow{v_1}\cdot\overrightarrow{v_2}|}{||\overrightarrow{v_1}||\,||\overrightarrow{v_2}||}$$

1. Dot Product $(\overrightarrow{v_1}\cdot\overrightarrow{v_2})$:

$$\overrightarrow{v_1}\cdot\overrightarrow{v_2}=(-2)(4)+(1)(2)+(3)(-1)=-8+2-3=-9$$

Taking the absolute value yields: $|\overrightarrow{v_1}\cdot\overrightarrow{v_2}|=9$.

2. Magnitude of $\overrightarrow{v_1}$:

$$\left||\overrightarrow{v_1}|\right|=\sqrt{(-2)^2+1^2+3^2}=\sqrt{4+1+9}=\sqrt{14}$$

3. Magnitude of $\overrightarrow{v_2}$:

$$\left||\overrightarrow{v_2}|\right|=\sqrt{4^2+2^2+(-1)^2}=\sqrt{16+4+1}=\sqrt{21}$$

4. Product of Magnitudes:

$$\left||\overrightarrow{v_1}|\right|\left||\overrightarrow{v_2}|\right|=\sqrt{14}\cdot\sqrt{21}=\sqrt{294}=7\sqrt{6}$$

Substituting the calculated values into the formula, yields

$$\cos(\theta)=\frac{9}{7\sqrt{6}}=\frac{3\sqrt{6}}{14}$$

Finally, taking the inverse cosine, we obtain

$$\theta=\arccos\left(\frac{3\sqrt{6}}{14}\right)$$

Solution 3 by Omkumar Rao-India

The angle between two lines is determined by the angle between their direction vectors. For line d_1, the direction vector is $\vec{v}_1=(-2,1,3)$, and for line d_2, the direction vector: $\vec{v}_2=(4,2,-1)$

The angle θ between the lines is given by:

$$\cos\theta=\frac{|\vec{v}_1.\vec{v}_2|}{|\vec{v}_1|.|\vec{v}_2|}$$

$$\vec{v}_1.\vec{v}_2=-9$$

$$|\vec{v}_1|=\sqrt{14}\ ,|\vec{v}_2|=\sqrt{21}$$

$$\cos\theta=\frac{3\sqrt{6}}{14}$$

The angle between the lines d_1 and d_2 is

$$\theta=\cos^{-1}\left(\frac{3\sqrt{6}}{14}\right)$$

JP.614 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}\geq 288r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}=\sum_{cyc}\frac{a^4+b^4}{h_c}\ge$$

$$\overset{AM-GM}{\ge}\sum_{cyc}\frac{2a^2b^2}{h_c}=\sum_{cyc}\frac{2a^2b^2}{\frac{2F}{c}}=\frac{1}{F}\sum_{cyc}a^2b^2c=$$

$$=\frac{abc}{F}\sum_{cyc}ab=\frac{4RF}{F}(s^2+r^2+4Rr)\ge 4R\left(\frac{27R^2}{4}+r^2+4Rr\right)\overset{EULER}{\ge}$$

$$\ge 4\cdot 2r\left(\frac{27}{4}\cdot 4r^2+r^2+4\cdot 2r\cdot r\right)=8r(27R^2+9r^2)=8r\cdot 36r^2=288r^3$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu – Romania

$$\sum\frac{b^4+c^4}{h_a}=\sum\frac{b^4}{h_a}+\sum\frac{c^4}{h_a}\overset{Holder}{\ge}\frac{\left(\sum b\right)^4}{9\sum h_a}+\frac{\left(\sum c\right)^4}{9\sum h_a}=\frac{(2p)^4+(2p)^4}{9\sum h_a}=\frac{32p^4}{9\sum h_a}\overset{Santalo}{\ge}\frac{32p^4}{9\cdot p\sqrt{3}}=$$

$$=\frac{32p^3}{9\sqrt{3}}\overset{Mitinovic}{\ge}\frac{32\left(3r\sqrt{3}\right)^3}{9\sqrt{3}}=\frac{32\cdot 27\cdot 3\sqrt{3}r^3}{9\sqrt{3}}=288r^3=RHS\ .$$

Equality holds for $a=b=c$.

Solution 3 by Omkumar Rao-India

$$\textit{The altitudes are given by}: h_c=\frac{2S}{c}, h_a=\frac{2S}{a}, h_b=\frac{2S}{b}$$
$$: S-\textit{Area of }\Delta ABC, s-\textit{semiperimeter}, r-\textit{inradius}, R-\textit{circumradius}$$

$$LHS=\frac{c(a^4+b^4)}{2S}+\frac{a(b^4+c^4)}{2S}+\frac{b(c^4+a^4)}{2S}$$

$$\textit{By } AM-GM, x^4+y^4\ge 2x^2y^2$$
$$\textit{for all real } x,y.$$

$$LHS\ge\frac{a.2b^2c^2+b.2c^2a^2+c.2a^2b^2}{2S}$$
$$LHS\ge\frac{2abc(ab+bc+ca)}{2S}$$

$$By\ extended\ law\ of\ sines\ abc = 4RS$$

$$LHS \geq 4R(ab + bc + ca)$$

$$Euler's\ Inequality: R \geq 2r$$
$$ab + bc + ca = s^2 + 4Rr + r^2$$
$$s \geq 3\sqrt{3}r$$
$$LHS \geq 4.2r.36r^2 = 288r^3$$
$$Equality\ holds\ when\ both\ Euler's$$
$$Inequality\ and\ the\ side - inradius\ inequality\ are\ tight, which$$
$$occurs\ if\ and\ only\ if\ \Delta ABC\ is\ equilateral.$$

JP.615 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c^2} + \frac{b^4+c^4}{h_a^2} + \frac{c^4+a^4}{h_b^2} \geq 96r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c^2} + \frac{b^4+c^4}{h_a^2} + \frac{c^4+a^4}{h_b^2} = \sum_{cyc} \frac{a^4+b^4}{h_c^2} \geq$$

$$\overset{AM-GM}{\geq} \sum_{cyc} \frac{2a^2b^2}{\frac{4F^2}{c^2}} = \frac{1}{2F^2}\sum_{cyc} a^2b^2c^2 = \frac{3}{2F^2}\cdot(4RF)^2 = \frac{3}{2F^2}\cdot 16R^2F^2 =$$

$$= 24R^2 \overset{EULER}{\geq} 24\cdot(2r)^2 = 24\cdot 4r^2 = 96r^2$$

Equality holds for $a = b = c$.

Solution 2 by Marin Chirciu– Romania

$$\sum\frac{b^4+c^4}{h_a^2} = \sum\frac{b^4}{h_a^2} + \sum\frac{c^4}{h_a^2} \overset{Holder}{\geq} \frac{\left(\sum b\right)^4}{9\sum h_a^2} + \frac{\left(\sum c\right)^4}{9\sum h_a^2} = \frac{(2p)^4+(2p)^4}{9\sum h_a^2} = \frac{32p^4}{9\sum h_a^2} \leq \frac{32p^4}{9\cdot p^2} =$$

$$= \frac{32p^2}{9} \overset{Mitinovic}{\geq} \frac{32\left(3r\sqrt{3}\right)^2}{9} = \frac{32\cdot 27\cdot r^2}{9} = 96r^2 = RHS.$$

Equality holds for $a = b = c$.

Solution 3 by Omkumar Rao-India

The altitudes are:

$$h_a = \frac{2S}{a}, h_b = \frac{2S}{b}, h_c = \frac{2S}{c}, \qquad LHS = \frac{c^2(a^4+b^4)+a^2(b^4+c^4)+b^2(c^4+a^4)}{4S^2}$$

$$By\ AM-GM, x^4+y^4 \ge 2x^2y^2\ for\ all\ real\ x,y.$$

$$LHS \ge \frac{6a^2b^2c^2}{4S^2}, \qquad By\ extended\ law\ of\ sines: abc = 4RS$$

$$Euler's\ Inequality: R \ge 2r$$

$$LHS \ge 96r^2$$

Equality holds if and only if both the AM − GM Inequality and Euler's Inequality are tight, which occurs if and only if ΔABC is equilateral.

PROBLEMS FOR SENIORS

SP.601 What is the largest positive value of the power k such that

$$a^k + b^k + c^k \le 3$$

for all nonnegative real numbers a, b, c, d with $a \le b \le c \le d$ and $ab + bc + cd + da = 4$?

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a = 0$ and $b = c = d = \sqrt{2}$, the constraints are satisfied and the inequality becomes

$$2 \cdot 2^{\frac{k}{2}} \le 3,$$

that is

$$k \le m = \frac{\ln 9}{\ln 2} - 2 \approx 1.17, \quad 2^{\frac{m}{2}} = \frac{3}{2}.$$

To show that m is the largest positive k, we need to prove the required inequality for $k = m$. Let's write it in the homogeneous form

$$\left(\frac{ab+bc+cd+da}{4}\right)^{\frac{m}{2}} \geq \frac{a^m+b^m+c^m}{3}.$$

Since $d \geq c$, it suffices to prove the inequality for $d = c$. For fixed b and c, we need to prove that $f(a) \geq 0$, where

$$f(a) = \left(\frac{ab+bc+ca+c^2}{4}\right)^{\frac{m}{2}} - \frac{a^m+b^m+c^m}{3}, \quad a \in [0,b],$$

with

$$\frac{1}{m}f'(a) = \frac{b+c}{8}\left(\frac{4}{ab+bc+ca+c^2}\right)^{1-\frac{m}{2}} - \frac{1}{3}a^{m-1}.$$

Since $f'(a)$ is decreasing, $f(a)$ is concave. So, it suffices to prove that $f(a) \geq 0$ for $a = 0$ and $a = b$.

Case 1: $a = 0$. We need to show that

$$\left(\frac{bc+c^2}{4}\right)^{\frac{m}{2}} \geq \frac{b^m+c^m}{3}.$$

Due to homogeneity, we may set $c = 1$. So, we need to prove that

$$\left(\frac{b+1}{4}\right)^{\frac{m}{2}} \geq \frac{b^m+1}{3}$$

for $0 \leq b \leq 1$. Since $b^m \leq b$, it suffices to show that

$$\left(\frac{b+1}{4}\right)^{\frac{m}{2}} \geq \frac{b+1}{3},$$

which is true if

$$3 \geq 2^m(b+1)^{1-\frac{m}{2}}.$$

We have

$$3 - 2^m(b+1)^{1-\frac{m}{2}} \geq 3 - 2^m \cdot 2^{1-\frac{m}{2}} = 3 - 2 \cdot 2^{\frac{m}{2}} = 3 - 2 \cdot \frac{3}{2} = 0.$$

Case 2: $a = b$. We need to show that

$$\left(\frac{b+c}{2}\right)^m \geq \frac{2b^m+c^m}{3}.$$

Due to homogeneity, we may set $c = 1$. So, we need to prove that

$$\left(\frac{b+1}{2}\right)^m \geq \frac{2b^m+1}{3}$$

for $0 \leq b \leq 1$. Since $b^m \leq b$, it suffices to show that

$$\left(\frac{b+1}{2}\right)^m \geq \frac{2b+1}{3}.$$

By Bernoulli's inequality, we have

$$\left(\frac{b+1}{2}\right)^m = \left(1-\frac{1-b}{2}\right)^m \geq 1-\frac{m}{2}(1-b) \geq 1-\frac{2}{3}(1-b) = \frac{2b+1}{3}.$$

The proof is completed. If $k=m$, then the equality occurs for $a=b=c=d=1$, as well as for $a=0$ and $b=c=d=\sqrt{2}$.

SP.602 Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that:

$$\sum_{i=1}^{n}\sqrt{\frac{n-a_i}{n-1+a_i}} \geq \sqrt{n(n-1)}$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

By Holder's inequality, we have

$$\left(\sum_{i=1}^{n}\sqrt{\frac{n-a_i}{n-1+a_i}}\right)^2 \sum_{i=1}^{n}(n-a_i)^2(n-1+a_i) \geq \left(\sum_{i=1}^{n}(n-a_i)\right)^3 = n^3(n-1)^3.$$

So, it suffices to show that

$$n^2(n-1)^2 \geq \sum_{i=1}^{n}(n-a_i)^2(a_i+n-1),$$

which is equivalent to $E \geq 0$, where

$$E = -\sum_{i=1}^{n}a_i^3 + (n+1)\sum_{i=1}^{n}a_i^2 - n^2.$$

For $n=2$, we have $E=0$, and for $n=3$, the inequality $E \geq 0$ reduces to the well-known inequality

$$(a_1+a_2+a_3)(a_1a_2+a_2a_3+a_3a_1) \geq 9a_1a_2a_3.$$

For $n \geq 4$, assume $a_1 = \max\{a_1, a_2, \dots, a_n\}$ and consider two cases.

Case 1: $a_1 \leq n-1$. We have

$$E = \sum_{i=1}^{n}\left[-a_i^3+(n+1)a_i^2-n\right] = \sum_{i=1}^{n}\left[-a_i^3+(n+1)a_i^2-(2n-1)a_i+n-1\right]$$

$$= \sum_{i=1}^{n} (1 - a_i)^2 (n - 1 - a_i) \geq 0.$$

Case 2: **$a_1 > n - 1$. From $a_1 + a_2 + \cdots + a_n = n$, it follows that $a_1 \in (n-1, n]$ and $a_2, \ldots, a_n < 1$. So,**

$$E \geq -a_1^3 + (n+1)a_1^2 - n^2 + a_2^2(1 - a_2) + \cdots + a_n^2(1 - a_n)$$
$$\geq -a_1^3 + (n+1)a_1^2 - n^2 = (n - a_1)(a_1^2 - a_1 - n).$$

Since $a_1^2 - a_1 - n > (n-1)^2 - (n-1) - n - n(n-4) + 2 > 0$, we have $E \geq 0$. The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n$, and also for $a_1 = n$ and $a_2 = \cdots = a_n = 0$ (or any cyclic permutation).

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Write the inequality as

$$\sum_{i=1}^{n} f(a_i) \geq nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

Where

$$f(x) = \sqrt{\frac{n - x}{n - 1 + x}}, \quad x \in [0, n].$$

For $x \in [0, 1]$, we have

$$f''(x) = \frac{(2n-1)(2n+1-4x)}{4\sqrt{(n-x)^3(n-1+x)^5}} \geq 0,$$

Then, f is convex on $[0, s]$. By the LHCF $-$ Theorem $[1]$, it suffice to prove that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for $x \geq 1 \geq y \geq 0$ such that $x + (n-1)y = n$. The last inequality can be rewritten as

$$\sqrt{\frac{n-x}{n-1+x}} + (n-1)\sqrt{\frac{n-y}{n-1+y}} \geq \sqrt{n(n-1)}$$

$$\Leftrightarrow \sqrt{\frac{(n-1)y}{2n-1-(n-1)y}} + (n-1)\sqrt{\frac{n-y}{n-1+y}} \geq \sqrt{n(n-1)}.$$

Let $t = \sqrt{\dfrac{ny}{2n-1-(n-1)y}} \in [0, 1]$, then $y = \dfrac{(2n-1)t^2}{n + (n-1)t^2}$.

The last inequality is equivalent to

$$\sqrt{\frac{(n-1)[n^2 + (n^2 - 3n + 1)t^2]}{n - 1 + nt^2}} \geq n - t$$

By squaring, we get the obvious inequality

$$nt(1-t)^2[2(n-1) - t] \geq 0,$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and for $a_1 = n, a_2 = \cdots = a_n = 0$, or any cyclic permutation.

[1] V. Cîrtoaje, *Mathematical Inequalities – Volume 4: Extensions and Refinements of Jensen's Inequality*, LAP LAMBERT Academic Publishing / Editura Universitatii Petrol – Gaze din Ploiesti, 2021, p. 4.

SP.603 **Prove that $\frac{5}{3}$ is the largest positive value of the power k such that**

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geq a^k+b^k+c^k+d^k$$

for all positive real numbers a, b, c, d with at most one of them smaller than 1 and $ab+ac+ad+bc+bd+cd=6$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a \in \left[1, \frac{5}{2}\right)$, $b = c = 1$ and $d = \frac{5-2a}{a+2}$, the constraints are satisfied and the inequality becomes $E(a) \geq 0$, where

$$E(a) = \frac{1}{a}+\frac{a+2}{5-2a}-a^k-\left(\frac{5-2a}{a+2}\right)^k.$$

We have

$$E'(a) = -\frac{1}{a^2}+\frac{9}{(5-2a)^2}-ka^{k-1}+\frac{9k}{(a+2)^2}d^{k-1},$$

$$E''(a) = \frac{2}{a^3}+\frac{36}{(5-2a)^3}-k(k-1)a^{k-2}-\frac{18k}{(a+2)^3}d^{k-1}-\frac{81k(k-1)}{(a+2)^4}d^{k-2}$$

Since $E(1) = E'(1) = 0$, the condition $E''(1) \geq 0$ is necessary to have $E(a) \geq 0$ for $a \in \left[1, \frac{5}{2}\right)$. From

$$E''(1) = 2+\frac{4}{3}-k(k-1)-\frac{2k}{3}-k(k-1) = \frac{2(5-3k)(1+k)}{3} \geq 0,$$

we get $k \leq \frac{5}{3}$. To show that $\frac{5}{3}$ is the largest value of k, we need to prove that $F \geq 0$ for $a \geq b \geq c \geq 1 \geq d$ such that $ab+ac+ad+bc+bd+cd=6$, where

$$F=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}-a^{\frac{5}{3}}-b^{\frac{5}{3}}-c^{\frac{5}{3}}-d^{\frac{5}{3}}.$$

For fixed a and b, assume that d and F are functions of c. We have

$$(a+b+c)d'+a+b+d=0$$

and

$$F'(c)=-c^{-2}-\frac{5}{3}c^{\frac{2}{3}}-\left(d^{-2}+\frac{5}{3}d^{\frac{2}{3}}\right)d'=-c^{-2}-\frac{5}{3}c^{\frac{2}{3}}+\left(d^{-2}+\frac{5}{3}d^{\frac{2}{3}}\right)\frac{a+b+d}{a+b+c}.$$

We claim that $F'(c)\ge 0$, that is

$$\frac{3d^{-2}+5d^{\frac{2}{3}}}{3c^{-2}+5c^{\frac{2}{3}}}\ge\frac{a+b+c}{a+b+d},\quad \frac{3d^{-2}+5d^{\frac{2}{3}}}{3c^{-2}+5c^{\frac{2}{3}}}-1\ge\frac{a+b+c}{a+b+d}-1,$$

$$\frac{3(c^2-d^2)-5c^2d^2\left(c^{\frac{2}{3}}-d^{\frac{2}{3}}\right)}{d^2\left(3+5c^{\frac{8}{3}}\right)}\ge\frac{c-d}{a+b+d}.$$

From : $6=ab+ac+ad+bc+bd+cd\ge 6\sqrt{abcd}\ge 6\sqrt{c^3d}$,

we have $c^3d\le 1$, hence

$$d^2\left(3+5c^{\frac{8}{3}}\right)-8cd=-3d(c-d)-5cd\left(1-c^{\frac{5}{3}}d\right)\le 0.$$

Since $cd\le 1, d^2\left(3+5c^{\frac{8}{3}}\right)\le 8cd$ and $a+b\ge 2c$, it suffices to prove the homogeneous inequality

$$\frac{3(c^2-d^2)-5(cd)^{\frac{2}{3}}\left(c^{\frac{2}{3}}-d^{\frac{2}{3}}\right)}{8cd}\ge\frac{c-d}{2c+d}.$$

For $c=x^3\ge 1$ and $d=1$, the inequality becomes

$$\frac{3(x^6-1)-5x^2(x^2-1)}{8x^3}\ge\frac{x^3-1}{2x^3+1},\quad (x-1)(A-B)\ge 0,$$

where

$$A=(2x^3+1)\left(3x^5+3x^4-2x^3-2x^2+3x+3\right),\ \ B=8x^3(x^2+x+1).$$

Since: $A-B=6x^8+6x^7-4x^6-9x^5+x^4-4x^3-2x^2+3x+3=$

$$=(x-1)\left(6x^7+12x^6+8x^5-x^4-4x^2-6x-3\right)\ge 0,$$

we have $F'(c) \geq 0, F(c)$ is increasing and has the minimum value when c is minimum, hence when $c = 1$. So, it suffices to consider this case, when we need to show that $G > 0$ for $a \geq b \geq 1 \geq d$ such that $ab + bd + da + a + b + d = 6$, where

$$G = \frac{1}{a} + \frac{1}{b} + \frac{1}{d} - a^{\frac{5}{3}} - b^{\frac{5}{3}} - d^{\frac{5}{3}}.$$

For fixed a, assume that d and G are functions of b. We have

$$(a + b + 1)d' + a + d + 1 = 0$$

and

$$G'(b) = -\frac{1}{b^2} - \frac{5}{3}b^{\frac{2}{3}} - \left(\frac{1}{d^2} + \frac{5}{3}d^{\frac{2}{3}}\right)d' = -\frac{1}{b^2} - \frac{5}{3}b^{\frac{2}{3}} + \left(\frac{1}{d^2} + \frac{5}{3}d^{\frac{2}{3}}\right)\frac{a + d + 1}{a + b + 1}.$$

We claim that $G'(b) \geq 0$, that is

$$\frac{3d^{-2} + 5d^{\frac{2}{3}}}{3b^{-2} + 5b^{\frac{2}{3}}} \geq \frac{a + b + 1}{a + d + 1}, \qquad \frac{3d^{-2} + 5d^{\frac{2}{3}}}{3b^{-2} + 5b^{\frac{2}{3}}} - 1 \geq \frac{a + b + 1}{a + d + 1} - 1,$$

$$\frac{3(b^2 - d^2) - 5b^2d^2\left(b^{\frac{2}{3}} - d^{\frac{2}{3}}\right)}{d^2\left(3 + 5b^{\frac{8}{3}}\right)} \geq \frac{b - d}{a + d + 1}.$$

From: $6 = ab + bd + da + a + b + d \geq 6\sqrt{abd} \geq 6\sqrt{b^2d}$,

we get $b^2d \leq 1$, hence

$$d^2\left(3 + 5b^{\frac{8}{3}}\right) - 8bd = -3d(b - d) - 5bd\left(1 - b^{\frac{5}{3}}d\right) \leq 0,$$

and from

$$6 = ab + bd + da + a + b + d \geq b^2 + 2bd + 2b + d =$$
$$= (b + d)^2 + 2(b + d) - d^2 - d \geq (b + d)^2 + 2(b + d) - 2,$$

we get

$$(b + d)^2 + 2(b + d) - 8 \leq 0, \quad b + d \leq 2,$$

hence

$$a + d + 1 \geq b + d = 1 \geq b + d + \frac{b + d}{2} = \frac{3(b + d)}{2}.$$

Since $bd \le 1, d^2\left(3+5b^{\frac{8}{3}}\right) \le 8bd$ and $a+d+1 \ge b+d+1 \ge b+d+\frac{b+d}{2}=\frac{3(b+d)}{2}$, it suffices to prove the homogeneous inequality

$$\frac{3(b^2-d^2)-5(bd)^{\frac{2}{3}}\left(b^{\frac{2}{3}}-d^{\frac{2}{3}}\right)}{8bd} \ge \frac{2(b-d)}{3(b+d)}.$$

For $b=x^3 \ge 1$ and $d=1$, the inequality becomes

$$\frac{3(x^6-1)-5x^2(x^2-1)}{8x^3} \ge \frac{2(x^3-1)}{3(x^3+1)}, \quad (x-1)(C-D) \ge 0,$$

where

$$C=3(x^3+1)\left(3x^5+3x^4-2x^3-2x^2+3x+3\right), \qquad D=16x^3(x^2+x+1).$$

Since

$$C-D=9x^8+9x^7-6x^6-13x^5+2x^4-13x^3-6x^2+9x+9$$
$$=(x-1)^2\left(9x^6+27x^5+39x^4+38x^3+39x^2+27x+9\right) \ge 0,$$

we have $G'(b) \ge 0, G(b)$ is increasing and has the minimum value when b is minimum, hence when $b=1$. So, it suffices to consider this case, when we need to show that $H \ge 0$ for $a \ge 1 \ge d$ such that $ad+2(a+d)=5$, where

$$G=\frac{1}{a}+\frac{1}{d}-a^{\frac{5}{3}}-d^{\frac{5}{3}}.$$

We have

$$d=\frac{5-2a}{a+2}, a<\frac{5}{2}.$$

Assume that d and G are function of a, and show that $G'(a) \ge 0$. Then, $G(a)$ is increasing and $G(a) \ge G(1)=0$ (because $a=1$ implies $d=1$). We have

$$(a+2)d'+d+2=0$$

and

$$G'(a)=-a^{-2}-\frac{5}{3}a^{\frac{2}{3}}-\left(d^{-2}+\frac{5}{3}d^{\frac{2}{3}}\right)d'=-a^{-2}-\frac{5}{3}a^{\frac{2}{3}}+\left(d^{-2}+\frac{5}{3}d^{\frac{2}{3}}\right)\frac{d+2}{a+2}.$$

Write the required inequality $G'(a) \ge 0$ as follows:

$$\frac{3d^{-2}+5d^{\frac{2}{3}}}{3a^{-2}+5a^{\frac{2}{3}}} \ge \frac{a+2}{d+2}, \quad \frac{3d^{-2}+5d^{\frac{2}{3}}}{3a^{-2}+5a^{\frac{2}{3}}}-1 \ge \frac{a+2}{d+2}-1,$$

$$\frac{3(a^2-d^2)-5a^2d^2\left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right)}{d^2\left(3+5a^{\frac{8}{3}}\right)} \geq \frac{a-d}{d+2}.$$

We will show that this inequality can be obtained by multiplying the inequality

$$8(d+2) \geq 3d^2\left(3+5a^{\frac{8}{3}}\right) \qquad (*)$$

and $3(a^2-d^2)-5a^2d^2\left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right) \geq \frac{8(a-d)}{3}$ (**)

By Bernoulli's inequality, we have

$$a^{\frac{8}{3}} = [1+(a^3-1)]^{\frac{8}{9}} \leq 1+\frac{8(a^3-1)}{9} = \frac{8a^3+1}{9}, \quad 3+5a^{\frac{8}{3}} \leq \frac{8(5a^3+4)}{9}.$$

So, to prove the inequality (*), it suffices to show that: $3(d+2) \geq d^2(5a^3+4)$,

which is equivalent to

$$\frac{3(5-2a)}{a+2}+6 \geq \frac{(5-2a)^2(5a^3+4)}{(a+2)^2},$$

$$-20a^5+100a^4-125a^3-16a^2+107a-46 \geq 0, \ \ (a-1)^2A \geq 0,$$

where

$$A = -20a^3+60a^2+15a-46.$$

The inequality (*) is true because

$$A \geq -20a^3+60a^2+15a-55 = (a-1)(-20a^2+40a+55)$$
$$\geq (a-1)(-20a^2+40a+25) = 5(a-1)(2a+1)(5-2a) \geq 0.$$

From $5 = ad+2(a+d) \geq ad+4\sqrt{ad}$ and $5 = ad+2(a+d) \leq \frac{(a+d)^2}{4}+2(a+d)$, we get $ad \leq 1$ and $a+d \geq 2$. So, to prove (**), it suffices to prove the homogeneous inequality

$$3(a^2-d^2)-5(ad)^{\frac{2}{3}}\left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right) \geq \frac{4(a-d)(a+d)}{3},$$

which is equivalent to

$$a^2-d^2-3(ad)^{\frac{2}{3}}\left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right) \geq 0, \quad \left(a^{\frac{2}{3}}-d^{\frac{2}{3}}\right)^3 \geq 0.$$

The proof is finished. If $k=\frac{5}{3}$, then the equality occurs for $a=b=c=d=1$.

SP.604 **Let be $A(2,1,0)$; $B(1,2,1)$; $C(3,3,3)$; $D(0,0,4)$. Find the distance from the point D to the real plan (ABC).**

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$(ABC):\begin{vmatrix} x & y & z & 1\\ 2 & 1 & 0 & 1\\ 1 & 2 & 1 & 1\\ 3 & 3 & 3 & 1\end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x-3 & y-3 & z-3 & 0\\ -1 & -2 & -3 & 0\\ -2 & -1 & -2 & 0\\ 3 & 3 & 3 & 1\end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x-3 & y-3 & z-3\\ -1 & -2 & -3\\ -2 & -1 & -2\end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x-3 & y-3 & z-3\\ 1 & 2 & 3\\ 2 & 1 & 2\end{vmatrix}=0$$

$$(ABC):4(x-3)+6(y-3)+(z-3)-4(z-3)-3(x-3)-2(y-3)=0$$

$$(ABC):x-3+4(y-3)-3(z-3)=0$$

$$(ABC):x+4y-3z-3-12+9=0$$

$$(ABC):x+4y-3z-6=0$$

$$d\big(D,(ABC)\big)=\frac{|x_D+4y_D-3z_D-6|}{\sqrt{1^2+4^2+(-3)^2}}=\frac{|0+4\cdot 0-3\cdot 4-6|}{\sqrt{1^2+4^2+(-3)^2}}$$

$$d\big(D,(ABC)\big)=\frac{18}{\sqrt{26}}=\frac{18\sqrt{26}}{26}=\frac{9\sqrt{26}}{13}$$

Solution 2 by Marin Chirciu-Romania

The equation of the real plan $(M_1M_2M_3)$, $M_1(x_1,y_1,z_1); M_2(x_2,y_2,z_2); M_3(x_3,y_3,z_3)$ is:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}=0.$$

2).The distance from $M_0(x_0,y_0,z_0)$ to $(P): ax+by+cx+d=0$ is:

$$d\left(M_0,(P)\right)=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

$$(ABC):\begin{vmatrix} x-2 & y-1 & z-0 \\ 1-2 & 2-1 & 1-0 \\ 3-2 & 3-1 & 3-0 \end{vmatrix}=0 \Leftrightarrow x+4y-3z-6=0.$$

The distance from $D(0,0,4)$ to $(ABC): x+4y-3z-6=0$ is:

$$d\left(D,(ABC)\right)=\frac{|1\cdot 0+4\cdot 0-3\cdot 4-6|}{\sqrt{1^2+4^2+(-3)^2}}=\frac{18}{\sqrt{26}}.$$

Solution 3 by José Luis Díaz – Barrero – Spain

We can find two vectors lying in the plane by using the points A, B, and C:

$$\overrightarrow{AB}=B-A=(1-2,2-1,1-0)=(-1,1,1)$$
$$\overrightarrow{AC}=C-A=(3-2,3-1,3-0)=(1,2,3)$$

The normal vector $\vec{n}$ to the plane is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which can be found using the cross product:

$$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\vec{n}=\vec{\imath}(1\cdot 3-1\cdot 2)-\vec{\jmath}\big((-1)\cdot 3-1\cdot 1\big)+\vec{k}\big((-1)\cdot 2-1\cdot 1\big)$$
$$\vec{n}=1\vec{\imath}+4\vec{\jmath}-3\vec{k}=(1,4,-3)$$

Using the point-normal equation of a plane $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ with the normal vector $\vec{n}=(1,4,-3)$ and the point $A(2,1,0)$, we get

$$1(x-2)+4(y-1)-3(z-0)=0 \Leftrightarrow x+4y-3z-6=0$$

The formula for the distance d from a point (x_0,y_0,z_0) to a plane $Ax+By+Cz+D=0$ is:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

Substitute the plane coefficients $(A=1, B=4, C=-3, D=-6)$ and coordinates of $D(0,0,4)$, to obtain

$$d=\frac{|1(0)+4(0)-3(4)-6|}{\sqrt{1^2+4^2+(-3)^2}}=\frac{18}{\sqrt{26}}=\frac{9\sqrt{26}}{13},$$

and the distance from the point D to the plane (ABC) is $\frac{9\sqrt{26}}{13}$.

SP.605 Let be $A(1,2,1); B(2,1,3); C(4,4,4)$. Find the distance from the point A to the line BC.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB}=(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}+(z_B-z_A)\vec{k}=$$

$$=(2-1)\vec{i}+(1-2)\vec{j}+(3-1)\vec{k}=\vec{i}-\vec{j}+2\vec{k}$$

$$\overrightarrow{AC}=(x_C-x_A)\vec{i}+(y_C-y_A)\vec{j}+(z_C-z_A)\vec{k}=$$

$$=(4-1)\vec{i}+(4-2)\vec{j}+(4-1)\vec{k}=3\vec{i}+2\vec{j}+3\vec{k}$$

$$BC=\sqrt{(x_C-x_B)^2+(y_C-y_B)^2+(z_C-z_B)^2}=$$

$$=\sqrt{(4-2)^2+(4-1)^2+(4-3)^2}=\sqrt{4+9+1}=\sqrt{14}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 2\\ 3 & 2 & 3\end{vmatrix}=-3\vec{i}+6\vec{j}+2\vec{k}+3\vec{k}-4\vec{i}-3\vec{j}=-7\vec{i}+3\vec{j}+5\vec{k}$$

$$A[ABC]=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{(-7)^2+3^2+5^2}=$$

$$=\frac{1}{2}\sqrt{49+9+25}=\frac{1}{2}\sqrt{58+25}=\frac{1}{2}\sqrt{83}$$

$$d(A,B,C)=\frac{2[ABC]}{BC}=\frac{2\cdot\frac{1}{2}\sqrt{83}}{\sqrt{14}}=\sqrt{\frac{83}{14}}$$

Solution 2 by Marin Chirciu-Romania

The distance from the point $A(x_1,y_1,z_1)$ **to the line** BC**,** $B(x_2,y_2,z_2)$**,** $C(x_3,y_3,z_3)$ **is:**

$$d(A,BC)=\frac{\sqrt{\begin{vmatrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{vmatrix}^2+\begin{vmatrix}y_2-y_1 & z_2-z_1\\ y_3-y_1 & z_3-z_1\end{vmatrix}^2+\begin{vmatrix}z_2-z_1 & x_2-x_1\\ z_3-z_1 & x_3-x_1\end{vmatrix}^2}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}.$$

The distance from the point $A(1,2,1)$ **to the line** BC**,** $B(2,1,3);C(4,4,4)$ **is:**

$$d(A,BC)=\frac{\sqrt{\begin{vmatrix}2-1 & 1-2\\ 4-1 & 4-2\end{vmatrix}^2+\begin{vmatrix}1-2 & 3-1\\ 4-2 & 4-1\end{vmatrix}^2+\begin{vmatrix}3-1 & 2-1\\ 4-1 & 4-1\end{vmatrix}^2}}{\sqrt{(4-2)^2+(4-1)^2+(4-3)^2}}=$$

$$= \frac{\sqrt{\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}^2 + \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix}^2 + \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}^2}}{\sqrt{4+9+1}} = \sqrt{\frac{83}{14}}$$

Solution 3 by Jose Luis Diaz Barrero-Spain

To find the distance from point $A(1,2,1)$ to the line passing through $B(2,1,3)$ and $C(4,,4,4)$ we can use the vector formula:

$$d = \frac{\left|\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|\right|}{\left|\left|\overrightarrow{BC}\right|\right|}$$

We find the components by subtracting the initial point from the terminal point:

$$\overrightarrow{BA} = (1-2, 2-1, 1-3) = (-1, 1, -2)$$

$$\overrightarrow{BC} = (4-2, 4-1, 4-3) = (2, 3, 1)$$

Using the determinant definition of the cross product:

$$\overrightarrow{BA} \times \overrightarrow{BC} = \det\begin{pmatrix} i & j & k \\ -1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$$

Expanding along the first row:

$$\overrightarrow{BA} \times \overrightarrow{BC} = i(1\cdot 1 - (-2)\cdot 3) - j\big((-1)\cdot 1 - (-2)\cdot 2\big) + k\big((-1)\cdot 3 - 2\cdot 2\big)$$

$$= i(1+6) - j(-1+4) + k(-3-2)$$

$$= (7, -3, -5)$$

Next, we find the lengths of the cross product vector and the direction vector $\overrightarrow{BC}$:

$$\left|\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|\right| = \sqrt{7^2 + (-3)^2 + (-5)^2} = \sqrt{49+9+25} = \sqrt{83}$$

$$\left|\left|\overrightarrow{BC}\right|\right| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$$

Substituting these values back into the distance formula gives:

$$d = \frac{\sqrt{83}}{\sqrt{14}} = \sqrt{\frac{83}{14}}$$

SP.606 **Let be $A(2,1,0)$; $B(0,1,2)$; $C(3,0,1)$; $D(4,4,4)$. Find the volume of the tetrahedron $ABCD$.**

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (0-2)\vec{i} + (1-1)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{k}$$

$$\overrightarrow{AC} = (x_i - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (3-2)\vec{i} + (0-1)\vec{j} + (1-0)\vec{k} = \vec{i} - \vec{j} + \vec{k}$$

$$\overrightarrow{AD} = (x_D - x_A)\vec{i} + (y_D - y_A)\vec{j} + (z_D - z_A)\vec{k}$$

$$\overrightarrow{AD} = (4-2)\vec{i} + (4-1)\vec{j} + (4-0)\vec{j} = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 2\vec{j} + 2\vec{k} + 2\vec{i} + 2\vec{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 2\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 2 = 4 + 12 + 8 = 24$$

$$V[ABCD] = \frac{1}{6}\left|\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right| = \frac{24}{6} = 4$$

Solution 2 by Marin Chirciu-Romania

$$V[ABCD] = \frac{1}{6}|\Delta|, \quad \Delta = \begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix}.$$

$$\Delta = \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 4 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & 3 & 4 & 0 \end{vmatrix} = -\begin{vmatrix} -2 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ 3 & 6 \end{vmatrix} = -24.$$

$$V[ABCD] = \frac{1}{6}|\Delta| = \frac{1}{6}|-24| = 4.$$

Solution 3 by Jose Luis Diaz Barrero-Spain

To find the volume of the tetrahedron with vertices $A(2,1,0)$; $B(0,1,2)$, $C(3,0,1)$, and $D(4,4,4)$, we use the scalar triple product of three vectors sharing a common vertex. The formula for the volume V is:

$$V=\frac{1}{6}\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\cdot\overrightarrow{AD}\right|$$

Choosing point A as the reference vertex, we subtract its coordinates from the other points:

$$\overrightarrow{AB}=(0-2,1-1,2-0)=(-2,0,2)$$
$$\overrightarrow{AC}=(3-2,0-1,1-0)=(1,-1,1)$$
$$\overrightarrow{AD}=(4-2,4-1,4-0)=(2,3,4)$$

The scalar triple product corresponds to the determinant of the matrix formed by these three vectors:

$$D=\det\begin{pmatrix}-2 & 0 & 2\\ 1 & -1 & 1\\ 2 & 3 & 4\end{pmatrix}$$

Expanding along the first row, we get

$$D=-2\cdot\det\begin{pmatrix}-1 & 1\\ 3 & 4\end{pmatrix}-0\cdot\det\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}+2\cdot\det\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$$
$$=-2\big((-1)(4)-(1)(3)\big)-0+2\big((1)(3)-(-1)(2)\big)=24$$

Plugging the absolute value of the determinant into our volume formula, we obtain

$$V=\frac{1}{6}\cdot|24|=\frac{24}{6}=4$$

Solution 4 by Omkumar Rao-India

$$\textit{The volume of tetrahedron } ABCD \textit{ is given by}:$$
$$V=\frac{1}{6}\left|\left[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right]\right|$$

$$\overrightarrow{AB}=\vec{B}-\vec{A}=(-2,0,2)$$
$$\overrightarrow{AC}=\vec{C}-\vec{A}=(1,-1,1)$$
$$\overrightarrow{AD}=\vec{D}-\vec{A}=(2,3,4)$$

$$V=\frac{1}{6}\begin{vmatrix}-2 & 0 & 2\\ 1 & -1 & 1\\ 2 & 3 & 4\end{vmatrix}$$

$$\begin{vmatrix}-2 & 0 & 2\\ 1 & -1 & 1\\ 2 & 3 & 4\end{vmatrix}=-2[(-1)(4)-(1)(3)]-0[(1)(4)-(1)(2)]+2[(1)(3)-(-1)(2)]$$

$$=24$$

$$V=\frac{1}{6}(24)=4\ cubic\ units$$

SP.607 **Let be the function $f:[0,1]\to\mathbb{R}$ integrable such that $f(1)=1$ and**

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big),\forall x,y\in[0,1]$$

Find:

$$I=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big),\forall x,y\in[0,1]\quad (*)$$

If in (*) we take $x=0$ and $y=x$, we get $\int_0^x f(t)\,dt=\frac{x}{2}f(x)$.

Let be $F(x)=\int_0^x f(t)\,dt$, how f is integrable function then f is bounded function and

$F(x)=\frac{x}{2}f(x)$ (1)

From $F'(x)=f(x)\Rightarrow xF'(x)=2F(x)$ (2)

On the other hand, $\left(\frac{F(x)}{x^2}\right)'=\frac{xF'(x)-2F(x)}{x^3}\overset{(2)}{=}0\Rightarrow F(x)=\alpha x^2,\alpha\in\mathbb{R}\Rightarrow$

$f(x)=\beta(x),\beta\in\mathbb{R},x\in(0,1]$ and from $f(1)=1$, we get $\beta=1$ and hence $f(x)=x$.

Therefore,

$$I=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx=\int_0^{\frac{\pi}{4}}\frac{x(1-\cos^2 x)}{\cos^2 x}dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{x}{\cos^2 x}dx-\int_0^{\frac{\pi}{4}}x\,dx=\left(x\tan x-\frac{\pi}{2}\right)\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan x\,dx=\frac{\pi}{4}-\frac{\pi^2}{32}-\frac{1}{2}\ln 2.$$

SP.608 If $a,b,c\in(0,1)$ and $x,y,z>0$ such that $a=(bc)^x, b=(ca)^y$, $c=(ab)^z$ and $xyz=1$ then holds:

$$\sqrt[n]{\sum_{cyc} a^n(y+z+2)^{2n-1}}\ge 6\cdot\sqrt[3]{abc}, n\in\mathbb{N}^*, n\ge 2$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

$$\sum_{cyc} a^n(y+z+2)^{2n-1}=\sum_{cyc}\frac{a^n(y+z+2)^n}{\frac{1}{(y+z+2)^{n-1}}}\overset{Radon}{\ge}$$

$$\ge\frac{[a(y+z+2)+b(z+x+2)+x(x+y+2)]^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}=$$

$$=\frac{[(b+c)x+(c+a)y+(a+b)x]^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}\overset{AM-GM}{\ge}$$

$$\ge\frac{\left(3\sqrt[3]{(a+b)(b+c)(c+a)xyz}\right)^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}\ge\frac{\left(6\sqrt[3]{abc}\right)^n}{\left(\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{x+y+2}\right)^{n-1}}\ (1)$$

Now, from $a=(bc)^x, b=(ca)^y, c=(ab)^z$, we get $x=\frac{-\ln a}{-\ln b-\ln c}, y=\frac{-\ln b}{-\ln c-\ln a}$

and $z=\frac{-\ln b}{-\ln a-\ln c}$. Let us denote $\begin{cases}-\ln a-\ln c=q\\-\ln b-\ln c=p\\-\ln a-\ln b=r\end{cases}$ with $p,q,r>0$, then

$\begin{cases}x=\frac{-p+q+r}{2p}\\y=\frac{p-q+r}{2q}\\z=\frac{p+q-r}{2r}\end{cases}$ and using $\frac{pq}{p+q}\le\frac{p+q}{2}$, we get: $\frac{1}{x+y+2}=\frac{2pq}{(p+q+r)(p+q)}\le\frac{p+q}{2(p+q+r)}$ (and analogs)

Hence, it follows that: $\frac{1}{x+y+2}+\frac{1}{y+z+2}+\frac{1}{z+x+2}\le\frac{p+q+q+r+r+p}{2(p+q+r)}=1$ (2)

From (1) and (2) we obtain:

$$a^n(y+z+2)^{2n-1}+b^n(z+x+2)^{2n-1}+c^n(x+y+z)^{2n-1}\ge\left(6\sqrt[3]{abc}\right)^n$$

Finally,

$$\sqrt[n]{\sum_{cyc} a^n(y+z+2)^{2n-1}} \ge 6\cdot\sqrt[3]{abc}, n\in\mathbb{N}^*, n\ge 2$$

SP.609 If $a,b,c>0$ then:

$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2}\ge\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=\sqrt{x^2+1}$

$f'(x)=\frac{x}{\sqrt{x^2+1}}; f''(x)=\frac{1}{(x^2+1)\sqrt{x^2+1}}>0$; f convex

By Jensen's inequality:

$$\frac{\sum_{cyc} abf\left(\frac{c}{ab}\right)}{\sum_{cyc} ab}\ge f\left(\frac{\sum_{cyc} ab\cdot\frac{c}{ab}}{\sum_{cyc} ab}\right)=f\left(\frac{\sum_{cyc} c}{\sum_{cyc} ab}\right)$$

$$\frac{\sum_{cyc} abf\left(\frac{c}{ab}\right)}{\sum_{cyc} ab}\ge f\left(\frac{a+b+c}{ab+bc+ca}\right)$$

$$abf\left(\frac{c}{ab}\right)+bcf\left(\frac{a}{bc}\right)+caf\left(\frac{b}{ca}\right)\ge(ab+bc+ca)f\left(\frac{a+b+c}{ab+bc+ca}\right)$$

$$\sum_{cyc} ab\sqrt{\left(\frac{c}{ab}\right)^2+1}\ge(ab+bc+ca)\sqrt{\left(\frac{a+b+c}{ab+bc+ca}\right)^2+1}$$

$$\sum_{cyc}\sqrt{c^2+a^2b^2}\ge\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

Equality holds for $a=b=c=1$.

Solution 2 by Jose Luis Diaz – Barrero – Spain

Minkowski's Inequality (the triangle inequality for vectors) states that for any real numbers:

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}\ge\sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}$$

We can strategically choose the components (x_i,y_i) from each radical term as follows:

- For $\sqrt{a^2b^2+c^2}$, let $x_1=ab$ and $y_1=c$.
- For $\sqrt{b^2c^2+a^2}$, let $x_2=bc$ and $y_2=a$.
- For $\sqrt{c^2a^2+b^2}$, let $x_3=ca$ and $y_3=b$.

Applying these to Minkowski's Inequality yields:

$$\sqrt{(ab)^2+c^2}+\sqrt{(bc)^2+a^2}+\sqrt{(ca)^2+b^2}\geq\sqrt{(ab+bc+ca)^2+(c+a+b)^2}$$

Equality holds if and only if the vectors (ab,c), (bc,a), and (ca,b) are linearly dependent (proportional):

$$\frac{ab}{bc}=\frac{c}{a}\Rightarrow\frac{a}{c}=\frac{c}{a}\Rightarrow a^2=c^2$$

Since $a,b,c>0$, this implies $a=c$. By symmetry across the other ratios, equality holds if and only if $a=b=c$.

Solution 3 by Omkumar Rao-India

$$\vec{u}=(ab,c),\vec{v}=(bc,a),,\vec{w}=(ca,b)\in R^2:$$

$$|\vec{u}|=\sqrt{a^2b^2+c^2},|\vec{v}|=\sqrt{c^2b^2+a^2}$$
$$|\vec{w}|=\sqrt{c^2a^2+b^2}$$

$$\textit{Minkowski Inequality}:|\vec{u}|+|\vec{v}|+|\vec{w}|\geq|\vec{u}+\vec{v}+\vec{w}|$$

$$\vec{u}+\vec{v}+\vec{w}=(ab+bc+ca,a+b+c):$$
$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2}\geq\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

$$\textit{Equality}:\vec{u}||\vec{v}||\vec{w}\Leftrightarrow\frac{ab}{c}=\frac{bc}{a}=\frac{ca}{b}\Leftrightarrow a=b=c$$

SP.610 If $a,b,c>0$ then:

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\geq\sqrt{(a+b+c)^2+9}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=\sqrt{x^2+1}$

$f'(x)=\frac{x}{\sqrt{x^2+1}}$; $f''(x)=\frac{1}{(x^2+1)\sqrt{x^2+1}}>0$; f **convex**

By Jensen's inequality:

$$\frac{\sum_{cyc} af\left(\frac{1}{a}\right)}{\sum_{cyc} a}\ge f\left(\frac{\sum_{cyc} a\cdot\frac{1}{a}}{\sum_{cyc} a}\right)=f\left(\frac{3}{\sum_{cyc} a}\right)$$

$$\sum_{cyc} af\left(\frac{1}{a}\right)\ge(a+b+c)f\left(\frac{3}{a+b+c}\right)$$

$$af\left(\frac{1}{a}\right)+bf\left(\frac{1}{b}\right)+cf\left(\frac{1}{c}\right)\ge(a+b+c)f\left(\frac{3}{a+b+c}\right)$$

$$a\sqrt{\frac{1}{a^2}+1}+b\sqrt{\frac{1}{b^2}+1}+c\sqrt{\frac{1}{c^2}+1}\ge(a+b+c)\sqrt{\left(\frac{3}{a+b+c}\right)^2+1}$$

$$\sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2}\ge\sqrt{9+(a+b+c)^2}$$

Equality holds for $a=b=c=1$.

Solution 2 by Jose Luis Diaz – Barrero – Spain

Minkowski's Inequality states that for any real numbers:

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}\ge\sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}$$

Choosing the components (x_i,y_i) to match the terms inside each radical on the left-hand side:

- **For $\sqrt{a^2+1}$, let $x_1=a$ and $y_1=1$.**
- **For $\sqrt{b^2+1}$, let $x_2=b$ and $y_2=1$.**
- **For $\sqrt{c^2+1}$, let $x_3=c$ and $y_3=1$.**

Substituting these components into Minkowski's Inequality gives:

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\ge\sqrt{(a+b+c)^2+(1+1+1)^2}=\sqrt{(a+b+c)^2+9}$$

Equality holds if and only if the vectors $(a,1),(b,1)$, and $(c,1)$ are proportional:

$$\frac{a}{1}=\frac{b}{1}=\frac{c}{1}\Rightarrow a=b=c$$

Thus, equality holds if and only if $a=b=c$.

SP.611 If $x, y, z > 0$ then:

$$\left(x^5 + y^5 + z^5\right)\left(x^6 + y^6 + z^6\right)\left(x^2 + y^2 + z^2\right)^5 \geq \left(x^3 + y^3 + z^3\right)^7$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Let be $f: (0, \infty) \to \mathbb{R}; f(x) = x^3,\ f'(x) = 3x^2; f''(x) = 6x > 0 \Rightarrow f$ convex

By Jensen's inequality: $p_1 f(x_1) + p_2 f(x_2) + p_3 f(x_3) \geq f(p_1 x_1 + p_2 x_2 + p_3 x_3)$ **(1)**

for $p_1, p_2, p_3 > 0; p_1 + p_2 + p_3 = 1$

Replace in (1):

$$p_1 = \frac{x^2}{x^2 + y^2 + z^2}; p_2 = \frac{y^2}{x^2 + y^2 + z^2}; p_3 = \frac{z^2}{x^2 + y^2 + z^2}$$

$$\sum_{cyc} \frac{x^2}{x^2 + y^2 + z^2} \cdot x^3 \geq f\left(\sum_{cyc} \frac{x^2}{x^2 + y^2 + z^2} \cdot x\right)$$

$$\frac{1}{x^2 + y^2 + z^2} \cdot \sum_{cyc} x^5 \geq \left(\sum_{cyc} \frac{x^3}{x^2 + y^2 + z^2}\right)^3$$

$$\frac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2} \geq \frac{(x^3 + y^3 + z^3)^3}{(x^2 + y^2 + z^2)^3}$$

$$\left(x^5 + y^5 + z^5\right)(x^2 + y^2 + z^2)^2 \geq (x^3 + y^3 + z^3)^3 \quad \textbf{(2)}$$

Let be $g: (0, \infty) \to \mathbb{R}; g(x) = x^4$

$g'(x) = 4x^3; g''(x) = 12x^2 > 0 \Rightarrow g$ convex

By Jensen's inequality:

$$q_1 g(x_1) + q_2 g(x_2) + q_3 g(x_3) \geq g(q_1 x_1 + q_2 x_2 + q_3 x_3)$$

Replace in (3):

$$q_1 = \frac{x^2}{x^2 + y^2 + z^2}; q_2 = \frac{y^2}{x^2 + y^2 + z^2}; q_3 = \frac{x^2}{x^2 + y^2 + z^2}$$

$$\sum_{cyc} \frac{x^2}{x^2 + y^2 + z^2} \cdot x^4 \geq g\left(\sum_{cyc} \frac{x^2}{x^2 + y^2 + z^2} \cdot x\right)$$

$$\frac{x^6+y^6+z^6}{x^2+y^2+z^2} \geq \frac{(x^3+y^3+z^3)^4}{(x^2+y^2+z^2)^4}$$

$$(x^6+y^6+z^6)(x^2+y^2+z^2)^3 \geq (x^3+y^3+z^3)^4 \quad \textbf{(4)}$$

By multiplying (2); (4): $(x^5+y^5+z^5)(x^6+y^6+z^6)(x^2+y^2+z^2)^5 \geq (x^3+y^3+z^3)^7$

Equality holds for $x=y=z$.

Solution 2 by Jose Luis Diaz – Barrero – Spain

Holder's Inequality for multiple sequences states that for positive numbers and weights that sum to 1 the following holds:

$$\prod_{j=1}^{m}\left(\sum_{i}^{n} a_{ji}\right)^{w_j} \geq \sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}^{w_j} \quad \text{where } \sum_{j=1}^{m} w_j = 1$$

We can interpret the LHS as a product of 7 brackets in total, assigning each an equal weight of $w_j = \frac{1}{7}$**. That is**

- **1 bracket of** $(x^5+y^5+z^5)$
- **1 bracket of** $(x^6+y^6+z^6)$
- **5 brackets of** $(x^2+y^2+z^2)$

Applying Holder's Inequality with these terms yields:

$$(x^5+y^5+z^5)^{\frac{1}{7}} \cdot (x^6+y^6+z^6)^{\frac{1}{7}} \cdot (x^2+y^2+z^2)^{\frac{5}{7}} \geq x^P+y^P+z^P$$

where the exponent P **for each variable is calculated by multiplying its original powers by their respective weights:**

$$P = \left(5\times\frac{1}{7}\right)+\left(6\times\frac{1}{7}\right)+\left(2\times\frac{5}{7}\right) = \frac{5+6+10}{7} = \frac{21}{7} = 3$$

Substituting $P = 3$ **back into the inequality gives:**

$$(x^5+y^5+z^5)^{\frac{1}{7}}(x^6+y^6+z^6)^{\frac{1}{7}}(x^2+y^2+z^2)^{\frac{5}{7}} \geq x^3+y^3+z^3$$

Since all terms are positive, we can raise both sides to the 7^{th} **power to obtain the final result:**

$$(x^5+y^5+z^5)(x^6+y^6+z^6)(x^2+y^2+z^2)^5 \geq (x^3+y^3+z^3)^7$$

Equality holds if and only if $x=y=z$.

Solution 3 by Marin Chirciu-Romania

$$LHS = \sum x^5 \sum x^6 \left(\sum x^2\right)^5 \overset{Holder}{\geq} \left(\sum \sqrt[7]{x^{5+6+2\cdot 5}}\right)^7 = \left(\sum x^3\right)^7 = RHS.$$

Equality holds for $x = y = z$.

SP.612 **Let be $A(1,2,3)$; $B(4,1,1)$; $C(0,3,2)$; $D(2,1,3)$. Find the volume of the parallelepiped builded on the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$.**

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (4-1)\vec{\imath} + (1-2)\vec{\jmath} + (1-3)\vec{k} = 3\vec{\imath} - \vec{\jmath} - 2\vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (0-1)\vec{\imath} + (3-2)\vec{\jmath} + (2-3)\vec{k} = -\vec{\imath} + \vec{\jmath} - \vec{k}$$

$$\overrightarrow{AD} = (x_D - x_A)\vec{\imath} + (y_D - y_A)\vec{\jmath} + (z_D - z_A)\vec{k}$$

$$\overrightarrow{AD} = (2-1)\vec{\imath} + (1-2)\vec{\jmath} + (3-3)\vec{k} = \vec{\imath} - \vec{\jmath}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -1 & -2 \\ -1 & 1 & -1 \end{vmatrix} = \vec{\imath} + 2\vec{\jmath} + 3\vec{k} - \vec{k} + 3\vec{\jmath} + 2\vec{\imath}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3\vec{\imath} + 5\vec{\jmath} + 2\vec{k}$$

$$\overrightarrow{AD} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 1 \cdot 3 - 1 \cdot 5 = -2$$

$$V[parallelipiped] = |-2| = 2$$

Solution 2 by Jose Luis Diaz Barrero-Spain

To find the volume V of the parallelepiped built on the vectors $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ from the points $A(1,2,3), B(4,1,1), C(0,3,2)$, and $D(2,1,3)$, w use the scalar triple product. The volume is given by the absolute value of the determinant of the matrix formed by these three vectors:

$$V = \left|\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right|$$

We find each vector by subtracting the coordinates of the initial point from the terminal point:

$$\overrightarrow{AB}=(4-1,1-2,1-3)=(3,-1,-2)$$
$$\overrightarrow{AC}=(0-1,3-2,2-3)=(-1,1,-1)$$
$$\overrightarrow{AD}=(2-1,1-2,3-3)=(1,-1,0)$$

The volume is the absolute value of the determinant of the 3×3 matrix containing these vectors as rows:

$$V=\left|\det\begin{pmatrix}3 & -1 & -2\\ -1 & 1 & -1\\ 1 & -1 & 0\end{pmatrix}\right|=|-2|=2$$

Solution 3 by Marin Chirciu-Romania

We use: $V[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}]=|(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD})|=|\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})|.$

In our case:

$$\overrightarrow{AB}=(4-1)\vec{\imath}+(1-2)\vec{\jmath}+(1-3)\vec{k}=3\vec{\imath}-\vec{\jmath}-2\vec{k}\Rightarrow$$
$$\Rightarrow\left||\overrightarrow{AB}|\right|=\sqrt{3^2+(-1)^2+(-2)^2}=\sqrt{14}$$
$$\overrightarrow{AC}=(0-1)\vec{\imath}+(3-2)\vec{\jmath}+(2-3)\vec{k}=-\vec{\imath}+\vec{\jmath}-\vec{k}\Rightarrow$$
$$\Rightarrow\left||\overrightarrow{AC}|\right|=\sqrt{(-1)^2+1^2+(-1)^2}=\sqrt{3}.$$
$$\overrightarrow{AD}=(2-1)\vec{\imath}+(1-2)\vec{\jmath}+(3-3)\vec{k}=\vec{\imath}-\vec{\jmath}\Rightarrow$$
$$\Rightarrow\left||\overrightarrow{AD}|\right|=\sqrt{1^2+(-1)^2}=\sqrt{2}$$
$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 3 & -1 & -2\\ -1 & 1 & -1\end{vmatrix}=\vec{\imath}+2\vec{\jmath}+3\vec{k}-\vec{k}+3\vec{\jmath}+2\vec{\imath}$$
$$\overrightarrow{AB}\times\overrightarrow{AC}=3\vec{\imath}+5\vec{\jmath}+2\vec{k}$$
$$\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=1\cdot 3-1\cdot 5=-2$$
$$V[parallelipiped]=|-2|=2$$

SP.613 **Let be $A(2,4,1)$; $B(4,2,5)$. Find the mediator plan of the segment $[AB]$.**

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$M\left(\frac{x_A+x_B}{2},\frac{y_A+y_B}{2},\frac{z_A+z_B}{2}\right)\Rightarrow M\left(\frac{2+4}{2};\frac{4+2}{2};\frac{1+5}{2}\right)$$

$$\Rightarrow M(3,3,3)$$

$$\overrightarrow{AB}=(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}+(z_B-z_A)\vec{k}$$

$$\overrightarrow{AB}=(4-2)\vec{i}+(2-4)\vec{j}+(5-1)\vec{k}$$

$$\overrightarrow{AB}=2\vec{i}-2\vec{j}+4\vec{k}$$

Let P be the mediator plan of the segment $[AB]$.

$$P:(x-x_M)\cdot 2+(y-y_M)\cdot(-2)+(z-z_M)\cdot 4=0$$

$$P:(x-3)\cdot 2+(y-3)\cdot(-2)+(z-3)\cdot 4=0$$

$$P: 2x-6-2y+6+4z-12=0$$

$$P: 2x-2y+4z-12=0$$

$$P: x-y+2z-6=0$$

Solution 2 by Marin Chirciu-Romania

M the middle of the segment $AB\Rightarrow$

$$M(x_M,y_M,z_M),x_M=\frac{x_A+x_B}{2},y_M=\frac{y_A+y_B}{2},z_M=\frac{z_A+z_B}{2}\Rightarrow$$

$$\Rightarrow x_M=\frac{2+4}{2},y_M=\frac{4+2}{2},z_M=\frac{1+5}{2}\Rightarrow x_M=3,y_M=3,z_M=3\Rightarrow M(3,3,3).$$

$\overrightarrow{AB}$ is the normal vector of the mediator plan of $[AB]$

$$(x_B-x_A)(x-x_M)+(y_B-y_A)(y-y_M)+(z_B-z_A)(z-z_M)=0\Leftrightarrow$$

$$\Leftrightarrow(4-2)(x-3)+(2-4)(y-3)+(5-1)(z-3)=0\Leftrightarrow 2(x-3)-2(y-3)+4(z-3)=0\Leftrightarrow$$

$$\Leftrightarrow 2(x-3)-2(y-3)+4(z-3)=0\Leftrightarrow x-y+2z-6=0$$

Solution 3 by Jose Luis Diaz Barrero-Spain

To find the mediator plane (the perpendicular bisector plane) of the segment $[AB]$ given the points $A(2,4,1)$ and $B(4,2,5)$, we need two elements:

1. A point on the plane, which is the midpoint M of the segment $[AB]$.
2. A normal vector $\vec{n}$ to the plane, which is the vector $\overrightarrow{AB}$.

The coordinates of the midpoint M are calculated by taking the average of the coordinates of A and B:

$$M=\left(\frac{x_A+x_B}{2},\frac{y_A+y_B}{2},\frac{z_A+z_B}{2}\right)$$

$$M=\left(\frac{2+4}{2},\frac{4+2}{2},\frac{1+5}{2}\right)=(3,3,3)$$

The normal vector $\vec{n}$ is directed along the segment AB:

$$\vec{n}=\overrightarrow{AB}=(x_B-x_A,y_B-y_A,z_B-z_A)$$

$$\vec{n}=(4-2,2-4,5-1)=(2,-2,4)$$

To simplify the equation of the plane, we cand divide the vector components by their greatest common divisor, 2, to obtain a simpler normal vector:

$$\vec{n}=(1,-1,2)$$

The standard equation of a plane with a normal vector $\vec{n}=(a,b,c)$ passing through a point $M(x_0,y_0,z_0)$ is:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Substituting $\vec{n}=(1,-1,2)$ and $M(3,3,3)$ into the formula, yields the Cartesian equation of the mediator plane of the segment $[AB]$:

$$x-y+2z-6=0$$

SP.614 Let be $A(1,3,2)$; $B(3,1,1)$; $C(4,2,0)$. Find the area of the parallelogram builded on $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB}=(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}+(z_B-z_A)\vec{k}$$

$$\overrightarrow{AB}=(3-1)\vec{i}+(1-3)\vec{j}+(1-2)\vec{k}$$

$$\overrightarrow{AB}=2\vec{i}-2\vec{j}-\vec{k}$$

$$\overrightarrow{AC}=(x_C-x_A)\vec{i}+(y_C-y_A)\vec{j}+(z_C-z_A)\vec{k}$$

$$\overrightarrow{AC}=(4-1)\vec{i}+(2-3)\vec{j}+(1-2)\vec{k}$$

$$\overrightarrow{AC}=3\vec{i}-\vec{j}-2\vec{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -2 & -1\\ 3 & -1 & -2\end{vmatrix}=4\vec{i}-3\vec{j}-2\vec{k}+6\vec{k}-\vec{i}+4\vec{j}=3\vec{i}+\vec{j}+4\vec{k}$$

Let be $ABCD$ the asked parallelogram.

$$\text{Area}[ABCD]=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{3^2+1^2+4^2}=\sqrt{26}$$

Solution 2 by José Luis Díaz – Barrero – Spain

The area of a parallelogram spanned by two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is equal to the magnitude (norm) of their cross product:

$$Area=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|$$

Now, we find the components of the vectors

$$\overrightarrow{AB}=(3-1,1-3,1-2)=(2,-2,-1)$$

$$\overrightarrow{AC}=(4-1,2-3,0-2)=(3,-1,-2)$$

The cross product is given by

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}i & j & k\\ 2 & -2 & -1\\ 3 & -1 & -2\end{vmatrix}=(3,1,4)$$

The area of the parallelogram S is the magnitude of this vector.

That is $S=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|=\sqrt{3^2+1^2+4^2}=\sqrt{26}$

Solution 3 by Marin Chirciu-Romania

We use: $S[\overrightarrow{AB},\overrightarrow{AC}]=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|$. In our case:

$$\overrightarrow{AB}=(3-1)\vec{i}+(1-3)\vec{j}+(1-2)\vec{k}=2\vec{i}-2\vec{j}-\vec{k}\Rightarrow$$

$$\Rightarrow\left|\left|\overrightarrow{AB}\right|\right|=\sqrt{2^2+(-2)^2+(-1)^2}=\sqrt{9}=3$$

$$\overrightarrow{AC}=(4-1)\vec{\imath}+(2-3)\vec{\jmath}+(0-2)\vec{k}=3\vec{\imath}-\vec{\jmath}-2\vec{k}\Rightarrow$$

$$\Rightarrow\left|\left|\overrightarrow{AC}\right|\right|=\sqrt{3^2+(-1)^2+(-2)^2}=\sqrt{14}$$

$$S=\left|\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\right|=\left|\left|\overrightarrow{AB}\right|\right|\cdot\left|\left|\overrightarrow{AC}\right|\right|\cdot\sin\angle\left(\overrightarrow{AB},\overrightarrow{AC}\right)=3\cdot\sqrt{14}\cdot\frac{\sqrt{13}}{3\sqrt{7}}=\sqrt{26},\text{ see:}$$

$$\cos\angle\left(\overrightarrow{AB},\overrightarrow{AC}\right)=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\left|\overrightarrow{AB}\right|\right|\cdot\left|\left|\overrightarrow{AC}\right|\right|}=\frac{\left(2\vec{\imath}-2\vec{\jmath}-\vec{k}\right)\cdot\left(3\vec{\imath}-\vec{\jmath}-2\vec{k}\right)}{3\cdot\sqrt{14}}=$$

$$=\frac{2\cdot3+(-2)(-1)+(-1)(-2)}{3\cdot\sqrt{14}}=\frac{10}{3\cdot\sqrt{14}}\Rightarrow\sin\angle\left(\overrightarrow{AB},\overrightarrow{AC}\right)=\frac{\sqrt{13}}{3\sqrt{7}}$$

SP.615 Let be $A(1,2,3)$ and the real plan: $P: 2x+y+z-5=0$

Find the parametrical equations of the perpendicular line from A to the real plan P.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$P: 2x+y+z-5=0$. The normal vector of P is $\vec{u}(2,1,1)$.

$$\vec{u}=2\vec{\imath}+\vec{\jmath}+\vec{k}$$

$$d:\frac{x-x_A}{2}=\frac{y-y_A}{1}=\frac{z-z_A}{1}$$

$$d:\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{1}=t;t\in\mathbb{R}$$

$$x-1=2t;y-2=t;z-3=t$$

$$d:\begin{cases}x=1+2t\\ y=2+t\\ z=3+t\end{cases};t\in\mathbb{R}$$

Solution 2 by José Luis Díaz – Barrero – Spain

To find the parametric equations of the line that is perpendicular to the plane P and passes through the point $A(1,2,3)$, we follow these steps:

1. Find the direction vector of the line. The equation of the given plane is:

$$P: 2x+y+z-5=0$$

The coefficients of x, y and z in the plane's equation give us its normal vector $\vec{n}$, which is perpendicular to the plane $\vec{n} = (2, 1, 1)$. Since the line we want to find is perpendicular to the plane, its direction vector $\vec{v}$ must be parallel to the plane's normal vector. Therefore, we can use the normal vector as the direct ion vector for our line $\vec{v} = (2, 1, 1)$.

2. Write the parametric equations. The parametric equations of a line passing through a point $A(x_0, y_0, z_0)$ with a direction vector $\vec{v} = (a, b, c)$ are given by:

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

where t is a real parameter $t \in \mathbb{R}$. Substituting the coordinates of the point $A(1, 2, 3)$ and the components of the direction vector $\vec{v} = (2, 1, 1)$, we get:

$$x = 1 + 2t$$

$$y = 2 + 1t$$

$$z = 3 + 1t$$

Thus, the parametric equations of the perpendicular line are:

$$\begin{cases} x = 1 + 2t \\ y = 2 + t \\ z = 3 + t \end{cases}, \quad \text{where } t \in \mathbb{R}.$$

UNDERGRADUATE PROBLEMS

UP.601 Let $\{L_n\}_{n\geq 0}$ be the Lucas' sequence defined by $L_0 = 2, L_1 = 1$ and for all $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Determine the sum of the lengths of the intervals, disjoint two by two, formed by all $f(x) = 1$, where

$$f(x) = \frac{L_1^2}{x + L_1^2} + \frac{L_2^2}{x + L_2^2} + \cdots + \frac{L_n^2}{x + L_n^2}$$

Proposed by Jose Luiz Diaz Barrero-Spain

Solution by proposer

To find the sum of the lengths of the intervals where $f(x) > 1$, we analyze the behavior of the function and find its roots using Vieta's formulas. The function is given by:

$$f(x) = \frac{L_1^2}{x + L_1^2} + \frac{L_2^2}{x + L_2^2} + \cdots + \frac{L_n^2}{x + L_n^2}$$

The poles (vertical asymptotes) of $f(x)$ occur at $x = -L_i^2$. Since the Lucas numbers grow strictly monotonically for $n \geq 1$, we can arrange these poles as:

$$-L_n^2 < -L_{n-1}^2 < \cdots < -L_2^2 < -L_1^2 < 0$$

Taking the derivative of $f(x)$ shows that it is strictly decreasing on every interval where it is defined:

$$f'(x) = -\sum_{i=1}^{n} \frac{L_i^2}{\left(x + L_i^2\right)^2} < 0$$

As x approaches any pole $-L_k^2$ from the right, $f(x) \to +\infty$. As it approaches from the left, $f(x) \to -\infty$. Thus, the equation $f(x) = 1$ has exactly n real roots, denoted as $x_0, x_1, \ldots, x_{n-1}$, situated in the following intervals:

$$x_0 \in \left(-L_1^2, \infty\right), x_1 \in \left(-L_2^2, -L_1^2\right), \ldots, x_{n-1} \in \left(-L_n^2, -L_{n-1}^2\right)$$

Consequently, the disjoint intervals where $f(x) \geq 1$ are:

$$\left(-L_1^2, x_0\right], \left(-L_2^2, x_1\right], \ldots, \left(-L_n^2, x_{n-1}\right]$$

The sum of the lengths of these intervals is:

Total Length $= \left(x_0 + L_1^2\right) + \left(x_1 + L_2^2\right) + \cdots + \left(x_{n-1} + L_n^2\right) = \sum_{k=0}^{n-1} x_k + \sum_{i=1}^{n} L_i^2$

To find $\sum_{k=0}^{n-1} x_k$, we look at the polynomial form of $f(x) = 1$:

$$\sum_{i=1}^{n} L_i^2 \prod_{j \neq i} \left(x + L_j^2\right) = \prod_{j=1}^{n} \left(x + L_j^2\right)$$

Expanding both sides to find the coefficients of x^n and x^{n-1}:

- **RHS:** $x^n + \left(\sum_{i=1}^{n} L_i^2\right) x^{n-1} + \cdots$
- **LHS:** $\left(\sum_{i=1}^{n} L_i^2\right) x^{n-1} + \cdots$

When subtracting the LHS from the RHS, we obtain

$$x^n + 0 \cdot x^{n-1} + \cdots = 0$$

By Vieta's formulas, the sum of the roots is equal to the negative of the coefficient of x^{n-1}, meaning $\sum_{k=0}^{n-1} x_k = 0$. Therefore, the total length simplifies to:

Total Length $= \sum_{i=1}^{n} L_i^2$

Using the recurrence relation for Lucas numbers, $L_k = L_{k+1} - L_{k-1}$, we rewrite each squared term to create a telescoping sum:

$$L_k^2 = L_k(L_{k+1} - L_k - 1) = L_k L_{k+1} - L_{k-1} L_k$$

Summing this from 1 to n, we obtain

$$\sum_{i=1}^{n} L_i^2 = (L_1 L_2 - L_0 L_1) + (L_2 L_3 - L_1 L_2) + \cdots + (L_n L_{n+1} - L_{n-1} L_n)$$

$$= L_n L_{n+1} - L_0 L_1 = L_n L_{n+1} - 2.$$

Thus, the sum of the lengths of the intervals is $L_n L_{n+1} - 2$.

UP.602 Find all non-negative integers for which

$$\sum_{k=0}^{n} \frac{k}{k+1} \binom{n}{k}^2$$

is and integer number.

Proposed by Jose Luiz Diaz Barrero-Spain

Solution by proposer

Since

$$\frac{k}{k+1} = \frac{(k+1) - 1}{k+1} = 1 - \frac{1}{k+1}$$

then the initial sum can be written as

$$\sum_{k=0}^{n} \binom{n}{k}^2 \left(1 - \frac{1}{k+1}\right) = \sum_{k=0}^{n} \binom{n}{k}^2 - \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}^2$$

We will evaluate each of these two sums individually.

1. The sum $\sum_{k=0}^{n}\binom{n}{k}^2$ is a standard combinatorial identity know as Vandermonde's Identity. It counts the number of ways to choose n objects out of $2n$ total objects by splitting them into two groups of size n. Its value is

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

2. To evaluate $\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}^2$. First, we use the identity for combinations to rewrite the term

$\frac{1}{k+1}\binom{n}{k}$ as

$$\frac{1}{k+1}\binom{n}{k}=\frac{1}{k+1}\cdot\frac{n!}{k!\,(n-k)!}=\frac{n!}{(k+1)!\,(n-k)!}$$

To turn this back into a binomial coefficient, we can multiply and divide by $n+1$:

$$\frac{1}{n+1}\cdot\frac{(n+1)!}{(k+1)!\,(n-k)!}=\frac{1}{n+1}\binom{n+1}{k+1}$$

Now, substitute this identity back into our second sum:

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n}{k}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{k}$$

Using the symmetric property $\binom{n}{k}=\binom{n}{n-k}$, we can rewrite the summation as:

$$\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{n-k}$$

By Vandermonde's Identity again, summing the product of these combinations over all k is equivalent to choosing $(k+1)+(n-k)=n+1$ elements out of a total pool of $(n+1)+n=2n+1$ elements:

$$\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n}{n-k}=\binom{2n+1}{n+1}$$

Thus, the second sum simplifies to:

$$\frac{1}{n+1}\binom{2n+1}{n+1}$$

Now, putting both components back together, we obtain

Total Sum $= \binom{2n}{n} - \frac{1}{n+1}\binom{2n+1}{n+1}$

We can expand $\binom{2n+1}{n+1}$ **in terms of** $\binom{2n}{n}$ **to simplify the expression:**

$$\binom{2n+1}{n+1} = \frac{2n+1}{n+1}\binom{2n}{n}$$

Substituting this back in yields:

Total Sum = $\binom{2n}{n} - \frac{1}{n+1}\left(\frac{2n+1}{n+1}\binom{2n}{n}\right) = \binom{2n}{n}\left[1 - \frac{2n+1}{(n+1)^2}\right]$

Finding a common denominator for the terms inside the brackets:

$$1 - \frac{2n+1}{(n+1)^2} = \frac{(n^2+2n+1)-(2n+1)}{(n+1)^2} = \frac{n^2}{(n+1)^2}$$

Multiplying it all together gives the final closed-form value:

$$\sum_{k=0}^{n} \frac{k}{k+1}\binom{n}{k}^2 = \frac{n^2}{(n+1)^2}\binom{2n}{n}$$

To find the values of n for which the sum is an integer, we analyze the closed – form expression:

$$\sum_{k=0}^{n} \binom{n}{k}^2 \frac{k}{k+1} = \frac{n^2}{(n+1)^2}\binom{2n}{n}$$

For this expression to evaluate to an integer, the denominator $(n+1)^2$ must completely divide the numerator $n^2\binom{2n}{n}$.

We start by examining the greatest common divisor (gcd) of the terms in the numerator and denominator. Since consecutive integers are always coprime, we have:

$$\gcd(n, n+1) = 1 \Rightarrow \gcd(n^2, (n+1)^2) = 1$$

Because n^2 shares no common factors with $(n+1)^2$ other than 1, the entire condition for divisibility falls on the central binomial coefficient. Thus, the total expression in an integer if and only if:

$$(n+1)^2 \,\Big|\, \binom{2n}{n}$$

Recall the definition of the n -th Catalan number, C_n, which is well-known to always be an integer for all $n \geq 0$:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

We can rewrite our original expression by factoring out C_n:

$$\frac{n^2}{(n+1)^2}\binom{2n}{n} = n^2 \cdot \frac{1}{n+1}\left[\frac{1}{n+1}\binom{2n}{n}\right] = n^2 \cdot \frac{C_n}{n+1}$$

Since $\gcd(n^2, n+1) = 1$, the expression yields an integer if and only if $n+1$ perfectly divides C_n:

$$(n+1)|C_n$$

Let us test the first few non-negative integers to see when this divisibility condition holds true (see table).

A known property regarding the divisibility of Catalan numbers states that $n+1$ divides C_n if and only if $n+1$ is a prime number p, or $n = 0$.

n	$n+1$	C_n	$\frac{C_n}{n+1}$	Is the Sum an Integer?
0	1	1	$\frac{1}{1} = 1$	**Yes** (Sum $= 0$)
1	2	1	$\frac{1}{2}$	No
2	3	2	$\frac{2}{3}$	No
3	4	5	$\frac{5}{4}$	No
4	5	14	$\frac{14}{5}$	No
5	6	42	$\frac{42}{6} = 7$	**Yes** (Sum $= 175$)

If $n+1 = p$ is a prime, then $n = p-1$. By Kummer's Theorem on the p-adic valuation of binomial coefficients, for all primes $p > 3$, the prime p does not divide C_{p-1}. Consequently, the denominator's factor of p^2 cannot be canceled out by the numerator for higher values of n. The only instances where the prime factors align to permit full cancellation are the small anomalies verified in our table.

Thus, we conclude that the only non-negative integer values of n for which the sum is an integer are:

$n=0$ and $n=5$

UP.603 Find all positive real values of the constant k such that

$$9(a^2+k)(b^2+k)(c^2+k)\le(1+k)^3(a+b+c)^2$$

for any nonnegative real numbers a,b,c with $ab+bc+ca=3.$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a=0$ and $b=c=\sqrt{3}$, the equality constraint is satisfied and the inequality becomes $f(k)\ge 0$, where

$$f(k)=k^3-6k^2-15k+4$$

So,

$$k\in\mathbb{I}=(0,k_1]\cup[k_2,\infty),$$

where $k_1\approx 0.2438$ and $k_2\approx 7.8467$ are the positive roots of the equation $f(k)=0$. To show that the range of values of k is $\mathbb{I}$, we need to prove the required inequality for $k\in\mathbb{I}$, that is for $k>0$ such that $f(k)\ge 0$. Denote $p=a+b+c$ and $r=abc$. For fixed p, we write the inequality as $F(r)\ge 0$, where

$$F(r)=-r^2+2kpr+C(p).$$

Since $F(r)$ is a quadratic concave function, it suffices to consider the cases when r has the minimum and maximum values. As it is known, the minimum and maximum values of r for fixed p and q are achieved when one of a,b,c is 0 or two of a,b,c are equal. So, it suffices to consider these cases.

Case 1: $a=0$. We need to prove that $bc=3$ implies $G\ge 0$, where

$$G=(1+k)^3(b+c)^2-9k(b^2+k)(c^2+k).$$

Denoting $S=b+c$, we have $S^2\ge 4bc=12$ and

$$G=(1+k)^3S^2-9k(kS^2+k^2-6k+9)=AS^2-9k(k^2-6k+9),$$

where $A=k^3-6k^2+3k=1$. Since

$$4A-f(k)=4(k^3-6k^2+3k+1)-(k^3-6k^2-15k+4)=3k(k-3)^2>0,$$

we have $4A>f(k)\ge 0$ and

$$G \geq 12A - 9k(k^2 - 6k + 9) = 3f(k) \geq 0.$$

Case 2: $b = c$. Denote $b = c := x$. From the equality constraint, we get $2ax + x^2 = 3$, with $x^2 \in (0,3]$. The required inequality becomes as follows:

$$[(3 - x^2)^2 + 4kx^2](x^2 + k)^2 \leq (1 + k)^3(x^2 + 1)^2,$$

$$(x^2 - 1)^2[x^4 + (6k - 4)x^2 - k^3 + 6k^2 - 3k - 1] \leq 0$$

Let $t = x^2, t \in (0,3]$. The inequality is true if $q(t) < 0$, where

$$g(t) = t^2 + (6k - 4)t - k^3 + 6k^2 - 3k - 1.$$

Since $g(t)$ is a quadratic convex function, it suffices to show that $g(0) \leq 0$ and $g(3) \leq 0$. Indeed, $g(0) = -k^3 + 6k^2 - 3k - 1 = -A < 0$ (see Case 1), and $g(3) = -f(k) \leq 0$. For $k \in (0, k_1] \cup [k_2, \infty)$, the equality occurs when $a = b = c = 1$. For $k = k_1$ and $k = k_2$, the equality also occurs when $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

UP.604 Prove that $\frac{3}{2}$ is the smallest positive value of the power k such that

$$\frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k} \geq a^2 + b^2 + c^2$$

for all positive real numbers a, b, c with at most one of them smaller than 1 and $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a \geq 1, b = 1$ and $c \in (0,1]$ such that $a + c + ac = 3$, the constraints are satisfied and the inequality becomes $E \geq 0$, where

$$E = \frac{1}{a^k} + \frac{1}{c^k} - a^2 - c^2.$$

Assume that a and E are functions of c. For $c = 1$, we have $a = 1$. By differentiating the equality constraint, we get

$$(1 + c)a' + 1 + a = 0, \quad a'(1) = -1,$$

$$(1 + c)a'' + 2a' = 0, \quad a''(1) = 1.$$

Therefore,

$$E'(c) = -\left(\frac{k}{a^{k+1}} + 2a\right)a' - \frac{k}{c^{k+1}} - 2c,\ \ E'(1) = 0$$

$$E''(c) = -\left(\frac{k}{a^{k+1}} + 2a\right)a'' + \left(\frac{k(k+1)}{a^{k+2}} - 2\right)(a')^2 + \frac{k(k+1)}{c^{k+2}} - 2,$$

$$E''(1) = (k+2)(2k-3)$$

Since $E(1) = E'(1) = 0$, the condition $E''(1) \geq 0$ is necessary to have $E(c) \geq 0$ for $c \in (0,1]$. From this condition, we get $k \geq \frac{3}{2}$. To show that $\frac{3}{2}$ is the smallest positive value of k, we need to prove that $F \geq 0$ for $a \geq b \geq 1 \geq c$ and $ab + bc + ca = 3$, where

$$F = a^{-\frac{3}{2}} + b^{-\frac{3}{2}} + c^{-\frac{3}{2}} - a^2 - b^2 - c^2.$$

For fixed c, assume that a and F are functions of b. We have

$$(b+c)a' + a + c = 0$$

and

$$F'(b) = -\left(\frac{3a^{-\frac{5}{2}}}{2} + 2a\right)a' - \frac{3b^{-\frac{5}{2}}}{2} - 2b = \left(\frac{3a^{-\frac{5}{2}}}{2} + 2a\right)\frac{a+c}{b+c} - \frac{3b^{-\frac{5}{2}}}{2} - 2b.$$

We claim that $F'(b) \geq 0$, that is

$$\frac{a+c}{b+c} \geq \frac{3b^{-\frac{5}{2}} + 4b}{3a^{-\frac{5}{2}} + 4a}, \qquad \frac{a+c}{b+c} - 1 \geq \frac{3b^{-\frac{5}{2}} + 4b}{3a^{-\frac{5}{2}} + 4a} - 1,$$

$$\frac{a-b}{b+c} \geq \frac{3\left(b^{-\frac{5}{2}} - a^{-\frac{5}{2}}\right) - 4(a-b)}{3a^{-\frac{5}{2}} + 4a}, \quad \frac{a-b}{b+c} \geq \frac{3\left(a^{\frac{5}{2}} - b^{\frac{5}{2}}\right) - 4a^{\frac{5}{2}}b^{\frac{5}{2}}(a-b)}{b^{\frac{5}{2}}\left(4a^{\frac{7}{2}}+3\right)}.$$

Denoting $x = a^{\frac{1}{2}}$ and $y = b^{\frac{1}{2}}$, we need to show that if $x \geq y \geq 1$, then

$$\frac{x^2 - y^2}{y^2 + c} \geq \frac{3(x^5 - y^5) - 4x^5y^5(x^2 - y^2)}{y^5(4x^7 + 3)}.$$

It is true if

$$\frac{x+y}{y^2+c} \geq \frac{3(x^4 + x^3y + x^2y^2 + xy^3 + y^4) - 4x^5y^5(x+y)}{y^5(4x^7+3)}.$$

Since

$$\frac{x+y}{y^2+c} \ge \frac{x+y}{y^2+1} \ge \frac{2}{y^2+1} \ge \frac{1}{y^2} \ge \frac{1}{y^5},$$

it suffices to show that

$$1 \ge \frac{3(x^4+x^3y+x^2y^2+xy^3+y^4)-4x^5y^5(x+y)}{4x^7+3},$$

that is

$$4x^7+3+4x^5y^5(x+y) \ge 3(x^4+x^3y+x^2y^2+xy^3+y^4).$$

It is true if

$$4x^7+3+4x^5(x+1) \ge 15x^4,$$

that is

$$4x^7+4x^6+4x^5-15x^4+3 \ge 0.$$

Indeed, we have

$$4x^7+4x^6+4x^5-15x^4+3 \ge 4x^6+4x^6+4x^4-15x^4+3 = 8x^6-11x^5+3$$
$$= (x^2-1)(8x^4-3x^2-3) \ge 0$$

From $F'(b) \ge 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b=1$. So, it suffices to consider the case $b=1$, when we need to show that

$$\frac{1}{a\sqrt{a}}+\frac{1}{c\sqrt{c}} \ge a^2+c^2$$

for $a+c+ac=3$. Denoting $ac=t$, we have $a+c=3-t$. From $3-t=a+c\ge 2\sqrt{t}$, we get $t\le 1$. By squaring, the required inequality becomes as follows

$$\left(a\sqrt{a}+c\sqrt{c}\right)^2 \ge a^3c^3(a^2+c^2)^2.$$

By Bernoulli's inequality, we have

$$t^{\frac{3}{2}} = [1+(t-1)]^{\frac{3}{2}} \ge 1+\frac{3(t-1)}{2} = \frac{3t-1}{2}, \qquad 2t^{\frac{3}{2}} \ge 3t-1.$$

Since $a^2+c^2=(a+c)^2-2ac=(3-t)^2-2t=9-8t+t^2$ and

$$\left(a\sqrt{a}+c\sqrt{c}\right)^2 = (a+c)^3-3ac(a+c)+2ac\sqrt{ac} \ge (3-t)^3-3t(3-t)+(3t-1)$$
$$= 26-33t+12t^2-t^3,$$

it suffices to show that

$$26-33t+12t^2-t^3 \geq t^3(9-8t+t^2)^2,$$

that is

$$(1-t)^2 f(t) \geq 0, f(t)=26+19t+24t^2-53t^3+14t^4-t^5.$$

It is true since

$$f(t) > 10+19t+24t^2-53t^3=(1-t)(10+29t+53t^2) \geq 0.$$

If $k=\frac{3}{2}$, then the equality occurs for $a=b=c=1$.

UP.605 We consider the function $u: \mathbb{R} \to \mathbb{R}$, periodic with period 2π. For the period $[0,2\pi]$ we have:

$u(x)=\cos(x)$ if $x \in \left[0,\frac{\pi}{2}\right)$; $u(x)=0$ if $\in \left[\frac{\pi}{2},\frac{3\pi}{2}\right)$; $u(x)=\cos(x)$ if $x \in \left[\frac{3\pi}{2},2\pi\right)$. Prove the equality:

$$\int_0^\infty \frac{u(x)}{1+x^2}dx=\frac{\pi}{4e}+\frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right).$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I=\int_0^\infty \frac{u(x)}{1+x^2}dx$$

The function $u(x)$ satisfies Dirichlet's conditions. Also, the function is even.

We expand the function in the Fourier series:

$$u(x)=A_0+\sum_{n=1}^{\infty} A_n \cos(nx)$$

where:

$$A_0=\frac{1}{2\pi}\int_0^{2\pi} u(x)\,dx;\ A_n=\frac{1}{\pi}\int_0^{2\pi} u(x)\cos(nx)\,dx$$

Calculating these integrals, we obtain:

$A_0=\frac{1}{\pi}$; $A_1=\frac{1}{2}$; $A_n=\frac{-1}{(n^2-1)\pi}\cdot\left[\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)\right]$ for $n \geq 2$

But:

$\cos\left(\frac{3n\pi}{2}\right) = \cos\left(\frac{n\pi}{2}\right)$ **for any** $n \geq 2$

Result:

$$A_n = -\frac{1}{n}\cdot\left[\frac{2\cos\left(\frac{n\pi}{2}\right)}{n^2-1}\right] = \frac{-1}{\pi}\cdot\left[\frac{\cos\left(\frac{n\pi}{2}\right)}{n-1} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}\right] = B_n + C_n$$

Where

$$B_n = \frac{-1}{\pi}\cdot\frac{\cos\left(\frac{n\pi}{2}\right)}{n-1};\ \ C_n = \frac{1}{\pi}\cdot\frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}$$

We have:

$B_n = 0$ **if** $n = 3,5,7,\dots;$ $B_n \neq 0$ **if** $n = 2,4,6,\dots;$

$C_n = 0$ **if** $n = 3,5,7,\dots;$ $C_n \neq 0$ **if** $n = 2,4,6,\dots;$

So:

$$u(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + [B_2\cos(2x) + B_4\cos(4x) + B_6\cos(6x) + B_8\cos(8x) + \cdots] +$$
$$+[C_2\cos(2x) + C_4\cos(4x) + C_6\cos(6x) + C_8\cos(8x) + \cdots]$$

$$u(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) + \left[\frac{1}{\pi}\cos(2x) - \frac{1}{3\pi}\cos(4x) + \frac{1}{5\pi}\cos(6x) - \frac{1}{7\pi}\cos(8x) + \cdots\right] +$$
$$+\left[-\frac{1}{3\pi}\cos(2x) + \frac{1}{5\pi}\cos(4x) - \frac{1}{7\pi}\cos(6x) + \frac{1}{9\pi}\cos(8x) + \cdots\right]$$

We now use the following relationship:

$\int_0^\infty \frac{\cos(mx)}{1+x^2}dx = \frac{\pi}{2}e^{-m}$**, where** $m > 0$

This relation is Laplace's integral and is well known.

It is easily proved for example using the properties of the Laplace transform.

We obtained the value of the integral I**:**

$$I = \frac{1}{\pi}\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2}e^{-1} + \left(\frac{1}{\pi}\frac{\pi}{2}e^{-2} - \frac{1}{3\pi}\frac{\pi}{2}\ e^{-4} + \frac{1}{5\pi}\frac{\pi}{2}\ e^{-6} - \frac{1}{7\pi}\frac{\pi}{2}\ e^{-8} + \cdots\right) +$$
$$+\left(-\frac{1}{3\pi}\frac{\pi}{2}\ e^{-2} + \frac{1}{5\pi}\frac{\pi}{2}\ e^{-4} - \frac{1}{7\pi}\frac{\pi}{2}e^{-6} + \frac{1}{9\pi}\frac{\pi}{2}e^{-8} + \cdots\right)$$

$$2I = 1 + \frac{\pi}{2}e^{-1} + \left(e^{-2} - \frac{1}{3}e^{-4} + \frac{1}{5}e^{-6} - \frac{1}{7}e^{-8} + \cdots\right) +$$

$$+\left(-\frac{1}{3}e^{-2}+\frac{1}{5}e^{-4}-\frac{1}{7}e^{-6}+\frac{1}{9}e^{-8}+\cdots\right)$$

We will now use the power series development of the following functions.

$\arctan(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots$ where $-1\le x\le 1$

$x\arctan(x)=x^2-\frac{1}{3}x^4+\frac{1}{5}x^6-\frac{1}{7}x^8+\cdots$ where $-1\le x\le 1$

$\frac{\arctan(x)}{x}=1-\frac{1}{3}x^2+\frac{1}{5}x^4-\frac{1}{7}x^6+\cdots$ where $-1\le x\le 1$

We have:

$$e^{-1}\arctan(e^{-1})=e^{-2}-\frac{1}{3}e^{-4}+\frac{1}{5}e^{-6}-\frac{1}{7}e^{-8}+\cdots$$

$$\frac{\arctan(e^{-1})}{e^{-1}}=1-\frac{1}{3}e^{-2}+\frac{1}{5}e^{-4}-\frac{1}{7}e^{-6}+\cdots$$

We obtain:

$$2I=\frac{\pi}{2}e^{-1}+e^{-1}\arctan(e^{-1})+\frac{\arctan(e^{-1})}{e^{-1}}$$

Or:

$$I=\frac{\pi}{4e}+\frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right)$$

Thus, the problem is solved.

UP.606 If $f:\mathbb{R}\to(1,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous functions, and $y_n=\sqrt[n]{n!\,F_n}$, $n\in\mathbb{N}^*-\{1\}$, where $(F_n)_{n\ge 0}$ is Fibonacci sequence, find:

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

Lemma 1. If $f:\mathbb{R}\to(1,\infty)$, $g:\mathbb{R}\to\mathbb{R}$ are continuous, and $(y_n)_{n\ge 1}$ is an increasing sequence, and

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx,\forall n\geq 2$$

then

$$\lim_{n\to\infty}I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n).$$

Proof.

If $a,b\in\mathbb{R},a<b$ and $I=\int_a^b\frac{\left(f(x-a)\right)^{g(b-x)}}{\left(f(b-x)\right)^{g(x-a)}+\left(f(x-a)\right)^{g(b-x)}}dx,$

$x=u(t)=a+b-t,u'(t)=-1,u(a)=b,u(b)=a$, we obtain that

$$I=\int_b^a\frac{\left(f(b-x)\right)^{g(x-a)}}{\left(f(x-a)\right)^{g(b-x)}+\left(f(b-x)\right)^{g(x-a)}}dx,$$

$2I=\int_a^b\frac{\left(f(b-x)\right)^{g(x-a)}+\left(f(x-a)\right)^{g(b-x)}}{\left(f(x-a)\right)^{g(b-x)}+\left(f(b-x)\right)^{g(x-a)}}dx=\int_a^b dx=b-a\Rightarrow I=\frac{1}{2}(b-a)$. So,

$$I_n=\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx=\frac{1}{2}(y_{n+1}-y_n),\forall n\geq 2.$$

Hence: $\lim_{n\to\infty}I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n)$.

Lemma 2.

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}\right)=\frac{\alpha}{e},\text{ where }\alpha=\frac{1+\sqrt{5}}{2}\text{ and}$$

$(L_n)_{n\geq 0},L_0=2,L_1=1,L_{n+2}=L_{n+1}+L_n,\forall n\in\mathbb{N}$ is Lucas sequence.

Proof.

$$L_{n+2}-L_{n+1}-L_n=0,\forall n\in\mathbb{N}^*\Leftrightarrow\frac{L_{n+2}}{L_n}-\frac{L_{n+1}}{L_n}-1=0\Leftrightarrow\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}-1=0$$

$$\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}-1\right)=0\Leftrightarrow x^2-x-1=0,$$

$x=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}$, so $x=\frac{1\pm\sqrt{5}}{2}$

Since $\frac{L_{n+1}}{L_n}>0,x=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}=\frac{\sqrt{5}+1}{2}=\alpha$.

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!\,L_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!\,L_n}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!\,L_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!\,L_n}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}$$

(1) $\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}=\sqrt[n]{n!\,L_n}(u_n-1)=$

$$=\sqrt[n]{n!\,L_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$$

$=\frac{\sqrt[n]{n!L_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\,,\forall n\ge 2$, where $u_n=\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{\sqrt[n]{n!L_n}}$.

$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{n!L_n}}\cdot\frac{n+1}{n}\right)=\frac{\alpha}{e}\cdot\frac{e}{\alpha}\cdot 1=1$, so $\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$;

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!\,L_{n+1}}{n!\,L_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!\,L_{n+1}}}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!\,L_{n+1}}}=$$

$$=\alpha\cdot\frac{e}{\alpha}=e$$

From (1): $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,L_{n+1}}-\sqrt[n]{n!\,L_n}\right)=\frac{\alpha}{e}\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=\frac{\alpha}{e}\cdot\ln e=\frac{\alpha}{e}\cdot 1=\frac{\alpha}{e}$.

Analogously, $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,F_{n+1}}-\sqrt[n]{n!\,F_n}\right)=\frac{\alpha}{e}$, where $\alpha=\frac{1+\sqrt{5}}{2}$.

By lemma 1, $\lim_{n\to\infty}I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n)=\frac{1}{2}\cdot\frac{\alpha}{e}=\frac{\alpha}{2e}$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and we taking account by lemma 2,

$$\lim_{n\to\infty}(y_{n+1}-y_n)=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,F_{n+1}}-\sqrt[n]{n!\,F_n}\right)=\frac{\alpha}{e}$$

UP.607 Calculate the integral:

$$\int_0^1\frac{x\ln x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by proposer

Let us denote: $A=\int_0^1\frac{x\ln x}{(x+1)(x^2+1)}dx$

We have, successively:

$$A=\int_0^1\frac{(1-x)x\ln x}{1-x^4}dx=\int_0^1\frac{x\ln x}{1-x^4}dx-\int_0^1\frac{x^2\ln x}{1-x^4}dx$$

$$A=\int_0^1\sum_{n=0}^{\infty}x^{4n+1}\ln x\,dx-\int_0^1\sum_{n=0}^{\infty}x^{4n+2}\ln x\,dx=$$

$$=\sum_{n=0}^{\infty}\left(\int_0^1 x^{4n+1}\ln x\,dx-\int_0^1 x^{4n+2}\ln x\,dx\right)$$

We will use the following relationship:

$\int_0^1 x^a\ln x\,dx=-\frac{1}{(a+1)^2}$, **where** $a\in\mathbb{R},a\geq 0$

We obtain:

$$A=\sum_{n=0}^{\infty}\left[\frac{1}{(4n+3)^2}-\frac{1}{(4n+2)^2}\right]=\sum_{n=0}^{\infty}\left[\frac{\frac{1}{16}}{\left(n+\frac{3}{4}\right)^2}-\frac{\frac{1}{16}}{\left(n+\frac{2}{4}\right)^2}\right]$$

We now use the following relationship

$$\Psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\Psi_1(x)$ **is the trigamma function.**

We obtained the value of the integral A**:**

$$A=\frac{1}{16}\left[\Psi_1\left(\frac{3}{4}\right)-\Psi_1\left(\frac{1}{2}\right)\right]$$

The following special value is known:

$$\Psi_1\left(\frac{1}{2}\right)=\frac{\pi^2}{2}$$

Result:

$$A=\frac{1}{16}\left[\Psi_1\left(\frac{3}{4}\right)-\frac{\pi^2}{2}\right]$$

Thus, the problem is solved.

Solution 2 by Omkumar Rao-India

$$I=\frac{1}{2}\int_0^1\left(\frac{-\ln x}{x+1}+\frac{\ln x}{x^2+1}+\frac{x\ln x}{x^2+1}\right)dx$$

$$A=\int_0^1\frac{\ln x}{x+1}dx,\qquad B=\int_0^1\frac{\ln x}{x^2+1}dx$$

$$C=\int_0^1\frac{x\ln x}{x^2+1}dx$$

$$I=\frac{1}{2}(-A+B+C)$$

$$A=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln x\,dx$$

$$A=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=-\frac{1}{2}\zeta(2)=-\frac{\pi^2}{12}$$

$$B=\sum_{r=0}^{\infty}(-1)^r\int_0^1 x^{2r}\ln x\,dx=-\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}=-G$$

$$\textit{where } G \textit{ denotes Catalan's constant.}$$

$$C=\int_0^1\frac{x\ln x}{x^2+1}dx\Rightarrow(x\longrightarrow x^2),\qquad C=\frac{1}{4}\int_0^1\frac{\ln x}{1+x}dx=\frac{1}{4}A=-\frac{\pi^2}{48}$$

$$I=\frac{1}{2}\left(\frac{\pi^2}{12}-G-\frac{\pi^2}{48}\right)$$

$$I=\frac{\pi^2}{32}-\frac{G}{2}$$

UP.608 If $f:\mathbb{R}\to(1,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous functions, and $(L_n)_{n\geq 0}$ is Lucas sequence, and $y_n=\sqrt[n]{(2n-1)!!\,L_n}, n\in\mathbb{N}^*-\{1\}$, find:

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx.$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

Lemma 1. If $f:\mathbb{R}\to(1,\infty)$, $g:\mathbb{R}\to\mathbb{R}$ are continuous, and $(y_n)_{n\geq 1}$ is an increasing sequence, and

$I_n=\int_{y_n}^{y_{n+1}}\frac{(f(x-y_n))^{g(y_{n+1}-x)}}{(f(y_{n+1}-x))^{g(x-y_n)}+(f(x-y_n))^{g(y_{n+1}-x)}}dx, \forall n\ge 2$, then:

$$\lim_{n\to\infty} I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n).$$

Proof.

If $a,b\in\mathbb{R}, a<b$ and $I=\int_a^b\frac{(f(x-a))^{g(b-x)}}{(f(b-x))^{g(x-a)}+(f(x-a))^{g(b-x)}}dx$,

$x=u(t)=a+b-t, u'(t)=-1, u(a)=b, u(b)=a$, we obtain that

$$I=\int_b^a\frac{(f(b-x))^{g(x-a)}}{(f(x-a))^{g(b-x)}+(f(b-x))^{g(x-a)}}dx,$$

$2I=\int_a^b\frac{(f(b-x))^{g(x-a)}+(f(x-a))^{g(b-x)}}{(f(x-a))^{g(b-x)}+(f(b-x))^{g(x-a)}}dx=\int_a^b dx=b-a\Rightarrow I=\frac{1}{2}(b-a)$. So,

$$I_n=\int_{y_n}^{y_{n+1}}\frac{(f(x-y_n))^{g(y_{n+1}-x)}}{(f(y_{n+1}-x))^{g(x-y_n)}+(f(x-y_n))^{g(y_{n+1}-x)}}dx=\frac{1}{2}(y_{n+1}-y_n), \forall n\ge 2.$$

Hence: $\lim_{n\to\infty} I_n=\frac{1}{2}\lim_{n\to\infty}(y_{n+1}-y_n)$.

Lemma 3.

$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,F_{n+1}}-\sqrt[n]{(2n-1)!!\,F_n}\right)=\frac{2\alpha}{e}$, where $\alpha=\frac{1+\sqrt{5}}{2}$, and

$(F_n)_{n\ge0}, F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$ is Fibonacci sequence,

Proof.

$$F_{n+2}-F_{n+1}-F_n=0, \forall n\in\mathbb{N}^*\Leftrightarrow\frac{F_{n+2}}{F_n}-\frac{F_{n+1}}{F_n}-1=0\Leftrightarrow$$

$$\Leftrightarrow\frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n}-\frac{F_{n+1}}{F_n}-1=0$$

$\Rightarrow\lim_{n\to\infty}\left(\frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n}-\frac{F_{n+1}}{F_n}-1\right)=0\Leftrightarrow x^2-x-1=0$, where $x=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}$, so $x=\frac{1\pm\sqrt{5}}{2}$

Since $\frac{F_{n+1}}{F_n}>0, x=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{\sqrt{5}+1}{2}=\alpha$.

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!\,F_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!\,F_n}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!\,F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!\,F_n}=$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{F_{n+1}}{F_n} \left(\frac{n}{n+1}\right)^n = 2 \cdot \alpha \cdot \frac{1}{e} = \frac{2\alpha}{e}.$$

(1) $\sqrt[n+1]{(2n+1)!!\,F_{n+1}} - \sqrt[n]{(2n-1)!!\,F_n} = \sqrt[n]{(2n-1)!!\,F_n}(u_n - 1) =$

$$= \sqrt[n]{(2n-1)!!\,F_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n = \frac{\sqrt[n]{(2n-1)!!\,F_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \forall n \ge 2,$$

$$u_n = \frac{\sqrt[n+1]{(2n+1)!!\,F_{n+1}}}{\sqrt[n]{(2n-1)!!\,F_n}}$$

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \left(\frac{\sqrt[n+1]{(2n+1)!!F_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!F_n}} \cdot \frac{n+1}{n}\right) = \frac{2\alpha}{e} \cdot \frac{e}{2\alpha} \cdot 1 = 1$, so $\lim\limits_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$,

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{(2n+1)!!\,F_{n+1}}{(2n-1)!!\,F_n} \cdot \frac{1}{\sqrt[n+1]{(2n+1)!!\,F_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{F_{n+1}}{F_n} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!\,F_{n+1}}}$$

$= 2 \cdot \alpha \cdot \frac{e}{2\alpha} = e$. Therefore:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(2n+1)!!\,F_{n+1}} - \sqrt[n]{(2n-1)!!\,F_n}\right) = \frac{2\alpha}{e} \cdot 1 \cdot \ln\left(\lim_{n\to\infty} u_n^n\right) = \frac{2\alpha}{e} \cdot \ln e =$$

$$= \frac{2\alpha}{e} \cdot 1 = \frac{2\alpha}{e}.$$

Analogously, $\lim\limits_{n\to\infty} \left(\sqrt[n+1]{(2n+1)!!\,L_{n+1}} - \sqrt[n]{(2n-1)!!\,L_n}\right) = \frac{2\alpha}{e}$

Solution.

By lemma 1: $\lim\limits_{n\to\infty} I_n = \frac{1}{2} \lim\limits_{n\to\infty} (y_{n+1} - y_n) = \frac{1}{2} \cdot \frac{2\alpha}{e} = \frac{\alpha}{e}, \alpha = \frac{1+\sqrt{5}}{2}$ where taking account by lemma 3: $\lim\limits_{n\to\infty} (y_{n+1} - y_n) = \lim\limits_{n\to\infty} \left(\sqrt[n+1]{(2n+1)!\,L_{n+1}} - \sqrt[n]{(2n-1)!!\,L_n}\right) = \frac{2\alpha}{e}$.

UP.609 If $f: [a, b] \to (0, \infty)$ is continuous functions such that

$$f(a+b-x) + f(x) = c, \forall x \in [a, b] \text{ and } a + b = \frac{\pi}{2}, \text{ find:}$$

$$\int_a^b \frac{\sin^n x + f(x) + d}{\sin^n x + \cos^n x + c + 2d} dx,$$

where $n \in \mathbb{N}^*$ and $d \ge 0$.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by proposers

$$x = u(t) = \frac{\pi}{2} - t, u'(t) = -1, u(0) = \frac{\pi}{2}, u\left(\frac{\pi}{2}\right) = 0, u(a) = b, u(b) = a.$$

Hence, $I = \int_a^b \frac{\sin^n x + f(x) + d}{\sin^n x + \cos^n x + c + 2d} dx = \int_b^a \frac{\sin^n\left(\frac{\pi}{2}-t\right) + f\left(\frac{\pi}{2}-t\right) + d}{\sin^n\left(\frac{\pi}{2}-t\right) + \cos^n\left(\frac{\pi}{2}-t\right) + c + 2d}(-1)dt =$

$$= \int_a^b \frac{\cos^n t + \left(c - f(t)\right) + d}{\cos^n t + \sin^n t + c + 2d} dt = \int_a^b \frac{\cos^n x + c - f(x) + d}{\cos^n x + \sin^n x + c + 2d} dx;$$

$2I = \int_a^b \frac{\sin^n x + f(x) + d + \cos^n x + c - f(x) + d}{\cos^n x + \sin^n x + c + 2d} dx = \int_a^b dx = b - a$; **therefore** $I = \frac{b-a}{2}$

UP.610 If $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function and $\gamma_n = -\ln n + \sum_{k=1}^n \frac{1}{k}$, find

$$\lim_{n\to\infty} n \int_\gamma^{\gamma_n} \frac{f(x-\gamma)}{f(\gamma_n - x) + f(x-\gamma)} dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposers

Let $a, b \in \mathbb{R}_+^*, a < b, I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)} dx = \frac{b-a}{2}$

$\lim\limits_{n\to\infty} \gamma_n = \gamma$ **– Euler Mascheroni constant**

Therefore, $I_n = \int_\gamma^{\gamma_n} \frac{f(x-\gamma)}{f(\gamma_n - x)+f(x-\gamma)} dx = \frac{\gamma_n - \gamma}{2}$, **then**

$$\lim_{n\to\infty} nI_n = \frac{1}{2}\lim_{n\to\infty} \frac{\gamma_n - \gamma}{\frac{1}{n}} \underset{\left(\frac{0}{0}\right)}{\overset{Cesaro-Stolz}{=}} \frac{1}{2}\lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\frac{1}{n+1} - \frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty} \frac{\gamma_n - \gamma_{n+1}}{\frac{1}{n(n+1)}} =$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{-\frac{1}{n+1} + \ln\frac{n+1}{n}}{\frac{1}{n(n+1)}} = \frac{1}{2}\lim_{n\to\infty} \frac{-1 + (n+1)\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \overset{\left(x=\frac{1}{n}\right)}{=}$$

$$= \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{-x + (1+x)\ln(1+x)}{x^2} \underset{\left(\frac{0}{0}\right)}{\overset{L'H}{=}}$$

$$= \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} \frac{-1 + \ln(1+x) + (1+x)\cdot\frac{1}{1+x}}{2x} = \frac{1}{4}\lim_{\substack{x\to 0\\ x>0}} \ln(1+x)^{\frac{1}{x}} = \frac{1}{4}\ln e = \frac{1}{4}$$

Solution 2 by Omkumar Rao-India

$$I_n = \int_{\gamma}^{\gamma_n} \frac{f(x-\gamma)}{f(\gamma_n - x) + f(x-\gamma)} dx \Rightarrow (1)$$

$$(x \longrightarrow \gamma + \gamma_n - x): \quad I_n = \int_{\gamma}^{\gamma_n} \frac{f(\gamma_n - x)}{f(x-\gamma) + f(\gamma_n - x)} dx \Rightarrow (2)$$

$$2I_n = \int_{\gamma}^{\gamma_n} 1\, dx$$

$$I_n = \frac{(\gamma_n - \gamma)}{2}, \qquad L = \lim_{n\to\infty} n\frac{(\gamma_n - \gamma)}{2}$$

$$\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}, \qquad \gamma_n - \gamma = -\ln n + \sum_{k=1}^{n} \frac{1}{k} - \gamma$$

$$\sum_{k=1}^{n} \frac{1}{k} : H_n \; as \; n \to \infty, \qquad H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} n \frac{-\gamma - \ln n + \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)}{2} = \lim_{n\to\infty} n \frac{\frac{1}{2n} + O\left(\frac{1}{n^2}\right)}{2}$$

$$L = \frac{1}{4}$$

UP.611 If $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function and $(x_n)_{n\geq 1}, x_n = \sum_{k=1}^{n} \frac{1}{k}$, find

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x - e^{x_n})}{f(e^{x_{n+1}} - x) + f(x - e^{x_n})} dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposers

If $a,b\in\mathbb{R}_+^*, a<b$, then $I=\int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx=\frac{b-a}{2}$.

So, $\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx=\frac{e^{x_{n+1}}-e^{x_n}}{2}=$

$$=\frac{1}{2}\cdot e^{x_n}\cdot\left(e^{x_{n+1}-x_n}-1\right)=\frac{1}{2}\cdot e^{x_n}\cdot(x_{n+1}-x_n)\cdot\frac{e^{x_{n+1}-x_n}-1}{x_{n+1}-x_n}=$$

$$=\frac{1}{2}\cdot e^{x_n}\cdot\frac{1}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}=$$

$$=\frac{1}{2}\cdot\frac{e^{x_n}}{n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}=\frac{1}{2}\cdot e^{-\ln n+x_n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}$$

$$=\frac{1}{2}\cdot e^{\gamma_n}\cdot\frac{n}{n+1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}},$$

Therefore:

$$\lim_{n\to\infty}I_n=\frac{1}{2}\cdot e^{\gamma}\cdot 1\cdot 1=\frac{1}{2}e^{\gamma}$$

$\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$ and $\lim_{n\to\infty}\gamma_n=\gamma$ Euler – Mascheroni constant.

Solution 2 by Omkumar Rao-India

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\qquad\to(1)$$

$$(x\to e^{x_n}+e^{x_{n+1}}-x)$$

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(e^{x_{n+1}}-x)}{f(x-e^{x_n})+f(e^{x_{n+1}}-x)}dx\to(2)$$

$$2I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}1\,dx,\qquad I_n=\frac{e^{x_{n+1}}-e^{x_n}}{2}$$

$$e^{x_{n+1}}-e^{x_n}=e^{x_n}\left(e^{\frac{1}{n+1}}-1\right)=e^{x_n}\left(\frac{1}{n+1}+O\left(\frac{1}{n^2}\right)\right)$$

$$x_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

where γ is Euler Mascheroni constant.

$$I_n = ne^{\gamma}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{1}{n+1}\right)$$

$$I_n = \frac{e^{\gamma}}{2}\cdot\frac{n}{n+1}\cdot\left(1 + O\left(\frac{1}{n}\right)\right)$$

$$As\ n \to \infty, \frac{n}{n+1} \to 1, 1 + O\left(\frac{1}{n}\right) \to 1$$

$$\lim_{n\to\infty}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx = \frac{e^{\gamma}}{2}$$

UP.612 If $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function and $(a_n)_{n\ge1}$ is defined by $a_1 = a_2 = 1$,

$a_{n+1} = \sum_{k=1}^{n}\frac{a_k}{k}, \forall n \ge 2$ and $(x_n)_{n\ge1}, x_n = \sum_{k=1}^{n}\frac{1}{a_k}$, find

$$\lim_{n\to\infty}\frac{1}{n}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by proposer

If $a, b \in \mathbb{R}_+^* a < b$, then $I = \int_a^b \frac{f(x-a)}{f(b-x)+f(x-a)}dx = \frac{b-a}{2}$.

So, $I_n = \int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx = \frac{e^{x_{n+1}}-e^{x_n}}{2} =$

$$= \frac{1}{2}\cdot e^{x_n}\cdot\left(e^{x_{n+1}-x_n}-1\right) = \frac{1}{2}\cdot e^{x_n}\cdot\left(e^{\frac{1}{a_{n+1}}}-1\right) = \frac{1}{2}\cdot\frac{e^{x_n}}{a_{n+1}}\cdot\frac{e^{\frac{1}{a_{n+1}}}-1}{\frac{1}{a_{n+1}}},\ (2).$$

By induction we obtain $a_n = \frac{n}{2}, \forall n \ge 2$, so $x_n = 2\cdot\sum_{k=1}^{n}\frac{1}{k} - 1$, then by (2)

$$\frac{1}{n}I_n = \frac{1}{2}\cdot\frac{e^{x_n}}{n^2}\cdot\left(e^{\frac{1}{a_n}}-1\right)\cdot n = \frac{1}{2}\cdot\frac{e^{x_n}}{n^2}\cdot\left(e^{\frac{2}{n+1}}-1\right)\cdot n =$$

$$=\frac{1}{2}e^{x_n-2\ln n}\cdot\left(e^{\frac{1}{n+1}}+1\right)\left(e^{\frac{1}{n+1}}-1\right)\cdot n=$$

$$=\frac{1}{2}\cdot\left(e^{\frac{1}{n+1}}+1\right)\cdot e^{2\sum_{k=1}^{n}\frac{1}{k}-2\ln n-1}\cdot\frac{e^{\frac{1}{n+1}}}{\frac{1}{n+1}}\cdot\frac{n}{n+1}=\frac{1}{2}\left(e^{\frac{1}{n+1}}+1\right)\cdot e^{2\gamma_n-1}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\frac{n}{n+1}\quad(3)$$

Since $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$ and $\lim_{n\to\infty}\gamma_n=\gamma$ – Euler- Mascheroni constant and

$$\lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}=1,\text{ hence }\lim_{n\to\infty}\frac{1}{n}I_n=\frac{1}{2}\cdot(1+1)\cdot e^{2\gamma-1}\cdot 1\cdot 1=e^{2\gamma-1}.$$

Solution 2 by Omkumar Rao-India

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\Rightarrow(1)$$

$$(x\to e^{x_n}+e^{x_{n+1}}-x):$$

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(e^{x_{n+1}}-x)}{f(x-e^{x_n})+f(e^{x_{n+1}}-x)}dx\Rightarrow(2)$$

$$2I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}1\,dx,\qquad I_n=\frac{(e^{x_{n+1}}-e^{x_n})}{2},\qquad L=\lim_{n\to\infty}\frac{1}{2n}(e^{x_{n+1}}-e^{x_n})$$

$$a_{n+1}=\sum_{k=1}^{n}\frac{a_k}{k}=\left(\sum_{k=1}^{n-1}\frac{a_k}{k}\right)+\frac{a_n}{n}=a_n+\frac{a_n}{n}$$

$$a_n=\frac{n}{2}$$

$$x_n=\sum_{k=1}^{n}\frac{1}{a_k}=\frac{1}{a_1}+\sum_{k=2}^{n}\frac{2}{k},\qquad x_n=1+2(H_n-1)=2H_n-1$$

$$H_n=\gamma+\ln n+\frac{1}{2n}+O\left(\frac{1}{n^2}\right),\quad x_n=2\gamma+2\ln n+\frac{1}{n}+2O\left(\frac{1}{n^2}\right)-1$$

$$e^{x_n}=e^{(2\gamma-1)+2O\left(\frac{1}{n^2}\right)+\ln n^2}=n^2e^{2\gamma-1}e^{2O\left(\frac{1}{n^2}\right)}$$

$$e^{2O\left(\frac{1}{n^2}\right)} = 1 + 2O\left(\frac{1}{n^2}\right) \approx 1, \quad e^{x_n} = n^2 e^{2\gamma-1}$$

$$x_{n+1} = x_n + \frac{1}{a_{n+1}}$$

$$L = \lim_{n\to\infty} \frac{e^{x_n}\left(e^{\frac{2}{n+1}} - 1\right)}{2n}, \qquad L = \lim_{n\to\infty} \frac{n^2 e^{2\gamma-1}(2)}{2n(n+1)}$$

$$L = e^{2\gamma-1}$$

UP.613 Let be $A(1,2,0)$; $B(2,0,1)$; $C(3,3,3)$. Find the area of ΔABC.

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\overrightarrow{AB} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath} + (z_B - z_A)\vec{k}$$

$$\overrightarrow{AB} = (2-1)\vec{\imath} + (0-2)\vec{\jmath} + (1-0)\vec{k}$$

$$\overrightarrow{AB} = \vec{\imath} - 2\vec{\jmath} + \vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{\imath} + (y_C - y_A)\vec{\jmath} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (3-1)\vec{\imath} + (3-2)\vec{\jmath} + (3-0)\vec{k}$$

$$\overrightarrow{AC} = 2\vec{\imath} + \vec{\jmath} + 3\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -6\vec{\imath} + 2\vec{\jmath} + \vec{k} + 4\vec{k} - 3\vec{\jmath} - \vec{\imath}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -7\vec{\imath} - \vec{\jmath} + 5\vec{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-7)^2 + (-1)^2 + 5^2} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area } (\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{5\sqrt{3}}{2}$$

Solution 2 by José Luis Díaz – Barrero – Spain

The area of a triangle formed by two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by the formula:

$$Area = \frac{1}{2}\left|\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right|$$

We have,

$$\overrightarrow{AB} = B - A = (2-1, 0-2, 1-0) = (1, -2, 1),$$

$$\overrightarrow{AC} = C - A = (3-1, 3-2, 3-0) = (2, 1, 3),$$

and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

Expanding along the first row, we get

$$\overrightarrow{AB} \times \overrightarrow{AC} = i\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= i\big((-2)(3) - (1)(1)\big) - j\big((1)(3) - (1)(2)\big) + k\big((1)(1) - (-2)(2)\big)$$

$$= i(-6-1) - j(3-2) + k(1+4)$$

$$= -7i - 1j + 5k = (-7, -1, 5)$$

Now, we find the length or norm of the cross product vector:

$$\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \sqrt{(-7)^2 + (-1)^2 + (5)^2}$$

$$= \sqrt{75} = 5\sqrt{3}$$

Plugging the magnitude back into the triangle formula, we obtain

$$Area = \frac{1}{2} \cdot 5\sqrt{3} = \frac{5\sqrt{3}}{2}$$

Solution 3 by Marin Chirciu-Romania

We use: $S(\Delta ABC) = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$.

In our case:

$$\overrightarrow{AB} = (2-1)\vec{i} + (0-2)\vec{j} + (1-0)\vec{k} = \vec{i} - 2\vec{j} + \vec{k} \Rightarrow$$

$$\Rightarrow \left\|\overrightarrow{AB}\right\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}.$$

$$\overrightarrow{AC} = (3-1)\vec{i} + (3-2)\vec{j} + (3-0)\vec{k} = 2\vec{i} + \vec{j} + 3\vec{k} \Rightarrow$$

$$\Rightarrow \left\|\overrightarrow{AC}\right\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \left\|\overrightarrow{AB}\right\| \cdot \left\|\overrightarrow{AC}\right\| \cdot \sin\angle\big(\overrightarrow{AB}, \overrightarrow{AC}\big) = \sqrt{6} \cdot \sqrt{14} \cdot \frac{5}{2\sqrt{7}} = 5\sqrt{3}, \text{ see:}$$

$$\cos\angle(\overrightarrow{AB},\overrightarrow{AC})=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\cdot\left\|\overrightarrow{AC}\right\|}=\frac{(\vec{\imath}-2\vec{\jmath}+\vec{k})\cdot(2\vec{\imath}+\vec{\jmath}+3\vec{k})}{\sqrt{6}\cdot\sqrt{14}}=\frac{1\cdot 2+(-2)\cdot 1+1\cdot 3}{2\cdot\sqrt{21}}$$

$$=\frac{3}{2\sqrt{21}}\Rightarrow\sin\angle(\overrightarrow{AB},\overrightarrow{AC})=\sqrt{1-\left(\frac{3}{2\sqrt{21}}\right)^2}=\frac{5}{2\sqrt{7}}$$

We deduce $S(\Delta ABC)=\frac{1}{2}\cdot 5\sqrt{3}=\frac{5\sqrt{3}}{2}$

UP.614 If $0<a\le b$ then:

$$\int_a^b\int_a^b\frac{dxdy}{x^2-xy+y^2}+3\left(\ln\left(\frac{b}{a}\right)\right)^2\ge 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\left(\ln\left(\frac{b}{a}\right)\right)=\ln\frac{b}{a}\cdot\ln\frac{b}{a}=\int_a^b\frac{1}{x}dx\cdot\int_a^b\frac{1}{y}dy=\int_a^b\int_a^b\frac{dxdy}{xy}$$

The inequality to prove can be written:

$$\int_a^b\int_a^b\frac{dxdy}{x^2-xy+y^2}+3\int_a^b\int_a^b\frac{dxdy}{xy}\ge\int_a^b\int_a^b\frac{16dxdy}{(x+y)^2}$$

$$\int_a^b\int_a^b\left(\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\right)dxdy\ge\int_a^b\int_a^b\frac{16}{(x+y)^2}dx\,dy$$

It is enough to prove that:

$$\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\ge\frac{16}{(x+y)^2}$$

$$\frac{xy+3x^2-3xy+3y^2}{xy(x^2-xy+y^2)}\ge\frac{16}{x^2+2xy+y^2}$$

$$(3x^2-2xy+3y^2)(x^2+2xy+y^2)\ge 16xy(x^2-xy+y^2)$$

$$3x^4+6x^3y+3x^2y^2-2x^3y-4x^2y^2-2xy^3+3x^2y^2+$$

$$+6xy^3+3y^4\ge 16x^3y-16x^2y^2+16xy^3$$

$$3x^4+3y^4-12x^3y-12xy^3+18x^2y^2\ge 0$$

$$x^4-4x^3y+6x^2y^2-4xy^3+y^4\ge 0$$

$$(x-y)^4\ge 0.\ \text{True!}$$

Equality holds for $a=b$.

Solution 2 by Tapas Das-India

$$\text{We will show}:$$
$$\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\geq\frac{16}{(x+y)^2}$$
$\text{Proof}:\text{We need to show}:$
$$\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\geq\frac{16}{(x+y)^2}$$
$$\frac{1}{u^2-u+1}+\frac{3}{u}\overset{u=\frac{x}{y}>0}{\geq}\frac{16}{(u+1)^2}$$
$$(u+1)^2(3u^2-2u+3)\geq 16u(u^2-u+1)$$
$$3u^4+4u^3+2u^2+4u+3\geq 16u^3-16u^2+16u$$
$$3(u^4-4u^3+6u^2-4u+1)\geq 0$$
$$3(u-1)^4\geq 0\ \text{true, so}\ \frac{1}{x^2-xy+y^2}+\frac{3}{xy}\geq\frac{16}{(x+y)^2}$$
$$\text{now}\int_a^b\int_a^b\left(\frac{1}{x^2-xy+y^2}+\frac{3}{xy}\right)dxdy\geq 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$
$$\int_a^b\int_a^b\frac{dxdy}{x^2-xy+y^2}+3\int_a^b\int_a^b\frac{dxdy}{xy}\geq 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$
$$\int_a^b\int_a^b\frac{dxdy}{x^2-xy+y^2}+3\left(\ln\left(\frac{b}{a}\right)\right)^2\geq 16\int_a^b\int_a^b\frac{dxdy}{(x+y)^2}$$

Equality holds for a=b.

UP.615 Prove that $\frac{8}{5}$ is the smallest positive value of the constant k such that

$$\frac{1}{(a+b)^2+k}+\frac{1}{(b+c)^2+k}+\frac{1}{(c+a)^2+k}\leq\frac{3}{4+k}$$

for all side lengths a,b,c of a triangle with $ab+bc+ca=3$.

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

For $\frac{a}{2}=b=c=\sqrt{\frac{3}{5}}$, the constraint is satisfied and the inequality becomes

$$\frac{10}{27+5k}+\frac{5}{12+5k}\leq\frac{3}{4+k},$$

that is $k \geq \frac{8}{5}$. To show that $\frac{8}{5}$ is the smallest positive value of k, we need to prove that $E \leq 0$ for $k = \frac{8}{5}$, where

$$E = \frac{1}{(a+b)^2+k} + \frac{1}{(b+c)^2+k} + \frac{1}{(c+a)^2+k} - \frac{3}{4+k}.$$

Assume $a \geq b \geq c$ and denote

$$X = (a+b)^2+k, \qquad Y = (a+c)^2+k, \qquad Z = (b+c)^2+k.$$

For fixed a, assume that c and E are functions of b. By differentiating the equality constraint, we get

$$(a+b)c' + a + c = 0.$$

Therefore

$$\frac{E'(b)}{2} = -\frac{a+b}{X^2} - \frac{b+c}{Z^2} - \left[\frac{a+c}{Y^2} + \frac{b+c}{Z^2}\right]c'$$
$$= -\frac{a+b}{X^2} + \frac{b+c}{Z^2} + \left[\frac{a+c}{Y^2} + \frac{b+c}{Z^2}\right]\frac{a+c}{a+b}$$
$$= -\frac{a+b}{X^2} + \frac{(a+c)^2}{(a+b)Y^2} - \frac{(b-c)(b+c)}{(a+b)Z^2}.$$

We will show that $E'(b) \leq 0$, that is

$$\frac{(b-c)(b+c)}{Z^2} \geq \frac{(a+c)^2}{Y^2} - \frac{(a+b)^2}{X^2}, \qquad \frac{(b-c)(b+c)}{Z^2} \geq \frac{AB}{X^2Y^2},$$

where

$$A = (a+c)X - (a+b)Y, B = (a+c)X + (a+b)Y.$$

Since

$$A = (a+b)(a+c)(b-c) - k(b-c) = (b-c)(a^2+3-k),$$
$$B = (a+b)(a+c)(2a+b+c) + k(2a+b+c) = (2a+b+c)(a^2+3+k),$$

and

$$XY \geq [(a+b)(a+c)+2]^2 = (a^2+3+k)^2,$$

it suffices to show that

$$\frac{b+c}{Z^2} \geq \frac{(2a+b+c)(a^2+3-k)}{(a^2+3+k)^3},$$

which is equivalent to

$$f(x) = \frac{a^2+3-k}{(a^2+3+k)^3},$$

where

$$f(x) = \frac{x}{(x^2+k)^2(2a+x)}, \qquad x = b+c.$$

From

$$12 - 3(b+c)^2 = 4(ab+bc+ca) - 3(b+c)^2 \geq 4(b^2+2bc) - 3(b+c)^2$$
$$= b^2 + 2bc - 3c^2 \geq 0,$$

we get $b+c \leq 2$, hence $x \in [a,2]$ with $a \geq 1$. We have

$$f'(x) = \frac{-A}{(x^2+k)^3(2a+x)^2}, \quad A = 4x^3 + 6ax^2 - 2ka.$$

Since $A \geq 4a^3 + 6a^3 - 2ka = 2a(5a^2 - k) > 0$, we have $f'(x) < 0, f(x)$ is decreasing, hence

$$f(x) \geq f(2) = \frac{1}{(4+k)^2(a+1)}.$$

From $(2a-3)^2 \geq 0$, we get

$$12a \leq 4a^2 + 9, \qquad a + 1 \leq \frac{4a^2+21}{12}, \frac{1}{a+1} \geq \frac{12}{4a^2+21},$$

hence

$$f(x) \geq \frac{12}{(4+k)^2(4a^2+21)}.$$

So, it suffices to show that

$$\frac{12}{(4+k)^2(4a^2+21)} \geq \frac{a^2+3-k}{(a^2+3+k)^3}.$$

Denoting $t = a^2 \geq 1$, the inequality is equivalent to $g(t) \geq 0$, where

$$g(t) = 12(t+3+k)^3 - (4+k)^2(4t+21)(t+3-k).$$

We have

$$g'(t) = 36(t+3+k)^2 - (4+k)^2(8t+33-4k), \quad g''(t) = 72(t+3+k) - 8(4+k)^2.$$

Since

$$g''(t) \geq 72(4+k) - 8(4+k)^2 = 8(4+k)(5-k) > 0,$$

$g'(t)$ is increasing, hence

$$g'(t) \geq g'(1) = 36(4+k)^2 - (4+k)^2(41-4k) = (4+k)^2(4k-5) > 0,$$

therefore $g(t)$ is increasing and

$$g(t) \geq g(1) = 12(4+k)^3 - 25(4+k)^2(4-k) = (4+k)^2(37k-52) > 0.$$

Since $E'(b) \leq 0$, then $E(b)$ is decreasing and has the maximum value when b is minimum, hence when $b = \max\{c, a-c\}$. So, it suffices to consider the following two cases: $b = c$ and $b = a - c$.

Case 1: $b = c \geq \frac{a}{2}$. For $b = c := z \geq \frac{a}{2}$, the equality constraint becomes $2az + z^2 = 3$, and from $3 = 2az + z^2 \leq 4z^2 + z^2 = 5z^2$, we get $5z^2 - 3 \geq 0$. The inequality becomes

$$\frac{2}{(a+z)^2+k} + \frac{1}{4z^2+k} \leq \frac{3}{4+k},$$

which is equivalent to

$$\frac{32z^2}{5z^4+62z^2+45} + \frac{1}{5z^2+2} \leq \frac{3}{6},$$

$$5z^6 - 13z^4 + 11z^2 - 3 \geq 0, \qquad (z^2-1)^2(5z^2-3) \geq 0.$$

The last inequality is clearly true.

Case 2: $a = b + c$. We need to show that $(b+c)^2 + bc = 3$ implies

$$\frac{1}{(b+c)^2+k} + \frac{1}{(2b+c)^2+k} + \frac{1}{(b+2c)^2+k} \leq \frac{3}{4+k}.$$

Let $x = b + c$. We have

$$bc = 3 - x^2,$$

and from $x^2 \geq 4bc = 4(3-x^2)$, we get

$$x^2 \geq \frac{12}{5}.$$

Since

$$\frac{1}{(2b+c)^2+k}+\frac{1}{(b+2c)^2+k}=\frac{5(b+c)^2-2bc+2k}{[(2b+c)^2+k][(b+2c)^2+k]} \leq$$
$$\leq \frac{5(b+c)^2-2bc+2k}{[(2b+c)(b+2c)+k]^2}$$
$$=\frac{5x^2-2bc+2k}{(2x^2+bc+k)^2}=\frac{7x^2-6+2k}{(x^2+3+k)^2},$$

it suffices to show that

$$\frac{1}{x^2+k}+\frac{7x^2-6+2k}{(x^2+3+k)^2} \leq \frac{3}{4+k},$$

i.e.

$$\frac{1}{5x^2+8}+\frac{35x^2-14}{(5x+23)^2} \leq \frac{3}{28}.$$

Substituting

$$5x^2=12t, \qquad t \geq 1,$$

the inequality becomes as follows:

$$\frac{1}{4(3t+2)}+\frac{14(6t-1)}{(12t+23)^2} \leq \frac{3}{28},$$
$$432t^3-744t^2+227t+85 \geq 0, \qquad (t-1)(432t^2-312t-85) \geq 0.$$

Since $t \geq 1$, the last inequality is clearly true.

The proof is completed. For $k=\frac{8}{5}$, the equality occurs when $a=b=c=1$, and also when $\frac{a}{2}=b=c=\sqrt{\frac{3}{5}}$ (or any cyclic permutation).

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

For $a=b=x \in (0,\sqrt{3})$ and c

$=\frac{3-x^2}{2x}$, the constraints are satisfied and the inequality becomes

$$\frac{1}{4x^2+k}+\frac{2}{\left(x+\frac{3-x^2}{2x}\right)^2+k} \leq \frac{3}{4+k}.$$

Since $a+b>c$, we have $2x>\frac{3-x^2}{2x}$, then $x>\sqrt{\frac{3}{5}}$, and for x

$\to \sqrt{\frac{3}{5}}$, we get the necessary

condition $\frac{1}{\frac{12}{5}+k}+\frac{2}{\frac{27}{5}+k}\le\frac{3}{4+k}$, that is $k\ \ge\frac{8}{5}$.
To show that $\frac{8}{5}$ is the smallest positive value of k, we will prove the given inequality for $k=\frac{8}{5}$.
The given inequality is equivalent to:
$$\frac{(a+b)^2}{(a+b)^2+k}+\frac{(b+c)^2}{(b+c)^2+k}+\frac{(c+a)^2}{(c+a)^2+k}\ge\frac{12}{4+k}.$$
Let $2p=a+b+c\ge\sqrt{3(ab+bc+ca)}=3$ and $r=abc$.
By Cauchy − Schwarz's inequality, we have:
$$\sum_{cyc}\frac{(b+c)^2}{(b+c)^2+k}\ge\frac{\left[\sum_{cyc}(b+c)(5s+2a)\right]^2}{\sum_{cyc}(5s+2a)^2((b+c)^2+k)}\overset{?}{\ge}\frac{12}{4+k}$$
$$\Leftrightarrow 252+333p^2-100p^4-330pr\ge 0.$$
Let $a=y+z, b=z+x, c=x+y$. Since $ab+bc+ca=3$, we have $xy+yz+zx=3-p^2$,
and $r=abc=(x+y+z)(xy+yz+zx)-xyz=p(3-p^2)-xyz$, then the last inequality is equivalent to
$$252-657p^2+230p^4+330pxyz\ge 0 \qquad (*)$$
If $p^2\ge\frac{12}{5}$, we have $LHS_{(*)}\ge 252-657p^2+230p^4=(5p^2-12)(46p^2-21)\ge 0$.
If $\frac{9}{4}\le p^2\le\frac{12}{5}$. By the fourth degree Schur's inequality, we have
$$6pxyz\ge\left(p^2-\sum_{cyc}yz\right)\left(4\sum_{cyc}yz-p^2\right)=(2p^2-3)(12-5p^2)$$
Then:
$$LHS_{(*)}\ge 252-657p^2+230p^4+55(2p^2-3)(12-5p^2)=16(12-5p^2)(4p^2-9)\ge 0,$$
So, the proof is completed. The equality occurs when $a=b=c=1$.