

COMMERCIAL MATHEMATICS

(For Three Year B.Com. Course)

By

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PREFACE

This book is intended for the students of Three Year Com. Course. It consists of three parts—Arithmetic, Algebra & Statistics.

The book is expected to serve as a self-educator. Numerous examples of varied types have been worked out in simple & concise methods. Various examples mostly culled from different examination papers have been graded and appended at the end of the Chapters.

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Notices of any error which might have inadvertently crept in and suggestions for improvement will be thankfully received.

S. K. Sur

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SYLLABUS FOR COMMERCIAL MATHEMATICS

I. Principally Arithmetical Operations :

(Help of elementary Algebraic or Geometric properties may be taken wherever found suitable).

1. Drills in rapid arithmetical calculations by application of artifices and short methods to practical illustrations. Fractions, Decimals, Decimalisation of concrete quantities. Square root. Metric System of Units.

2. Approximation and Errors :

Degrees of working approximation in different contexts—Significant figures—Errors (absolute and relative). Contracted Method of multiplication and division.

Illustrations from some ready rules of conversion of weights and measures, and errors therein with reference to English, Metric and existing Indian measures.

3. Recapitulation of ideas on Ratio, Proportion, Proportional parts, Variation, Rule of Three (Simple and compound) and chain Rule ; Percentage—Working rules for ready applications. *Percentage calculations on proportional, step and slab basis and computation of Effective Rates.*

Typical Business Calculations : (students are expected to calculate arithmetical problems on the following subjects) :—

(a) Elements of Manufacturing cost—

Direct and Indirect cost, Prime cost, Works cost, Allocation of Overcost—Total cost, Costing of articles where some items are fixed and some variable. Standard cost and cost budgeting.

(b) Price fixing and quotations—Total cost, Profit, Discount. Catalogue price.

(c) Calculations relating to Commission and Brokerage, freight, rebates, insurance premium and claims.

(d) Problems of Gains and Losses, Mixture problems.

(e) Partnership problems—Apportionment of Partnership Incomes and Profits.

(f) Profit in relation to turnover and costs—Operating ratio in Sales, Cost price of goods sold during a given period and average cost of stock held—Rapidity of Turnover. Inter-relation of capital, turnover, working expenses, gross and net profits, taxation, dividend and retained profits with particular reference to Companies where necessary.

II. Algebraic and Other Operations :

- (a) Laws of Indices : Operations with exponential radicals.
- (b) Equations : Simultaneous and Quadratic- Rule of Cross Multiplication.
- (c) Series : Arithmetic and Geometric Progression. Simple applications of Formulae.
- (d) Permutation and Combinations : Simple application of formulae.
- (e) Elementary notions of Probability and Chance calculation.
- (f) Logarithms : Basic formulae and use of Tables.
- (g) Compound interest and Immediate Annuities. Finding the term of annuity, calculation of instalment amount. Application of Payments. Problems on Population, Depreciation and Investments.
- (h) Bill Discounting—True and Banker's Discount—Average Due date and Equated Payments.
- (i) Foreign Exchange, Quotation, Calculations—Methods of Arbitrated Rate, Short and long rates, Discounting of Foreign Bills.
- (j) Simple Mensuration : Triangle, Parallelogram, Trapezium, Rectangle, Square and Circle, Rectangular Parallelopiped, Prism, Cylinder, Pyramid, Cone and Sphere.
- (k) Dealings in Stocks and Shares.

III. Elementary Statistical Concepts and Graphical Representations :

- (a) Numerical data including elementary notions of Sampling.
- (b) Classification and Tabulations—Frequency Distribution.
- (c) Measures of Central Tendency, Mean, Mode and Median. Measures of Dispersion (elementary idea only).

(Median computation from Frequency Table to be excluded)

(d) Computation of Index Number with special reference to the principal Index Numbers of Production and Prices as computed by the Government of India.

(e) Graphical Representation of quantitative data—Linear Bar, Histogram, Pie-chart, Logarithmic Graphs (Elementary only). Graphics as applied to budgeting and business performance and presentation of results.

(f) Students will be expected to have a knowledge of the Principal sources of main statistical data in major fields of economic interest, e.g., Population, Employment, Production, Trade, Income and Prices.

Chapter I

Rapid Arithmetical Calculations

1. Addition *Ex. 1.* Add together

	<i>T</i>	<i>H</i>	<i>T</i>	<i>U</i>		
Rs	6	4	2	9	↓	Add together the unit column. <i>Tick off</i> for every complete 10. Put below the addition line excess over 10. Count the number of ticks and <i>carry it over</i> to the next column and proceed as before.
	8'	4	4	8'		
	7'	0	8'	5'		
	9'	6'	0	6		
	3	1	5	6	8	

Thus

<i>Column</i>	<i>tick</i>	<i>excess</i>	<i>carry</i>
Unit	2	8	2
Ten	1	6	1
Hundred	1	5	1
Thousand	3	1	3

Ex. 2. Add together

£	s	d		
4	2	14	3	↓
1	8'	9'	10'	
4	'5	16	8	
2	9'	9'	2	
1	3	6	9	11

Add together the pence column. Tick off for every complete 12 d. Put below the addition line excess over 12. Count the number of ticks and carry it over to

the next Column. Add this column and tick off for every complete 20s or pound. Put the excess below the line and carry the number of ticks over to the next column.

Thus	<i>Column</i>	<i>tick</i>	<i>excess</i>	<i>carry</i>
	Pence (d)	1	11	1
	Shilling (s)	2	9	2
	Pound (£)	{ 2 1	{ 6 3	{ 2 1 }

2. Long and Cross Tots. By such device, correctness of the totals of a large number of figures can be verified by the grand total.

Ex. 3. Add up vertically (long totals) and horizontally (cross totals).

1 2 3	9 1 4	3 1 2	5 7 0	4 8 3 = 2 4 0 2
4 6 8'	7'2 9'	4 6 3	1 2 8	1 2'4 = 1 9' 1 2
6'3'1	2 0 7	1 8'9'	3'5'1	7'2 1 = 2 0 9'9'
4 3 2	1'9'2'	3'0 5	1 6'9'	9'8'2' = 2 0 8 0
1 6 5 4	2 0 4 2	1 2 6 9	1 2 1 8 2 3 1 0	= 8 4 9 3

3. Addition and Subtraction. Addition and Subtraction may be combined together as in balancing cash account.

Ex. 4. Balance the following cash account.

Receipts	Rs	nP	Payments	Rs	nP
To Balance of cash			By capital Expen-		
on 1st. Jan, 1960	4000	34	diture	2918	60
" Interest on Secu-			" Rent	1234	50
rities	789	96	" Salary	2680	00
" Goods	2268	50	" Printing	780	25
" Subscriptions			" Balance of Cash		
received	1800	78	on 31. 12. 1961	1246	23
	8859	58		8859	58

Total of first column of R.H.S. is 5 nP. which is deducted directly from 8 nP. of total and the difference 3 nP. is put down against balance. The total of second column is 1 tick and residue 3. 3 is subtracted from 5 giving 2 which is at once put down. Carry 1 and add to the unit column of rupees and proceed as usual. The use of ticks makes calculation easier and quicker.

4. Multiplication. Multiplication can be performed as follows :

Ex. 5. Multiply 438 by 576

First Method

$$\begin{array}{r}
 438 \dots (\text{multiplicand}) \\
 576 \dots (\text{multiplier}) \\
 \hline
 2628 \\
 3066 \\
 2190 \\
 \hline
 252288 \dots (\text{product})
 \end{array}$$

Second Method

$$\begin{array}{r}
 438 \\
 576 \\
 \hline
 219000 \text{ product of } 500 \\
 30660 \text{ " " } 70 \\
 2628 \text{ " " } 6 \\
 \hline
 252288 \text{ " " } 576.
 \end{array}$$

Third Method (multiplication in one line)

438	carry	place
576		

Process.	8×6	$= 4$	8
	$8 \times 7 + 6 \times 3 + 4$	$= 7$	8
	$8 \times 5 + 6 \times 4 + 3 \times 7 + 7$	$= 9$	2
	$7 \times 4 + 3 \times 5 + 9$	$= 5$	2
	$4 \times 5 + 5$	$= 2$	5

The above process can be generalised.

5. Short Methods of Multiplication.

(1) To multiply by nines only :

$$467 \times 9 = 467 \times (10 - 1) = 4670 - 467$$

$$467 \times 99 = 467 \times (100 - 1) = 46700 - 467 \text{ etc.}$$

Put as many zeroes as there are nines after the multiplicand and subtract from it the multiplicand itself.

(2) To multiply by numbers like 98, 97, 998, 997, 101, 102, 1005,.....etc.

$$467 \times 98 = 467 \times (100 - 2) = 46700 - 934 = 45766$$

$$467 \times 998 = 467 \times (1000 - 2) = 467000 - 934 = 466066$$

$$467 \times 101 = 467 \times (100 + 1) = 46700 + 467 = 47167 \text{ etc.}$$

(3) To find squares

$$(78)^2 = (80 - 2)^2 = (80)^2 + (2)^2 - 2.80.2 = 6084$$

$$\text{or} = (70+8)^2 = (70)^2 + (8)^2 + 2.70.8 = 6084$$

$$[(a+b)^2 = a^2 + b^2 + 2ab, (a-b)^2 = a^2 + b^2 - 2ab]$$

$$(746)^2 - (741)^2 = (746+741)(746-741) = 1487 \times 5 = 7435$$

$$[a^2 - b^2 = (a+b)(a-b)] \quad \text{etc.}$$

6. Casting out nines to check a product.

Ex. 6. Verify $438 \times 576 = 252288$.

Multiplicand. Sum of digits $= 4+3+8=15$; $15 \div 9$, remainder $= 6$. The above can be simplified thus

$$4+3+8=15. \quad 1+5=6. \quad \text{remainder} = 6 \dots \dots (1)$$

$$\text{Multiplier. } 5+7+6=18; \quad 1+8=9. \quad \text{remainder} = 0 \dots (2)$$

$$\text{Product } (1) \times (2) = 6 \times 0 = 0. \quad \text{remainder} = 0 \dots (3)$$

$$\text{Actual product } 2+5+2+2+8+8=27; \quad 2+7=9, \\ \text{remainder} = 0 \dots (4)$$

For correctness of the result, the remainders (3) & (4) must agree.

7. Division. Tests of divisibility.

A number is divisible (divided without remainder) by

(1) 2 if its last digit is 0 or even

(2) 5 " " " " " 0 or 5

(3) 3 if the sum of the digits is divisible by 3

(4) 9 " " " " " by 9

(5) 4 if the last two digits are 0 or divisible by 4

(6) 8 " " three " " " 8

(7) 11 if the difference between the sums of digits in the odd and even places is 0 or divisible by 11.

8. Division by nines

Ex. 7. Divide 45627 by (i) 9 (ii) 99 etc.

(i) 45627 ÷ 9	4 5 6 2 7	To divide by one 9 line off
(dividend) (divisor)	4'5'6' 2	one digit to the right.
	4 5'6'	Next shift the number one
	4 5	place to the right and ignore
	4'	the right hand digit. Proceed
	5 0 6 9 4	thus. Add them up. The
		left hand number is the quo-

tient. The right hand number with number of ticks gives remainder.

$$\text{Quotient} = 5069, \text{ remainder} = 4 + 2 = 6$$

(ii) $45627 \div 99$. 4 5 6 | 27 To divide by two nines (99)
 4' 56' line off *two* digits. Next
 4 shift the number *two* places
 4 6 0 | 77 to the right. Proceed as
 above.

$$\text{Quotient} = 460, \text{ remainder} = 77 + 1 = 78 \quad \text{etc.}$$

9. Division by powers of 10.

Ex. 8. Divide 4967 by 10, 100, 1000 etc.

Shift the decimal to the left by as many places as there are zeroes.

$$\text{Thus } 4967 \div 10 = 496.7, 4967 \div 100 = 49.67, \text{ etc.}$$

10. Division by powers of 5

Ex. 9. Divide 61258 by 5, 25, 125...etc.

$$61258 \div 5 = \frac{61258 \times 2}{10} = 6125.8 \times 2 = 12251.6$$

$$61258 \div 25 = \frac{61258 \times 4}{100} = 612.58 \times 4 = 2450.32$$

$$61258 \div 125 = \frac{61258 \times 8}{1000} = 61.258 \times 8 = 490.064$$

11. Division by Factors

Ex. 10. Divide 4796 by 1155

$$1155 = 3 \times 5 \times 7 \times 11$$

(1st D)...3 | 4796

(2nd D)...5 | 1598 + 2... (1st R)

(3rd D).....7 | 319 + 3... (2nd R)

(4th D).....11 | 45 + 4... (3rd R)

 4 + 1... (4th R)

Quotient = 4.

Remainder = $2 + 3 \times 3$

+ $4 \times (3 \times 5) + 1 \times (3 \times 5 \times 7)$

= $2 + 9 + 60 + 105 = 176$.

Remainder = 1st R + 2nd R $\times$ 1st D + 3rd R $\times$ 1st D $\times$ 2nd D etc.

Fractions and Decimals

1. Conversion of fractions into decimals and vice versa.

$$\frac{3}{4} = .75,$$

$$\frac{8}{9} = .8888\dots = .\dot{8}$$

$$\frac{2}{7} = .285714285714\dots = .\dot{2}8571\dot{4}$$

$$\frac{5}{17} = .294117647058823\dot{5}$$

Again, $.75 = \frac{75}{100} = \frac{3}{4}$.

$$.\dot{8} = \frac{8}{9}$$

$$.3\dot{8} = \frac{438 - 43}{90} = \frac{395}{90} = \frac{79}{18} = 4\frac{7}{18}$$

$$\text{or } = 4\frac{38 - 3}{90} = 4\frac{35}{90} = 4\frac{7}{18}$$

$$2.\dot{6}4\dot{5} = 2\frac{645}{999} = 2\frac{215}{333} \quad \text{etc.}$$

2. *An ingenious method of conversion if the denominator ends with 9.*

$$\frac{9}{13} = \frac{9 \times 3}{13 \times 3} = \frac{27}{39} = .\dot{6}9230\dot{7}$$

(1) Multiply both the numerator and denominator by 3.

(2) Add 1 to denominator and divide the sum by 10.

This gives 4 as quotient which is called the multiplying factor.

(3) The next steps consist in multiplying the numerator by 4, *beginning from the right* as follows till the numerator is repeated.

	Place	Carry
$4 \times 7 + 2 = 30$	7 (of 27)	2
$4 \times 0 + 3 = 3$	0	3
$4 \times 3 + 0 = 12$	3	nil
$4 \times 2 + 1 = 9$	2	1
$4 \times 9 + 0 = 36$	9	nil
$4 \times 6 + 0 = 24$	6	3
$4 \times 6 + 3 = 27$	7	2

The numerator 27 is now repeated. $\therefore \frac{9}{13} = .\dot{6}9230\dot{7}$

3. Multiplication & Division of decimals.**Ex. 1.** Multiply 8.972×3.89 .*First Method* $8.972 \dots$ number of decimal places 3 $3.89 \dots$ " " " 2 80748 71776 26916 $34.90108 \dots$ " " " $(3+2)=5$ *Second Method* 8.972 3.89

Place the unit digit of

 26.916

the multiplier below

 7.1776

the last digit of the

 $.80748$

multiplicand.

 34.90108 **Ex. 2.** Divide 87.62926 by 1.25 .

Make the divisor a whole number by shifting decimal to 2 places to the right. Remove the decimal 2 places to right in the dividend also and proceed as in the ordinary way. Put the decimal point in the quotient as soon as the first digit after the decimal point is brought down. The division may be continued even after the last digit is brought by bringing down zeroes *only one at a time*.

 $125 \overline{) 8762.926} \left(70.1034 \dots$ 875 129 125 426 375 510 500 10 **4. L. C. M. & H. C. F**

L. C. M of fractions = $\frac{\text{L. C. M. of their numerators}}{\text{H. C. F. of their denominators}}$

H. C. F of fractions = $\frac{\text{H. C. F. of their numerators}}{\text{L. C. M. of their denominators}}$

L. C. M $\times$ H. C. F = product of two fractions.

Ex. 3. Find the H. C. F and L. C. M. of $\frac{6}{16}, \frac{9}{20}, \frac{3}{10}, \frac{12}{15}$.

First reduce the fractions to their simplest forms.

$$\frac{6}{16} = \frac{3}{8}, \quad \frac{12}{15} = \frac{4}{5}.$$

H.C.F of numerators 3, 9, 3, 4 = 1

L.C.M of denominators 8, 20, 10, 5 = 40

$\therefore$ H.C.F of fractions = $\frac{1}{40}$.

L.C.M of numerators = 36, H.C.F of denominators = 1

$\therefore$ L.C.M of fractions = $\frac{36}{1} = 36$.

SQUARE ROOT

1. Definition. The square root of a given number is a number which multiplied by itself produces the given number. The symbol $\sqrt{\quad}$ indicates the square root.

2. Square root by factors.

Ex. 1. Find the square root of 144

$$\sqrt{144} = \sqrt{12 \times 12} = 12.$$

$$\sqrt{6,724} = \sqrt{2 \times 2 \times 41 \times 41} = 2 \times 41 = 82 \text{ etc.}$$

3. The common method of square root.

The following method is based on the algebraical formula $(a+b)^2 = a^2 + 2ab + b^2 = a^2 + (2a+b)b$.

1. Mark off from the decimal point to the left or to the right or both ways digits in pairs forming different periods.

2. Find the greatest square root in the period to the extreme left.

3. Bring down only a pair of digits at a time.

4. Double the quotient as a trial divisor.

5. Find out a suitable digit (x) to the right hand side of the new divisor such that enlarged divisor multiplied by x is just less or equal to the new dividend. etc.

Ex. 2. Find the square root of 6,724.

$$\begin{array}{r} \leftarrow \leftarrow \\ 67 \ 24 \overline{)82} \\ 64 \\ \hline 162 \overline{)324} \\ 324 \\ \hline \end{array} \quad \text{Reqd. sq. root} = 82$$

4. In the absence of any further digit, bring down zeroes by pairs after putting the decimal point.

Ex. 3. Find the square root of (i) 2 (ii) 18634·265.
(C.U. 1949)

$$\begin{array}{r} 2 \overline{)1 \cdot 414} \\ 1 \\ \hline 24 \overline{)100} \\ 96 \\ \hline 281 \overline{)400} \\ 281 \\ \hline 2824 \overline{)11900} \\ 11296 \\ \hline 604 \end{array} \quad \begin{array}{r} \leftarrow \leftarrow \leftarrow \\ 1 \ 8 \ 6 \ 3 \ 4 \cdot 2 \ 6 \ 5 \overline{)136 \cdot 507 \dots} \\ 1 \\ \hline 23 \overline{)86} \\ 69 \\ \hline 266 \overline{)1734} \\ 1596 \\ \hline 2725 \overline{)13826} \\ 13625 \\ \hline 273007 \overline{)2015000} \\ 1911049 \\ \hline 103951 \end{array}$$

Ex. 4. Find the square root of the fraction $\frac{693}{1837}$ to 2 decimal places.

First Method. Reduce the fraction to a decimal

$$\begin{array}{r} 1837 \overline{)6930 \cdot 377250 \dots} \\ 5511 \\ \hline 14190 \\ 12859 \\ \hline 13310 \\ 12859 \\ \hline 4510 \\ 3674 \\ \hline 9360 \\ 9185 \\ \hline 1750 \end{array} \quad \begin{array}{r} \rightarrow \rightarrow \rightarrow \\ \cdot 37 \ 72 \ 50 \overline{) \cdot 614} \\ 36 \\ \hline 121 \overline{)172} \\ 121 \\ \hline 1224 \overline{)5150} \\ 4896 \\ \hline 254 \end{array}$$

Second Method.

$$\sqrt{\frac{693}{1837}} = \frac{\sqrt{693}}{\sqrt{1837}} = \frac{26 \cdot 323 \dots}{42 \cdot 861 \dots} = \cdot 614$$

5. Decimalisation of Concrete Quantities.

1. £. s. d. f.

(1) 4 farthings (f) = 1 penny (d) $\therefore$ divide farthings by 4 and reduce them to pence (d)

(2) Divide the total amount of pence by 12 and reduce them to shillings (s)

(3). Divide the total shillings by 20 and reduce them to pounds (£). This may be done by multiplying the shillings by 5 and shifting the decimal 2 places to the left.

Ex. 1. Decimalise £ 1 11s 3d 2f.Steps (1) $2f = \frac{2}{4}d = .5d$

$$(2) 3 + .5d = \frac{3.5}{12}s = .29166\dots s.$$

$$(3) 11 + .29166\dots = £ \frac{11.29166}{20}s = £ \frac{1.129166}{2} = £.564583\dots$$

Answer £ 1.5645.....

Ex. 2. Convert £12.5645 into £. s. d.

$$(1) £12 = £12$$

$$(2) £.5645 = .5645 \times 20s = 11.290s.$$

$$(3) .290s = .29 \times 12d = 3.48d$$

$$(4) .48d = .48 \times 4f = 1.92f$$

Answer £12 11s 3d 1.92f

Ex. 3. Convert (i) 13as 4p to naye paise (nP) and (r) 36nP to annas.

$$\text{Re } 1 = 100\text{nP} = 16\text{as}.$$

(i) $13\text{as } 4p = 13\frac{1}{3}\text{ as} = \text{Rs. } \frac{40}{3 \times 16} = \frac{40}{48} \times 100 \text{ nP} = 83\frac{1}{3}\text{nP}$
 $= 83\text{nP}$, (app) (In rounding off to nP. any fraction less than or equal to $\frac{1}{2}$ nP is ignored).

$$(ii) 56 \text{ nP} = \text{Rs. } \frac{56}{100} = \frac{56}{100} \times 16 \text{ as} = 8\frac{24}{25} \text{ as} = 9\text{as (app)}$$

Ex. 4. Find out the cost of 25 yards of shirtings at 15as 6 pies per yard in terms of rupees and naye paise.

Cost of 1 yd = 15as 6 pies = $15\frac{1}{2}$ as

$$\therefore \text{Cost of 25 yds} = 15\frac{1}{2} \times 25 \text{as} = \text{Rs} \frac{775}{32} = \text{Rs. } 23\frac{7}{32} \\ = \text{Rs. } 24 \cdot 22.$$

Note. It is incorrect to obtain the cost by converting 15 as 6 pies to naye paise *first* and then multiplying it by 25. The cost in such case is Rs. 24·25.

Exercise 1

- Find the H.C.F. & L.C.M of $2\frac{1}{3}$, $3\frac{1}{4}$, $4\frac{1}{5}$.
- Reduce $\frac{2\frac{2}{5}}{2 + \frac{1}{1 + \frac{1}{5}}}$ of $1 \cdot 1\bar{8} \div \frac{1}{\cdot 8680\bar{5}}$ to fraction in lowest terms.
- Find the square root of (i) 27019204 (ii) $\frac{6934}{18372}$ to nearest 3 decimal places. (C.U. 1946, 1948)
- Find the square root of $11 + 6\sqrt{2}$
- Simplify $\frac{(1 \cdot 2)^2 \times (\cdot 05)^2 \div (\cdot 25)^2}{(\cdot 2)^2 \div (\cdot 01)^2} \div \cdot 00288$
- Determine the smallest quantity that must be added to the following to make the sum the square of a whole number
 $1\frac{3}{8} \times \frac{19\frac{1}{2} + 1\frac{3}{10}}{\frac{5}{8} \text{ of } (\frac{11}{14} - \frac{7}{11})} \div \frac{0 \cdot 7\bar{8} \text{ of } 3 \cdot \bar{5}7142\bar{8}}{3 \cdot \bar{8}4615\bar{8}}$ (C.U.)
- The area of a square field is 2.45 hectares. Calculate, to the nearest metre, the length of the sides (1 hectare = 10,000 sq. metres) (C.C.S, 1951)
- Express 7 hours 7 minutes and 7 seconds as decimal of a day. (C.U. 1947)
- A triangular field has its sides 111 yds 2 ft ; 119 yds 4 inches and 134 yds. If supports are fixed at equal distances round the field, find the greatest distance

apart of the supports in order that one should be placed at each corner and also find the least number of such supports.

10. In a long division the dividend is 529565 and the first two remainders (before taking down the next figure from the dividend) are 246 and 222, and the third and final remainder is 542. Find the divisor and the quotient.

11. The area of square grass plot is $915\cdot0625$ sq. yds. Find its perimeter.

12. Decimalise £ 2 9s 2d 1f

13. Convert 14 as 9p into naye paise.

Chapter II

Metric System

The *Metric System* of weights & measures was first introduced in France in 1795. In this system each set of measurements has one unit and others are expressed as multiples or submultiples of 10 of this unit.

The unit of length is called *metre*, that of weight is grammic or *gram* (gm) and that of volume is *litre*.

A metre = 39.370113 inches (app).

Units of higher denominations are named *deka* (10 times) *hecta* (100 times), *kilo* (1000 times) & *myria* (10,000 times)

Units of smaller values are called *deci* (tenth), *centi* (hundredth) & *milli* (thousandth).

1. The unit of area is an *Are* which is equal to one square dekametre.

A larger unit is a *hectare* (=100 ares)

The unit of capacity for measuring liquids is the *Litre* which is equal to one cubic decimetre. The larger unit is a *hectolitre* (=100 litres).

The unit of volume for measuring solids is a *Stere* which is equal to one cubic metre.

The unit of weight is *gram* which is the weight of one cubic centimetre (1 c.c.) of pure water at 4°C. The larger units of weights are *quintal* (=100 kg) and *tonne* (=10 quintals)

2. Indian Standards

The metric system has been adopted in India. Since 1st April, 1957 new system of coinage has been introduced. The rupee as the basic unit of monetary transactions is divided into 100 *naye paise*. In converting old currency of

annas into naya paise, fractions equal to or less than half are ignored, but fractions over one-half have been reckoned as one nP.

Thus 2 annas = Rs. $\frac{1}{8} = \frac{1}{8} \times 100 \text{ nP.} = 12\frac{1}{2} \text{ nP} = 12 \text{ nP.}$

3 annas = Rs. $\frac{3}{16} = \frac{3}{16} \times 100 \text{ nP} = 18\frac{3}{4} \text{ nP} = 19 \text{ nP.}$

As in the metric system, the unit of length is *metre* (=39·37 in). Larger unit is *kilometre* ($=\frac{5}{8}$ mile)

The following gives basic units of money, length, area, weight and capacity both old units and new decimal units to which India has changed over.

3. Basic Units of Money

<i>Old units</i>	<i>New units</i>
3 pies = 1 paisa	100 naye paise = 1 rupee
4 paise = 1 anna	(unit being naya paise)
16 annas = 1 rupee	

Some Conversion Factors

1 paisa	= 1·56250 naye paise
1 naya paise	= 0·64 paisa (exact)

4. Basic units of Length

<i>Old units</i>	<i>New units</i>
12 in = 1 foot	10 millimetres = 1 centimetre
3 ft = 1 yard	(mm) (cm)
220 yards = 1 furlong	10 centimetres = 1 decimetre
8 furlongs = 1 mile	10 decimetres = 1 metre
1760 yds = 1 mile	10 metres = 1 dekametre
5½ yds = 1 rod or pole	10 dekametres = 1 hectometre
40 poles = 1 furlong	10 hectometres = 1 kilometre

Some Conversion Factors

1 foot	= 30·48 centimetres (exact)
1 yard	= 0·914 metre
1 mile	= 1·609 kilometres
1 metre	= 1·094 yards
1 kilometre	= 0·621 mile

5. Basic Units of Area.

<i>British Units</i>		<i>Metric Units</i>	
144 sq. in	= 1 sq. ft.	100 sq. millimetres	= 1 sq. centimetre
9 sq. ft.	= 1 sq. yd.	(sq. mm)	(sq. cm)
4,840 sq. yards	= 1 acre	100 sq. cm	= 1 sq. dm
640 acres	= 1 sq. mile	100 sq. dm	= 1 sq. metre
30½ sq. yards	= 1 sq. pole	100 sq. metres	= 1 are or 1 square dekametre
40 sq. poles	= 1 rood	100 ares	= 1 hectare or 1 square hectometre
4 roods	= 1 acre	100 hectares	= 1 sq. kilometre

Some Conversion Factors

1 sq. yard	= 0·836 sq. metre
1 sq. mile	= 2·59 sq. kilometres
1 sq. metre	= 1·196 sq. yards
1 sq. kilometre	= 0·386 sq. mile

6. Basic Units of Weight.

<i>British Units</i>		<i>Metric Units</i>	
16 drams	= 1 ounce (oz)	10 milligrams	= 1 centigram
16 ounces	= 1 pound (lb)	(mg)	
28 pounds	= 1 quarter (qr)	10 centigrams	= 1 decigram
4 quarters	= 1 hundred-weight (cwt)	10 decigrams	= 1 gram
20 hundred-weights	= 1 ton	10 grams	= 1 dekagram
		10 dekagrams	= 1 hectogram
		10 hectograms	= 1 kilogram
		10 kilograms	= 1 myriogram
		10 myriograms	= 1 quintal
80 tolas	= 1 seer	10 quintals	= 1 metric tonne
40 seers	= 1 maund		

Some Conversion Factors

1 kilogram	= 1·072 seers
1 metric tonne	= 26·792 maunds.
1 ton	= 1·016 metric tonnes
1 maund	= 0·373 quintal.

7. Basic Units of Capacity

<i>British units</i>	<i>Metric units</i>
4 gills = 1 pint	10 millilitres (ml) = 1 centilitre
2 pints = 1 quart (qr)	10 centilitres = 1 decilitre
4 quarts = 1 gallon (Imp)	10 decilitres = 1 litre
8 pints	10 litres = 1 dekalitre
36 gallons = 1 barrel	10 dekalitres = 1 hectolitre
64 gallons = 1 quarter	10 hectolitres = 1 kilolitre

Some Conversion Factors

1 gallon (Imp)	= 4.546 litres
1 litre	= 0.220 gallon (Imp)
1 litre	= 61 cu. inches

Ex. 3. Convert (i) 28 yards 1 foot 10 inches into metres
(ii) 237 miles 7 furlong into kilometres
(iii) 138 metres into yards feet and inches, given

1 metre = 39.37 inches, 1 kilometre = 0.62 miles

$$(i) \quad 28 \text{ yds } 1 \text{ ft } 10 \text{ inches} = 1030 \text{ inches} = \frac{1030}{39.37} \text{ metres} \\ = 26.16 \text{ metres}$$

$$(ii) \quad 237 \text{ m } 7 \text{ fur} = 237\frac{7}{8} \text{ miles} = \frac{237\frac{7}{8}}{0.62} \text{ kilometres} \\ = \frac{190300}{62 \times 8} = 383.67 \text{ kilometre}$$

$$(iii) \quad 138 \text{ metres} = 138 \times 39.37 \text{ inches} = 150 \text{ yd } 2 \text{ ft } 9.06 \text{ in}$$

Ex. 4. Having given that a metre is 39.3708 ins, show the difference between 43 yards and 32 metres and express the difference in terms of centimetres. (C.U. 1945)

$$1 \text{ metre} = 39.3708 \text{ in}$$

$$32 \text{ metres} = 32 \times 100 \text{ cm} = 3200 \text{ cm.}$$

$$43 \text{ yards} = 43 \times 3 \times 12 \text{ in} = \frac{43 \times 3 \times 12}{39.3708} \text{ m} = \frac{43 \times 36 \times 100}{39.3708} \text{ cm} \\ = 3931.9 \text{ cm.}$$

$$\therefore 43 \text{ yards} - 32 \text{ metres} = (3931.9 - 3200) \text{ m} = 731.9 \text{ cm.}$$

Ex. 5. If a metre equals to 3.29 ft, how many miles will a man walk in 9 hrs 46 min 40 sec at the rate of 6 kilometres per hour ? (C.U. 1947)

$$9\text{h } 46\text{m } 40\text{s} = 9\text{h } 46\frac{2}{3}\text{m} = 9\frac{140}{3 \times 60}\text{h} = 9\frac{7}{9}\text{h}.$$

$$\begin{aligned} 6 \text{ kilometres} &= 6 \times 1000 \text{ metres} = 6000 \times 3.29 \text{ ft} \\ &= \frac{6000 \times 3.29}{3 \times 1760} \text{ m} = \frac{164.5}{44} \text{ m}. \end{aligned}$$

Hours	Miles	
1	$\frac{164.5}{44}$	$? = \frac{164.5}{44} \times \frac{88}{9} \text{ miles}$
$9\frac{7}{9}$	?	$= 36\frac{5}{9} \text{ miles}.$

Ex. 6. One yard equals 0.914 metre. Find approximately the equivalent in cubic centimetres of a cubic foot. How much larger is a square centimetre ? (C.U. 1944)

$$1 \text{ yard} = 0.914 \text{ metre}.$$

$$\therefore 1 \text{ ft} = \frac{0.914}{3} \times 100 \text{ cm} = \frac{91.4}{3} \text{ cm}$$

$$\therefore 1 \text{ cu. ft} = \left(\frac{91.4}{3}\right)^3 \text{ c. cm} = 28279.7 \text{ c. cm}.$$

$$\text{Also } 1 \text{ in.} = \frac{91.4}{3 \times 12} \text{ cm} = \frac{45.7}{18} \text{ cm}.$$

$$\therefore 1 \text{ sq. in.} = \left(\frac{45.7}{18}\right)^2 \text{ sq. cm} = 6.446 \text{ sq. cm}.$$

Hence 1 sq. in. is 6.446 times a sq. cm.

Exercise 2.

1. If a mile is equal to 1600 metres, find the number of square metres in $7\frac{1}{2}$ acres. (C.U. 1948)

2. A metre is defined to be the ten-millionth part of a quarter of the circumference of the earth and is equal to 39.37079 inches. Find the circumference of the earth in miles.

3. Given that an English gallon is 277·274 cu. in. and that 6 U.S. gallons are equal to 5 English gallons and a litre is 61 cu. in, express the U.S. gallon in litres correct to two decimal places. (C.U. 1935)

4. The velocity of light is 3×10^8 metres per second. Calculate the distance of the sun from the earth in miles, being given that light takes 8 minutes to travel the distance. (1 metre = 39·3708 inches)

5. Express 4 km 6 hm 9 m 4 dm into
(i) kilometres (ii) metres (iii) centimetres.

6. The correct measurements of a rectangle are 4·3 yds by 3·2 yds. These are recorded as 4·3 metres by 3·2 metres. Find the error in square yards and also the percentage error, given that 1 metre = 39·37 inches. (C.U. 1951)

7. If a metre is 39·3708 inches, find the difference between 430 yards and 320 metres in terms of centimetres.

8. Find (i) kilometre equivalent of 475 miles 6 furlongs
(ii) square metre equivalent of 444 square yards and 5 square feet
(iii) equivalent of 483 kilograms in seers and tolas.

Chapter III

Ratios

1. A *ratio* is the relationship between two quantities of the same kind and expresses how many times one quantity is contained in the other.

A ratio is expressed as a *fraction* whose numerator is the first quantity and denominator is the second quantity. The two quantities of a ratio are called its *terms*, the first term being called *antecedent* and the second the *consequent*.

The ratio between a and b is written as a/b . or $a : b$

Thus, the ratio of 14 yds and 2 yds $= \frac{14 \text{ yds}}{2 \text{ yds}} = 7$ (not 7 yds)

The ratio of Rs. 2 and 70 nP $= \frac{\text{Rs. } 2}{70 \text{ nP}} = \frac{20}{7}$ (not Rs, nor nP)

Note. Both rupees and nP are to be expressed in the same unit

Thus, a ratio is *independent of any unit* and is a pure number.

2. Being a fraction, a ratio obeys all the rules application to a fraction.

(i) A ratio remains unaltered if its terms are multiplied or divided by the same number and can be reduced to its lowest terms.

$$\text{Thus, } \frac{2}{3} = \frac{4}{6} = \frac{8}{12} = \frac{200}{300} = \dots$$

$$\frac{1000}{1250} = \frac{200}{250} = \frac{20}{25} = \frac{4}{5} \dots$$

(ii) A ratio is increased or decreased in value if the same number is added to or subtracted from its terms.

$$\text{Thus, } \frac{4}{8} < \frac{5}{8} < \frac{6}{8} < \frac{7}{8} \dots$$

$$\frac{10}{8} > \frac{9}{8} > \frac{8}{8} > \frac{7}{8} \dots$$

Usually the ratio between two quantities a and b is written in the form $a : b$, after dropping the line ($-$) of the division sign ($\div$). The ratio ' $a : b$ ' is read as ' a is to b '.

3. Inverse ratio. The inverse ratio of a/b is b/a .

If two ratios are inverse to each other, their product is unity.

Thus, $a/b \times b/a = 1$, $\frac{3}{4} \times \frac{4}{3} = 1$ etc.

4. To increase or decrease a ratio. To increase a quantity P in the ratio a/b , a being greater than b , multiply P by a/b . To decrease it in the ratio b/a , multiply P by b/a .

Ex. 1. The length and breadth of a rectangle are increased in the ratio $6 : 5$ and $9 : 8$ respectively. By what ratio is the area increased?

Let l and b be the original length and breadth respectively.

Increased length $= \frac{6}{5}l$, increased breadth $= \frac{9}{8}b$.

$\therefore$ increased area $= \frac{6}{5}l \times \frac{9}{8}b$.

$\therefore$ the area is increased in the ratio $\frac{\frac{6}{5}l \times \frac{9}{8}b}{l.b} = \frac{27}{20} = 27 : 20$

Ex. 2. An employee whose salary is Rs. 1840 per annum is given an increase of Rs. 5 per week. By what ratio is his salary increased?

1 year = 52 weeks.

Annual increase in salary = Rs. 5×52 = Rs. 260

$\therefore$ increased annual salary = Rs. $1840 + 260$ = Rs. 2100

$\therefore$ the required ratio $= \frac{\text{Rs. } 2100}{\text{Rs. } 1840} = \frac{2100}{1840} = \frac{105}{92} = 105 : 92$

Ex. 3. Two trains A and B travel 30 and 45 miles per hour respectively. Show that the times taken by them to travel a certain distance are in the ratio $3 : 2$.

$\therefore$ A travels 30 miles in 1 hour

$\therefore$ A " 1 mile in $\frac{1}{30}$ hours.

B " 45 miles in 1 hour

$\therefore$ B " 1 mile in $\frac{1}{45}$ hours.

$\therefore$ the times taken are in the ratio $\frac{\frac{1}{30}}{\frac{1}{45}} = \frac{45}{30} = 3 : 2$

Exercise 3

1. The length of a rectangle is decreased in the ratio 3 : 4 and its breadth is increased in the ratio 5 : 4. By what ratio will the area be increased or decreased ?

2. An employer decreases the number of his employees in the ratio 9 : 10 and increases their wages in the ratio 12 : 11. In what ratio will the wage bill increased or decreased ?

What will be the new wage bill if it was previously Rs. 275 ?

3. Three trains travel at 25, 30, 40 miles per hour respectively. Show that the times taken by them to travel a certain distance are in the ratios 24 : 20 : 15

4. A photograph is $5\frac{1}{4}'' \times 3\frac{1}{2}''$. It is enlarged and its length is 1 foot. What will be the enlarged breadth ? What will be the ratio of the areas of the photograph ?

5. The weights of equal volumes of three substances are in the ratios 2 : 3 : 4. What are the ratios of the weights of cubes of these substances, if the lengths of the edges of the cubes are in the ratios 3 : 2 : 1 ?

6. A trader increased his profit per article in the ratio 4 : 3 so that the number of articles sold decreased in the ratio 5 : 6. By what ratio did his profit increase or decrease ?

7. Three one-pint decanters contain milk and water in the ratios 12 : 11, 13 : 10, 15 : 8. The contents of the 3 decanters are poured into a 3-pint decanter. Find the ratio of milk and water in the resulting mixture.

8. A man's property is to be shared between A, B and C. The ratio of A's share to B's is 7 : 5 and the ratio of B's share to C's is also 7 : 5. If B receives £ 564. 6s. 3d more than C, find how much more A receives than B. Find also the value of the estate. (L.C.C. 1950)

Chapter IV

Proportion and Rule of Three

1. Four quantities are said to be in proportion if the ratio of the first to the second is equal to the ratio of the third to the fourth.

The proportion is written in the form

$$a : b :: c : d \text{ or } a : b = c : d \text{ or } \frac{a}{b} = \frac{c}{d}$$

and read as 'a is to b' as 'c is to d'.

The first and fourth terms in the proportion are called the extremes and the second and third the means.

Since $ad=bc$, the product of the extremes=product of the means.

The fourth term is called the *fourth proportional*.

2. **Direct and Inverse Proportion.** When two quantities of different kinds in a proportion are so related that if one increases or decreases, the other also increases or decreases, then the proportion is said to be *direct*. If, however, if one increases, the other decreases and if one decreases the other increases, the proportion is said to be *inverse*.

For example, if 4 books cost Rs. 40, 3 books will cost Rs. 30 and 5 books will cost Rs. 50. Thus less books cost less and more books cost more. Here the proportion is direct.

Again, if 10 men can do a piece of work in 20 days, 20 men will do it in 10 days. Thus more men requires less days and vice versa. Here the proportion is inverse.

3. **Continued Proportion.** Quantities $a, b, c, d, \dots$ are said to be in continued proportion if

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \dots$$

In a proportion *two* quantities forming a ratio must be similar. But in a continued proportion *all* the quantities must necessarily be similar.

4. Multiplying Ratio (M.R.). This is best illustrated as follows.

Ex. 1. If 8 fans cost Rs. 1600, how much will 3 fans cost ?

Less fans cost less. Here the proportion is *direct*.

Hence the ratio of the required cost and Rs. 1600 is equal to the *direct ratio* of 3 fans and 8 fans.

$\therefore$ the required cost $= \frac{3}{8} \times \text{Rs. } 1600 = \text{Rs. } 600$.

The ratio $\frac{3}{8}$ is called the *multiplying ratio* and may be denoted by M.R.

Thus the required cost $= \text{M.R.} \times \text{given cost}$.

Ex. 2. If 10 men can reap a field in 15 days, how long will it take 15 men to do the same work ?

More men will take less time. Here the proportion is *inverse*.

Therefore, the ratio of the required men and 15 days is equal to the *inverse ratio* of 15 men and 10 men.

$\therefore$ the required time $= \frac{10}{15} \times 15 \text{ days} = 10 \text{ days}$.

The ratio $\frac{10}{15}$ is called the *multiplying ratio* and may be denoted by M.R.

$\therefore$ the required time $= \text{M.R.} \times \text{given days}$.

Note. If the expected answer is less, M.R. is less than 1, but if the expected answer is more, M.R. is more than 1.

5. Compound Proportion.

In a compound proportion all the ratios are multiplied together to form a new ratio.

If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots$ are several ratios, then the compound

$$\text{proportion} = \frac{a_1}{b_1} \times \frac{a_2}{b_2} \times \frac{a_3}{b_3} \times \dots = \frac{a_1 \times a_2 \times a_3 \dots}{b_1 \times b_2 \times b_3 \dots}$$

Ex. 3. If 1000 men can build an embankment 800 yds. long in 45 days working 8 hours per day, in how many days would 1500 men build a similar embankment 900 yds long working 1 hour overtime?

Let the required number of days be X .

Arrange the data as follows :—

men	length	days	hours
1000	800	45	8
1500	900	X	9

6. Working Rule.

1. Place the data in two rows, the *top row* containing the items of the *known part* of the problem and the *bottom row* containing the *unknown item* X and other items associated with it.

2. Place similar items under the same head (*i.e.*, in the same column)

3. Compare separately each pair of similar items with the pair containing the unknown item X independently of other pairs.

4. Find mentally whether the expected answer is larger or smaller. If the expected answer is larger, the corresponding multiplying ratio (M. R) is greater than 1, but if the expected answer is smaller, the corresponding multiplying ratio (M. R) is less than 1.

5. The required answer is equal to the product of all these multiplying ratios and the given item of the pair containing X

In the above example, the pair containing the unknown X consists of days.

Each pair of similar items is compared with the pair containing X independently of the other items.

Thus, 1000 men can finish the given work in 45 days.

∴ 1500 men can finish the same in less days.

i.e., more men require *less* days

∴ the expected answer as found by commonsense is less.

So that the multiplying ratio (M. R) is less than 1

$$\therefore M. R = \frac{1000}{1500} \text{ (less than 1) } \dots \dots (1)$$

Similarly, more length of embankment (i.e., more work) requires more days.

i.e., more length requires *more* days.

∴ the expected answer is more.

$$\therefore M. R = \frac{900}{800} \text{ (more than 1) } \dots \dots (2)$$

More hours require *less* days

$$\therefore M. R = \frac{8}{9} \text{ (less than 1) } \dots \dots (3)$$

$$\therefore X = \frac{1000}{1500} \times \frac{900}{800} \times \frac{8}{9} \times 45 \text{ days} = 30 \text{ days.}$$

Ex. 4. It took 72 men working 8 hours a day to construct a wall 250 yds long, 24 ft. high and 6 ft. wide. How many men should be engaged to fulfil a contract for constructing a wall 275 yds. long, 25 ft. high and 9 ft. wide, if they are to work 9 hours a day and finish the work in the same time as the first? (C.U. 1955)

$$\text{Volume of the 1st. wall} = 250 \times 8 \times 2 \text{ cu. yds} = 4000 \text{ cu. yds.}$$

$$\text{Volume of the 2nd. wall} = 275 \times \frac{25}{8} \times 3 \text{ cu. yds} = 6875 \text{ cu. yds.}$$

Arrange the data as follows :—

men	hours	volume
72	8	4000
<i>X</i>	9	6875

$$\text{More hours, less men} \quad \therefore M.R = \frac{8}{9} \text{ (less than 1)}$$

$$\text{More volume, more men} \quad \therefore MR = \frac{6875}{4000} \text{ (more than 1)}$$

$$\therefore X = \frac{8}{9} \times \frac{6875}{4000} \times 72 \text{ men} = 110 \text{ men.}$$

Ex. 5. A contractor undertook to make 15 miles of road-way in 40 weeks. In 10 weeks 3 miles were completed by 180 men working 8 hours a day; then the men agreed to work one hour a day overtime, and some boys being engaged to assist them, the work was finished in the stipulated time. How many boys were employed, if the work of 3 boys was equal to that of 2 men? (C.U. 1941)

Arrange the data as follows:—

men	miles	weeks	hours
180	3	10	8
X	12	30	9

Here (15-3) or 12 miles are to be completed in (40-10) or 30 weeks working (8+1) or 9 hours a day.

More miles, *more* men $\therefore$ M.R = $\frac{12}{3}$ (greater than 1)

More weeks, *less* men $\therefore$ M.R = $\frac{10}{30}$ (less than 1)

More hours, *less* men $\therefore$ M.R = $\frac{8}{9}$ (less than 1)

$$\therefore X = \frac{12}{3} \times \frac{10}{30} \times \frac{8}{9} \times 180 \text{ men} = \frac{640}{3} \text{ men.}$$

$$\therefore \text{ number of extra men required} = \frac{640}{3} - 180 = \frac{100}{3}$$

Again, men boys

2 3

$\frac{100}{3}$ X

More men, *more* boys $\therefore$ MR = $\frac{\frac{100}{3}}{2} = \frac{100}{6}$ (greater than 1)

$$\therefore X = \frac{100}{6} \times 3 \text{ boys} = 50 \text{ boys.}$$

Note. We could first convert 180 men into boys and then proceed as above.

Ex. 6. A can do a piece of work in $3\frac{1}{2}$ days which B can do in $4\frac{1}{2}$ days. If A's wages are Rs. 27.28 a week

and B's wages are Rs. 21·19 a week, what would A have charged for doing a piece of work for which B was paid Rs. 158 ?
(C.U. 1948)

Arrange the data as follows :—

	days	wages per week	total wages (Rs.)
B	$4\frac{1}{3}$	21·19	158.
A	$3\frac{1}{2}$	27·28	X

Less days, *less* wages $\therefore$ M.R. = $\frac{3\frac{1}{2}}{4\frac{1}{3}}$ (less than 1)

More wages per week, *more* total wages

$\therefore$ M.R. = $\frac{27\cdot28}{21\cdot19}$ (greater than 1)

$$\therefore X = \frac{3\frac{1}{2}}{4\frac{1}{3}} \times \frac{27\cdot28}{21\cdot19} \times \text{Rs. } 158 = \text{Rs. } 164\cdot29$$

Ex. 7. 20 electric fans and 82 lamps consume electricity, the average bill for consumption being Rs. 63·25 per month. It is proposed to use 6 additional fans and 19 additional lights in the building. Find the increase in the average cost per month assuming that on an average a fan consumes $2\frac{5}{8}$ times as much current as a lamp. (C.U. 1954)

$$1 \text{ fan} = 2\frac{5}{8} \text{ lamps}$$

$$\therefore 20 \text{ fans} = 20 \times 2\frac{5}{8} \text{ or } 56\frac{2}{3} \text{ lamps}$$

$$\therefore 20 \text{ fans and } 82 \text{ lamps} = 56\frac{2}{3} + 82 \text{ or } 138\frac{2}{3} \text{ lamps}$$

$$\text{Similarly, } 6 \text{ fans and } 19 \text{ lamps} = 6 \times 2\frac{5}{8} + 19 \text{ or } 36 \text{ lamps.}$$

lamps	cost (Rs.)
$138\frac{2}{3}$	63·25
36	X

Less lamps, *less* cost $\therefore$ M.R. = $\frac{36}{138\frac{2}{3}}$ (less than 1)

$$\therefore X = \frac{36}{138\frac{2}{3}} \times \text{Rs. } 63\cdot25 = \text{Rs. } 16\cdot42.$$

Ex. 8. If 5 men, 10 women and 30 boys can do a piece of work in 81 days working 8 hours a day, in how many days will 10 men, 10 women and 10 boys do two-

thirds of the work working one hour overtime assuming that their capacity of doing work to be in the ratios 3 : 2 : 1.

1 man does as much work as 3 boys

1 woman does as much work as 2 boys

$\therefore$ 5 men, 10 women and 30 boys do as much work as $5 \times 3 + 10 \times 2 + 30$ or 65 boys.

$\therefore$ 10 men, 10 women and 10 boys do as much work as $10 \times 3 + 10 \times 2 + 10$ or 60 boys.

Arrange the data as follows :—

boys	days	hours	work
65	81	8	1
60	X	9	$\frac{2}{3}$

Less boys, more days $\therefore$ M.R. = $\frac{65}{60}$ (greater than 1)

More hours, less days $\therefore$ M.R. = $\frac{8}{9}$ (less than 1)

Less work, less days $\therefore$ M.R. = $\frac{\frac{2}{3}}{1}$ (less than 1)

$$\therefore X = \frac{65}{60} \times \frac{8}{9} \times \frac{2}{3} \times 81 \text{ days} = 52 \text{ days.}$$

Ex. 9. Two workmen A and B are engaged in a piece of work for which they are to receive Rs. 28. One workman works for 3 days of 9 hours each and the other 3 days of 8 hours each, but the second does as much work in 2 hours as the first does in 3 hours. How should the amount be divided between the two workmen?

(C.U. 1938)

A works for 3 days of 9 hours each *i.e.*, for 27 hours

B " " 3 " 8 " " *i.e.*, for 24 hours.

Now B's 2 hours $\equiv$ A's 3 hours

$\therefore$ " 24 " $\equiv$ " $\frac{2}{3} \times 24$ or 36 hours.

$\therefore$ A's working hours : B's working hours = 27 : 36 = 3 : 4

$\therefore$ A gets $\frac{3}{3+4} \times \text{Rs. } 28$ or Rs. 12

B gets $\frac{4}{3+4} \times \text{Rs. } 28$ or Rs. 16

Ex. 10. The value of diamond varies as the square of its weight. A piece of diamond worth Rs. 7290 is dropped and broken into 2 pieces, the weights of which are in the ratio 4 : 5. Find the loss incurred due to breakage.

Let the weights of the two pieces be 4 and 5, then the weight of the unbroken stone was 9.

weight	worth
9^2	Rs. 7290
4^2	X

Less weight, less worth $\therefore$ M.R. = $\frac{4^2}{9^2}$ (less than 1)

$\therefore$ worth of this piece = $\frac{4^2}{9^2} \times$ Rs. 7290 = Rs. 1440

Similarly, worth of the second piece = $\frac{5^2}{9^2} \times$ Rs. 7290 = Rs. 2250

$\therefore$ total worth of two pieces = Rs. 1440 + Rs. 2250 = Rs. 3690

$\therefore$ loss due to breakage = Rs. 7290 - Rs. 3690 = Rs. 3600.

Exercise 4

1. In a refugee camp of 1500 men, the cost of maintaining the refugees for 14 days amounted to Rs. 24,000. What will be the cost of maintenance of 1200 men for 17 days at the same rate of maintenance? (C.U. 1947)

2. The proprietor of a hotel charged Rs. 597.19 for boarding and lodging a party of 13 people for 7 days. How much he would charge for a party of 10 people who stayed for 18 days? (C.U. 1956)

3. If it costs so much to feed 2 men as it does for 5 boys, and for 4 boys the cost is Rs. 25 per week, how much per week will it cost to feed 36 men?

4. The cost of gas for 7 burners, burning 5 hours a day, for 20 days was 8s. 3d. For how many hours every

day may 70 burners be lighted for 31 days for £8. 10s. 6d ?

(P.U. 1948)

5. A transport company has an average monthly income of Rs 80,000 from 10 motor lorries doing an average of $229\frac{1}{2}$ miles a day each for 6 days in the week. What would be the estimated monthly income with 85 lorries doing $202\frac{1}{2}$ miles a day each with a rest of 2 days in the week ? (G. C.)

6. In a besieged garrison 10,000 soldiers had provisions for 4 weeks at the rate of $9\frac{1}{2}$ oz per head per day. After one week 9500 soldiers left the garrison and the daily allowance per head per day was reduced to 5 oz. How long did the provisions last ?

7. A Steamer Company conveys parties of 20 and above at a reduced rate of single fare and half for return journey, but for parties of smaller number the rate is $1\frac{3}{4}$ of single journey. If it costs Rs 213.50 for a party of 18, what would it cost for a party of 30 ?

8. If 75 persons can do a piece of work in 12 days of 10 hours each, how many men will perform a piece of work twice as great in a tenth part of the time, if they work the same number of hours in a day, supposing that two of the second set can do as much work in an hour as three of the first set ? (C.U. 1940)

9. If 8 men or 17 boys can do a piece of work in 26 days, how many days will 4 men and 24 boys take to do a piece of work 5 times as great ? (P.U. 1950)

10. If 5 men, 20 women and 40 boys can do a piece of work in 120 days, working 6 hours a day, in how many days will 10 men, 10 women and 20 boys do the same work working 9 hours a day (supposing the amount of work done by man, woman and boy being in the proportion of 4, 2 and 1). (C.U. 1950)

11. A can do a piece of work in $11\frac{1}{2}$ days which B can do in $7\frac{2}{3}$ days. If A's wages are Rs 4.9as per week

and B's wages are Rs 6. 11as per week, what would A have charged for doing some work for which B was paid Rs. 35. 8as ?

12. An engine while driving machinery burns coal at the rate of $3\frac{1}{2}$ tons in 8 hours. When the machinery is not in motion the consumption of coal is $\frac{1}{5}$ of this rate. How much coal would the engine consume in 1080 hours during $\frac{1}{4}$ of which the machinery is not in motion ?

13. A contractor undertakes to execute a certain piece of work in a given time and engages 60 men who work 8 hours daily. When three-fourths of the time has expired it is found that only two-fifths of the work has been done. How many additional men, all working 10 hours daily, must he engage now so as to finish the work in the stipulated time ?

14. In a printing Press, 40 men were employed and the working hours were 45 hours per week. The weekly wages amounted to Rs 691-4-0. If 8 extra men were employed and all the 48 men worked for 9 hours per week overtime for which they were paid $1\frac{1}{8}$ times the usual rate, calculate the increase of the total amount paid in wages per week. (C.U. 1951)

[Hints : 9 hours overtime work = $9 \times 1\frac{1}{8}$ or $10\frac{1}{2}$ hours normal work].

15. Two horses plough as much as three oxen in the same time and the daily cost of four oxen is equal to that of three horses. A certain area is ploughed by three horses in 8 days. Find the cost of ploughing it by oxen in 6 days. The horse ploughing costs Rs 5 per day per horse. (C.U. 1946)

[Hints : 3 horses for 8 days = 4 horses for 6 days = 6 oxen for 6 days].

16. A precious stone worth Rs 7100 is accidentally dropped and broken into 3 pieces the weights of which are in the ratio of 5 : 7 : 8. The value of the stone is proportionate to the square of the weight. Calculate the loss thus incurred by the breakage. (C.U. 1951)

Chapter V

Proportional Parts

1. If a quantity x be divided into two parts in the ratio $p : q$, then the parts are $\frac{p}{p+q} \times x$, $\frac{q}{p+q} \times x$ respectively.

Similarly, if a quantity x be divided into n parts in the proportion $p_1 : p_2 : p_3 : \dots : p_n$, then the n parts are respectively

$$\frac{p_1 \cdot x}{p_1 + p_2 + \dots + p_n}, \frac{p_2 \cdot x}{p_1 + p_2 + \dots + p_n}, \dots, \frac{p_n \cdot x}{p_1 + p_2 + \dots + p_n}.$$

Ex. 1. Divide Rs. 132 between A , B and C in the proportion $2 : 3 : 6$.

Total number of parts $= 2 + 3 + 6 = 11$

Out of 11 parts, A gets 2, B gets 3 and C gets 6 parts

$$\therefore A's \text{ share} = \frac{2}{11} \times \text{Rs. } 132 = \text{Rs. } 24$$

$$B's \text{ share} = \frac{3}{11} \times \text{Rs. } 132 = \text{Rs. } 36$$

$$C's \text{ share} = \frac{6}{11} \times \text{Rs. } 132 = \text{Rs. } 72.$$

Ex. 2. Divide £ 800 between A , B , C , and D such that A 's and B 's shares are as $2 : 1$; B 's and C 's shares are as $3 : 2$; C 's and D 's shares are as $4 : 3$.

$$A : B = \frac{2}{1}; B : C = \frac{3}{2} = \frac{1}{\frac{2}{3}}; C : D = \frac{4}{3} = \frac{1}{\frac{3}{4}} = \frac{\frac{2}{3}}{\frac{3}{4}} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$

(Make the numerator of each ratio = denominator of the previous ratio)

$$\therefore A : B : C : D = 2 : 1 : \frac{8}{3} : \frac{8}{9} = 12 : 6 : 4 : 3.$$

Total number of parts $= 12 + 6 + 4 + 3 = 25$

$$\therefore A's \text{ shares} = \frac{12}{25} \times \text{£ } 800 = \text{£ } 12 \times 32 = \text{£ } 384$$

$$B's \text{ share} = \frac{6}{25} \times \text{£ } 800 = \text{£ } 6 \times 32 = \text{£ } 192$$

$$C's \text{ share} = \frac{4}{25} \times \text{£ } 800 = \text{£ } 4 \times 32 = \text{£ } 128$$

$$D's \text{ share} = \frac{3}{25} \times \text{£ } 800 = \text{£ } 3 \times 32 = \text{£ } 96$$

Ex. 3. Three men rent a field for Rs. 1424. One puts in 10 sheep for 21 weeks, another 50 sheep for 15 weeks, and the third 80 sheep for 18 weeks. What should each pay towards the rent?

10 sheep for 21 weeks $\equiv 10 \times 21$ or 210 sheep for 1 week

50 " " 15 " $\equiv 50 \times 15$ or 750 " " "

80 " " 18 " $\equiv 80 \times 18$ or 1440 " " "

$\therefore$ they should pay the rent in the proportion

$$210 : 750 : 1440 \text{ or } 7 : 25 : 48.$$

$$\text{Total number of shares} = 7 + 25 + 48 = 80$$

$$\therefore \text{ first man's share} = \frac{7}{80} \times \text{Rs. } 1424$$

$$= \text{Rs. } 7 \times 17.8 = \text{Rs. } 124.60$$

$$\text{Second man's share} = \frac{25}{80} \times \text{Rs. } 1424$$

$$= \text{Rs. } 25 \times 17.8 = \text{Rs. } 445$$

$$\text{Third man's share} = \frac{48}{80} \times \text{Rs. } 1424$$

$$= \text{Rs. } 48 \times 17.8 = \text{Rs. } 854.40.$$

Ex. 4. A man having 3 sons left £ 9,656 to be divided among them in proportion to their ages at the time of his death. When he died their ages were 25, 22 and 21 years respectively. What was the share of each and what difference would it have made to each of his sons, if the father had lived a year longer? (C.U. 1940)

First Part

£ 9,656 is to be divided in the proportion 25 : 22 : 21

$$\text{Total parts} = 25 + 22 + 21 = 68$$

$$\therefore \text{ Eldest son's share} = \frac{25}{68} \times \text{£ } 9,656$$

$$= 25 \times \text{£ } 142 = \text{£ } 3,550$$

$$\begin{aligned}\text{Second son's share} &= \frac{22}{68} \times \text{£ } 9,656 \\ &= 22 \times \text{£ } 142 = \text{£ } 3,124\end{aligned}$$

$$\begin{aligned}\text{Youngest son's share} &= \frac{21}{68} \times \text{£ } 9,656 \\ &= 21 \times \text{£ } 142 = \text{£ } 2,982\end{aligned}$$

Second Part

£ 9,656 is to be divided in the proportion 26 : 23 : 22.

$$\text{Total parts} = 26 + 23 + 22 = 71$$

$$\begin{aligned}\therefore \text{Eldest son's share} &= \frac{26}{71} \times \text{£ } 9,656 \\ &= 26 \times \text{£ } 136 = \text{£ } 3,536\end{aligned}$$

$$\begin{aligned}\text{Second son's share} &= \frac{23}{71} \times \text{£ } 9,656 \\ &= 23 \times \text{£ } 136 = \text{£ } 3,128\end{aligned}$$

$$\begin{aligned}\text{Youngest son's share} &= \frac{22}{71} \times \text{£ } 9,656 \\ &= 22 \times \text{£ } 136 = \text{£ } 2,992\end{aligned}$$

$\therefore$ Eldest son loses £ 3550 - £ 3536 or £ 14

Second son gains £ 3128 - £ 3124 or £ 4

Youngest son gains £ 2992 - £ 2982 or £ 10

Exercise 5

1. Divide Rs. 338.25 among 29 boys of whom 17 are boy scouts such that each boy receives twice that of a scout.

2. A, B, C, D and E club together and rent an acre of land for Rs. 450 per annum for the purpose of forming allotments. The sizes of the allotments are 850 sq yds, 720 sq yds, 560 sq yds and 980 sq yards. What amount should each subscribe for the rent?

3. A, B, C are working partners and agree to divide half the annual profits equally and the other half in pro-

portion to amounts invested. If they contribute Rs. 2500, Rs. 5000, Rs. 7500 and the profit in a year amounts to Rs. 1620, what should each have received ?

4. Two workmen are engaged on a piece of work for which they receive Rs 108. *A* works for 9 days of 8 hours each and *B* for 6 days of 9 hours each. But *B* does as much in 3 hours as *A* does in 4 hours. Divide the amount between *A* and *B*.

5. A man left $\frac{2}{3}$ ths of his property to his wife, $\frac{1}{3}$ th to his son and the remainder was divided between his grandson and grand daughter as 10 : 5. The grand daughter received Rs 10,000. Find how much each of the others received.

(C.U. 1949)

6. *A* and *B* enter into a partnership. *A* furnishes 25 men for 4 days in addition to 18 men for 5 days. *B* furnishes 15 men for 8 days and 18 men for 10 days. *A* receives Rs 262. What should *B* receive ?

(P.U. 1950)

7. A man having three sons left £ 4,828 to be divided among them in proportion to their ages at the time of his death ; when he died their ages were 25, 22 and 21 years respectively. What was the share of each, and what difference would it have made to each of the sons if the father had lived a year longer ?

8. A prize valued at Rs 32,160 is taken by a cruiser and it is proposed to divide the sum amongst the ship's company in proportion to their pay and the time each had been on board. If the Captain has served 20 months at Rs 90 a month, 2 Chief Mates at Rs 45 a month for 15 months, 3 Second Mates 12 months at Rs 40 a month and 157 Sailors 27 months at Rs 30 a month, calculate the share of each.

(C.U. 1938)

Chapter VI

Variation

1. Direct Variation. One quantity A is said to vary *directly* as another quantity B , if they are so related that when one is changed, the other is also changed in the same ratio, *i.e.*, if B is doubled, A is also doubled ; if B is halved, A is also halved, etc.

The word 'directly' is generally omitted and by ' A varies as B ' we mean ' A varies directly as B '.

Symbollically it is written as $A \propto B$, where the symbol ' $\propto$ ' denotes 'varies as'.

From definition it follows that if $a_1, a_2, a_3 \dots$ and $b_1, b_2, b_3, \dots$ be the corresponding values of A and B , then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots\dots\dots$$

Denoting each of these ratios by $k(k \neq 0)$, we have

$$a_1 = b_1 k, \quad a_2 = b_2 k, \quad a_3 = b_3 k \dots\dots$$

This fact is expressed as $A = kB$, where k is some constant, called the constant of variation.

Conversely, if $A = kB$, then $A \propto B$.

For example, the circumference of a circle is $2\pi r$, when π is some constant and we may say that the circumference of a circle varies directly as its radius.

2. Given any pair of corresponding values of A and B , the constant of variation (k) can be determined.

Thus if $A \propto B$ and $A=8$, when $B=4$, we have

$$A = kB, \text{ and } 8 = k \cdot 4 \text{ or } k = 2.$$

3. Inverse Variation. One quantity A is said to vary *inversely* as B , if A varies *directly* as the *reciprocal* of B .

i.e., if $A \propto \frac{1}{B}$ or $A = \frac{k}{B}$, where k is some constant.

Since $A.B=k$, the product of A and B is constant.

Thus for a given piece of work, the greater is the number of men employed, the less is the time required and vice versa.

4. Joint Variation. One quantity is said to vary jointly as a number of other quantities, when it varies directly as their product.

Thus, if A varies jointly as B and C , then $A \propto B \times C$.
i.e., $A = k.BC$, where k is some constant.

Similarly, if A varies jointly as B , C , and D then $A \propto B.C.D$.

For example, the area of a triangle $= \frac{1}{2} \text{ base} \times \text{altitude}$,
i.e., the area of a triangle varies jointly as its base and altitude.

One quantity A is said to vary as another quantity B and inversely as a third quantity C , when it varies jointly as B and reciprocal of C .

i.e., $A \propto B \times \frac{1}{C}$ or $A = k \cdot \frac{B}{C}$, where k is some constant.

For example, the time taken for doing a piece of work varies directly as the volume of work and inversely as the number of men engaged.

5. Theorem on Joint Variation.

If A varies as B when C is constant and A varies as C when B is constant, then A varies as $B.C$ when both B and C vary.

Let a_1 be the value of A , when b_1 that of B and c_1 that of C . Further let a_2 be the value of A , when b_2 that of B and c_2 that of C .

Then a_1 , b_1 , c_1 and a_2 , b_2 , c_2 are two sets of simultaneous values of A , B , and C .

- Evidently, the change of A from a_1 to a_2 is due to
- (1) the change of B from b_1 to b_2 , C retaining the value c_1
 - (2) the change of C from c_1 to c_2 , B retaining the value b_2 .
 - (1) Let A change from a_1 to some value a different from a_2 , when B changes from b_1 to b_2 .

$$\therefore \frac{a_1}{a} = \frac{b_1}{b_2} \dots \dots \dots (1)$$

- (2) Let now C change from c_1 to c_2 , then A changes from a to a_2

$$\therefore \frac{a}{a_2} = \frac{c_1}{c_2} \dots \dots \dots (2)$$

From (1) & (2)

$$\frac{a_1}{a} \times \frac{a}{a_2} = \frac{b_1}{b_2} \times \frac{c_1}{c_2}, \text{ i.e., } \frac{a_1}{a_2} = \frac{b_1 c_1}{b_2 c_2}$$

$\therefore A \propto BC$, when both B and C vary.

For example, the area of a triangle varies (1) as the base, when the height is constant, and varies (2) as the height, when the base is constant.

Hence, the area of a triangle varies directly as the base and height, when both base and height vary.

Cor. If $A \propto B$, when C is constant, and $A \propto \frac{1}{C}$, when

B is constant, then $A \propto \frac{B}{C}$ when both B & C vary.

Ex. 1. Apply the principle of variation to find how many men working 10 hours a day can complete a piece of work in 16 days, if 10 men take 10 days of 8 hours each to complete it.

Let m =number of men, d =number of days, h =number of hours.

Clearly m varies inversely as d , when h is constant and varies inversely as h , when d is constant.

$$\therefore m \propto \frac{1}{d} \cdot \frac{1}{h} \text{ when both } d \text{ and } h \text{ vary}$$

$$\therefore m = \frac{k}{dh}, \text{ where } k \text{ is some constant.}$$

Now $m=10$, when $d=10$, $h=8$

$$\therefore 10 = \frac{k}{10 \times 8} \text{ or } k=800 \text{ so that } m = \frac{800}{dh}.$$

Hence when $d=16$, $h=10$,

$$\text{we have } m = \frac{800}{16 \times 10} = 5.$$

Ex. 2. Three solid globes whose radii are 3, 4, & 5 inches are melted and formed into a new solid globe. Find the radius of the new globe, given that the volume of a globe varies as the cube of its radius.

Let V_1 , V_2 , V_3 be the volumes of the globes whose radii are 3, 4, 5 inches and V the volume and R the radius of the new globe.

$\therefore V_1 = k \cdot 3^3$, $V_2 = k \cdot 4^3$, $V_3 = k \cdot 5^3$, where k is some constant.

$$\text{But } V = k \cdot R^3 = V_1 + V_2 + V_3 = k(3^3 + 4^3 + 5^3)$$

$$\therefore R^3 = 3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$$

$$\therefore R = 6 \text{ inches.}$$

Ex. 3. The mass m of a body varies as its density ρ , when the volume v is constant, and varies as the volume v , when the density is constant. If the unit of mass be defined as the mass of a body of unit volume and unit density, show that $m = \rho \cdot v$.

We have, $m \propto \rho$, when v is constant

and $m \propto v$, when ρ is constant

$$\therefore m \propto \rho \cdot v, \text{ when both } \rho \text{ and } v \text{ vary.}$$

$$\therefore m = k\rho \cdot v, \text{ where } k \text{ is some constant.....(1)}$$

When $v=1$, $\rho=1$, we have $m=1$

$$\therefore 1=k \cdot 1 \cdot 1 \quad \text{or} \quad k=1$$

$$\therefore \text{from (1)} \quad m=\rho \cdot v.$$

Ex. 4. The intensity of illumination from a source of light varies inversely as the square of the distance. A picture is 9 ft. off the source. How much nearer to the light must it be so as to receive thrice as much light?

Let I be the intensity of illumination at a distance d from the source. Then

$$I \propto \frac{1}{d^2} \quad \text{or} \quad I = \frac{k}{d^2} \quad \text{where } k \text{ is some constant.}$$

The intensity of illumination at a distance 9 ft is

$$I = \frac{k}{9^2}.$$

Let d_1 be the distance when I increases to $3I$

$$\therefore 3I = \frac{k}{d_1^2}.$$

$$\text{Hence } \frac{I}{3I} = \frac{d_1^2}{9^2} \quad \text{or} \quad d_1^2 = 27 \quad \text{or} \quad d_1 = 3\sqrt{3}.$$

$\therefore$ it must be nearer by $9 - 3\sqrt{3}$ ft.

6. Part-Variation. In manufacture there is a fixed initial cost and variable supplementary cost which directly varies as the production of commodities. Salary of commercial agents is composed of two parts—a fixed sum and a percentage commission on the value of orders secured. Similarly, royalty paid by publishers to authors may consist of a fixed sum and a fixed percentage on sold copies.

Ex. 5. A traveller gets a fixed salary and also a commission of $1\frac{3}{4}\%$ on all orders he secures. He thus receives an annual income of £ 670 per week on an average. What is his fixed salary per year? (C.U. 1942)

$$\text{Amount of order per year} = £407 \times 52.$$

$$\begin{aligned} \text{Commission on order per year} &= 1\frac{3}{4}\% \text{ of } £407 \times 52 \\ &= £370.37 \end{aligned}$$

$$\therefore \text{fixed salary per year} = £ 670 - £370.37 = £300 \text{ nearly.}$$

Ex. 6. An employer pays dearness allowance to his employees at a fixed rate on the first hundred rupees and at a lower rate on the balance of the basic pay. An employee with a basic pay of Rs. 190 receives Rs. 340, and another with a basic pay of Rs. 130 receives Rs. 250 as total emoluments in a month. Determine the rates of dearness allowance. (C.U. 1952)

$$\begin{aligned} \text{D.A. of the first employee} &= \text{Rs. } (340 - 190) = \text{Rs. } 150, \\ &\quad (\text{basic pay being Rs. } 190) \\ \text{" second " } &= \text{Rs. } (250 - 130) = \text{Rs. } 120, \\ &\quad (\text{basic pay being Rs. } 130) \end{aligned}$$

$$\begin{aligned} \text{Total D.A.} &= \text{D.A. on first Rs. } 100 + \text{D.A. on balance} \\ \therefore \text{Rs. } 150 &= \text{D.A. on first Rs. } 100 + \text{D.A. on Rs. } 90. \\ \text{Rs. } 120 &= \text{" " " " " + D.A. on Rs. } 30. \end{aligned}$$

By subtraction,

$$\text{Rs. } 30 = \text{D.A. on Rs. } 60.$$

$$\therefore \text{rate of D.A. on balance of basic pay} = \frac{30}{60} \times 100 \text{ or } 50\%$$

$$\begin{aligned} \text{Hence, D.A. on first Rs. } 100 \text{ of basic pay} \\ &= \text{Rs. } 150 - \text{D.A. or Rs. } 90 \\ &= \text{Rs. } 150 - 50\% \text{ of Rs. } 90 \\ &= \text{Rs. } 150 - \text{Rs. } 45 = \text{Rs. } 105 \end{aligned}$$

$$\therefore \text{rate of D.A. on first Rs. } 100 \text{ of basic pay} = 105\%$$

7. Uniform Growth.

Ex.7. 18 cows can eat a crop of growing grass in 15 days, and 22 cows can eat it in 10 days. Assuming that all the time the grass is growing uniformly, find how many cows can eat the same in 20 days.

$$\begin{aligned} \text{Let original crop} &= x \text{ units} \\ \text{growth per day} &= y \text{ " } \\ \text{daily consumption per cow} &= t \text{ " } \end{aligned}$$

Since original crop + growth in 15 days can be eaten by 18 cows in 15 days, we have

$$x + 15y = 18 \times 15 \times t = 270t \dots\dots\dots(1)$$

Similarly, $x+10y=10 \times 22 \times t=220t$(2)

$\therefore$ by subtraction, $5y=50t$ or $y=10t$.

Hence from (2), $x+10 \times 10t=220t$ or $x=120t$

If n be the required number of cows, we have

$$x+20y=20 \times n \times t$$

$\therefore 120t+20 \times 10t=20 \times n \times t$, whence $n=16$ cows.

Exercise 6

1. Apply the principle of variation to find (a) how much 13 men will earn in 20 days, if 20 men earn Rs. 600 in 13 days? (b) how long 25 men will take to plough 30 acres, if 5 men take 9 days to plough 10 acres of land?

2. If m men take n days working h hours a day to make an embankment l miles long, how many men working h_1 hours a day will make an embankment l_1 miles long in n_1 days?

3. The area of a circle varies as the square of its diameter. If the area of a circle of diameter 14 in be 154 sq. in, find the area of the circle whose diameter is 8 in.

4. The volume of a sphere varies as the cube of its radius. If the volume of a sphere of radius $3\frac{1}{2}$ in be $179\frac{3}{8}$ cu in, find the radius of the sphere whose volume is $113\frac{1}{4}$ cu in.

5. Two globes of gold have their radii equal to r and r' . They are melted and formed into a single globe. Find its radius assuming that the volume of a globe varies as the cube of its radius. (C.U. 1931)

6. If the volume of a cone whose height is 10 in and base 15 square inch be 50 cubic inch, find the volume of another cone whose height is 18 in and base 2 square foot, assuming that the volume of a cone varies as the height and area of the base.

7. The volume of a cylinder varies as the square of the radius of the base and its height. If a drum whose base is of diameter 6 inches and height 72 inches contains a liquid weighing 16 lbs, what will be height of another cylinder whose base is 9 inches in diameter and contains 20 lbs of the same liquid?

8. (i) If $x \propto y$, show that $x^2 + y^2 \propto x^2 - y^2$

(ii) If $x^3 + y^3 \propto x^3 - y^3$, show that $x \propto y$ and $x + y \propto x - y$.
(C.W.A.)

9. If y varies as the sum of two quantities, one of which varies as x , the other inversely as x and if $y=13$, when $x=4$; and $y=8$, when $x=2$, find the relationship between x and y .

10. The pressure in a liquid varies as the depth when the density is constant and varies as the density when the depth is constant. The pressure is 1 when the depth is 32 and density is 1. Find the depth at which the pressure is 2, when the density is 16.

11. The weight of a sphere varies jointly as the cube of its radius and density of the material of which it is composed. The weights of two spheres are as 9 : 2 and their radii as 3 : 2. Find the ratio of their densities.

12. The expenses of a boarding house are partly constant and partly vary as the number of boarders. In two months the expenses amount to Rs. 2100 and Rs. 2400, when the boarders are 30 and 35 respectively. Find the expenses for 42 boarders.

13. An author received a sum of money for a book, and in addition, a fixed sum for every copy sold. When 500 copies had been sold, he had received £ 85, 16s, 8d, and when 1350 copies had been sold, he had received altogether £ 142, 10s. How much will he have received when 10,000 copies had been sold?
(C.U. 1931)

14. The salary of a traveller is made up of a fixed sum and a commission on orders secured. In two consecu-

tive years he obtained orders to the value of Rs. 31,500 and Rs. 45,000 respectively, and he received Rs. 2,545 and Rs. 2,950 respectively. If his salary for the next year was Rs. 3,200, what was the value of the orders obtained? If he had obtained orders to the value of Rs. 60,000, what would have been his salary? (C.U. 1932)

15. A manufacturer has priced certain pieces of furnitures. The largest sells at Rs. 400 and the smallest at Rs. 90. He wishes to alter the price so that the largest will sell at Rs. 260 and the smallest at Rs. 60. Assuming that the new price P and the old price Q are connected by the relation $P=a+bQ$, find the new prices of the furniture valued at Rs. 125, Rs. 150 and Rs. 175.

16. A photographer offered to supply 3 copies of a photograph for Rs. 12 and 12 copies for Rs. 30 and he estimated that his profit would be Rs. 4 and Rs. 9 respectively. For what sum should he supply 39 copies in order to gain Rs. 18 as profit? (C.U. 1951)

[$C=P+Q.n$, C =total cost, P =constant cost,
 Q =variable cost, n =number of copies]

17. A mercantile firm pays dearness allowance to its staff at a fixed rate on the first hundred rupees and at a different fixed rate on the balance of the basic salary. An employee whose basic salary is Rs. 270 is entitled to get altogether Rs. 435 per month before deduction of income-tax and another married employee with a basic salary of Rs. 160 gets altogether Rs. 270. Find the rate of dearness allowance paid by the firm. (C.U.B. Com.I, 1962)

18. The electric current passing through a wire is, for a constant voltage, proportional to the square of the diameter and inversely proportional to the length of the wire. If 440 amperes of current will pass through wire of 0.15 inch diameter and 3 miles in length, calculate the current that will pass through a wire $2\frac{1}{2}$ miles in length and 0.4 inch in diameter. (C.U. 1960)

Chapter VII

Chain Rule

1. It is a powerful arithmetical device to calculate an unknown quantity from a series of known quantities mutually linked up by two. Problems of proportion, simple or compound, and foreign exchange can be easily tackled by this method.

2. Working Rule

1. Arrange each link or pair so that the unknown quantity denoted by x is put on the left side of the first link.

2. Arrange other links so that the left side of any other link has the same unit as the right side of the previous link.

3. The required quantity x is of the same unit as the right side of the last link.

4. Multiply the right hand figures and the left hand figures separately and equate the products. This gives x .

Ex. 1. Express shillings per pound (lb) into rupees per kg, given the cost of 1 lb = 25s, Re 1 = 1s 6d, 1 kg = 2.205 lbs.

Let Rs. x = cost of 1 kg, then

Rs. x = 1 kg,

1 kg = 2.205 lb

1 lb = 25s

1½s = Re 1.

$$\therefore x \times 1 \times 1 \times 1\frac{1}{2} = 1 \times 2.205 \times 25 \times 1$$

$$\therefore x = \text{Rs. } \frac{2.205 \times 25}{1\frac{1}{2}} = \text{Rs. } \frac{110.25}{3} = \text{Rs. } 36.75.$$

Ex. 2. If 5 lambs are worth 2 sheep, 8 sheep are worth 3 cows, 4 cows are worth 3 horses, how many lambs are worth 18 horses?

Let x = number of lambs, then

$$x \text{ lambs} = 18 \text{ horses}$$

$$3 \text{ horses} = 4 \text{ cows}$$

$$3 \text{ cows} = 8 \text{ sheep}$$

$$2 \text{ sheep} = 5 \text{ lambs}$$

$$\therefore x \times 3 \times 3 \times 2 = 18 \times 4 \times 8 \times 5$$

$$\therefore x = \frac{18 \times 4 \times 8 \times 5}{3 \times 3 \times 2} = 160 \text{ lambs.}$$

3. The Chain Rule may also be illustrated as follows.

Ex. 3. Find a suitable multiplier to convert 100 francs per metre into rupees per yard, given that
Re 1 = 1s 6d, £ 1 = 125 fr, 1 metre = 39·37 inch

Here the price of 1 yard is to be expressed in rupees.

Arrange the data so that the *yard* forms the *first* and *rupees* the *last* items of the different links.

The links are arranged not *all* in the same row but one below another under proper heading.

Equate the *continued product* of the right hand members and the *continued product* of the left hand members of the different links that make the whole chain.

yard	inch	metre	fr	£	s	Rs.
------	------	-------	----	---	---	-----

$$1 \longrightarrow 36$$

$$39 \cdot 37 \longrightarrow 1$$

$$1 \longrightarrow 100$$

$$125 \longrightarrow 1$$

$$1 \longrightarrow 20$$

$$\frac{3}{2} \longrightarrow 1$$

$$\therefore \text{cost of 1 yard} = \text{Rs. } \frac{36 \times 100 \times 20}{39 \cdot 37 \times 125 \times \frac{3}{2}} = \text{Rs. } 9 \cdot 75$$

Exercise 7

1. Gold is being sold in London at £7 0s 6d per ounce. Find the value of 21 grammes of gold in dollars when 1 oz = 28.35 gm, and £ 1 = \$ 4.86 (P.U. 1946)

2. If 20 lambs are worth 15 sheep, 25 sheep are worth 6 cows and 15 cows are worth 8 horses, find the cost of a lamb if a horse is worth Rs 750.

3. Express the speed of 40 m. p. h. in kilometres per second, given 1 metre = 39.37 inches.

4. Given 1 metre = 39.3 inch, £ 1 = 25 francs, quote the cost in shillings per yard of silk worth 8½ fr per metre.

5. If 19 dollars = 80 marks, 16.1 marks = 100 fr, 25 fr = £ 1, 1s 4d = Re 1, how many rupees are equivalent to 3,059 dollars? (P.U. 1946)

6. Find a suitable multiplier to convert

(i) rupees per yard into francs per metre.

(ii) shillings per lb. into rupees per kilogram, having given Re 1 = 1s 6d, £ 1 = 125 fr, 2.2 lb = 1 kg, 1 metre = 39.37 inch.

7. Given that an English gallon is 277.274 cu. in., that 6 U. S. gallons are equal to 5 English gallons, and that a litre is 61 cu. in, express 10 U.S. gallons in litres correct to 2 decimal places.

8. Safety razor blades are quoted at German port at R. M. 175 per kgm. Find the imported cost of blades per gross, being given £ 1 = 19.50 R. M., Re 1 = 1s 6d, 1 kgm = 2.2 lb, weight of a blade = 2.20 grains, average cost of transport, insurance, etc., = 8 annas 7.4 pies per gross (given 1 lb = 7,000 grains) (P.U. 1945)

9. Coal is sold in England f. o. b. for 17s a ton and in France the price is 20 francs per metric ton

(1000 kg). What is the total amount in francs per metric ton of freight and profit ? $\text{£ } 1 = 25.23 \text{ fr.}$

(C.U. 1943)

10. A man gave 25 twenty-franc pieces in exchange for 20 sovereigns. Calculate his gain in English money and in French money. What was his percentage gain ? (Assume $\text{£ } 1 = 25.225 \text{ fr.}$)

11. An English merchant ordered silk in France at 40 francs per metre. When the silk entered England, the merchant had to pay duty 25 per cent of the total value of the silk. Find to the nearest penny, the final cost per yard of the silk to the merchant, taking $80 \text{ francs} = \text{£ } 1$ and $1 \text{ metre} = 39.37 \text{ inches.}$

Chapter VIII

Percentage

1. **Percent** or **per centum** means 'for every hundred.' Therefore, 3 per cent means 3 for every hundred and is written as 3 per cent, or 3 p. c, or 3%.

The **rate per cent** is a fraction whose denominator is 100 and the numerator is the rate per cent.

$$\text{Thus } 3\% = \frac{3}{100}, 4\% = \frac{4}{100} \text{ etc.}$$

Every rate per cent can always be expressed as a *fraction* and reduced to its lowest terms.

$$\text{Thus } 12\frac{1}{2}\% = \frac{12\frac{1}{2}}{100} = \frac{1}{8}; 33\frac{1}{3}\% = \frac{33\frac{1}{3}}{100} = \frac{1}{3} \text{ etc.}$$

Conversely, every fraction can be expressed as a rate per cent by multiplying it by 100.

$$\text{Thus } \frac{2}{5} = \frac{2 \times 20}{5 \times 20} = \frac{40}{100} = 40\%$$

$$\text{Again } \frac{2}{5} \times 100 = 40 \text{ etc.}$$

2. Illustration.

$$66\frac{2}{3}\% = \frac{66\frac{2}{3}}{100} = \frac{2}{3} \therefore 66\frac{2}{3}\% \text{ of } 300 = \frac{2}{3} \times 300 = 200$$

$$100\% = \frac{100}{100} = 1 \therefore 100\% \text{ of } 300 = 1 \times 300 = 300$$

$$10\% = \frac{10}{100} = \frac{1}{10} \therefore 10\% \text{ of } 200 = \frac{1}{10} \times 200 = 20$$

$$6\frac{1}{2}\% = \frac{6\frac{1}{2}}{100} = \frac{1}{16} \therefore 6\frac{1}{2}\% \text{ of } 40 = \frac{1}{16} \times 40 = 2.50$$

$$\frac{1}{2}\% = \frac{1}{100} = \frac{1}{200} \therefore \frac{1}{2}\% \text{ of } 1000 = \frac{1}{200} \times 1000 = 5$$

3. Decimal Method. Division of any number by 100 is effected by shifting the decimal two places to the left.

$$10\% \text{ of } 200 = 10 \times 1\% \text{ of } 200 = 10 \times 2 = 20 \text{ etc.}$$

4. Distinction between 2as per cent and 2 per cent etc.

(i) 2as per cent means 2 annas for every hundred rupees.

$$\therefore 2\text{as per cent} = \frac{2\text{as}}{\text{Rs } 100} = \frac{2}{1600} = \frac{1}{800} = \frac{\frac{1}{8}}{100} = \frac{1}{8}\%$$

$$\text{But } 2 \text{ per cent} = \frac{2}{100} = \frac{1}{50}$$

(ii) 25s per cent and 25 per cent.

25s per cent means 25 shillings for every hundred £

$$\therefore 25\text{s per cent} = \frac{25\text{s}}{\text{£}100} = \frac{25}{100 \times 20} = \frac{1}{80} = \frac{\frac{5}{4}}{100} = \frac{5}{4}\%$$

5. To increase a quantity Q by r per cent

If the original quantity is 100, the increased quantity is $100+r$

If the original quantity is Q , the increased quantity is

$$Q \times \left(\frac{100+r}{100} \right)$$

Thus, to increase Q by $r\%$, multiply by $\frac{100+r}{100}$

If the quantity Q be increased n times by $r\%$, the increased quantity $= Q \times \left(\frac{100+r}{100} \right)^n$

6. To decrease a quantity Q by r per cent, multiply by $\frac{100-r}{100}$.

If the quantity Q be decreased n times by $r\%$, the decreased quantity $= Q \times \left(\frac{100-r}{100} \right)^n$

Ex. 1. The price of coal is increased by 12.5% and the present price per maund is Rs. 2.50. What was it previously?

Original price	Increased price
100	112.5
?	Rs. 2.50

$$? = \text{Rs. } \frac{2.50 \times 100}{112.5} = \text{Rs. } 2 \frac{2}{9} = \text{Rs. } 2.22.$$

Ex. 2. The annual increase in the population of a town is 25 per thousand. If the present population is 4,09,600, what will it be in 4 years time? What was it 2 years ago?

$$25 \text{ per thousand} = 2.5\%$$

$$\begin{aligned} \text{Population 4 years after} &= 4,09,600 \times \left(\frac{100+2.5}{100} \right)^4 \\ &= 4,09,600 \times \left(\frac{41}{40} \right)^4 = 4,52,122 \end{aligned}$$

$$\begin{aligned} \text{Population 2 years ago} &= 4,09,600 \times \left(\frac{100}{100+2.5} \right)^2 \\ &= 4,09,600 \times \left(\frac{40}{41} \right)^2 = 3,89,863 \end{aligned}$$

Ex. 3. A man spends 25% of his annual income for food, 15% for rent, 12% for education, 33% for other expenses and his annual saving is Rs. 540. Find his annual income.

Let his annual income = 100

Then his annual expenditure = 25 + 15 + 12 + 33 = 85.

∴ his saving = 100 - 85 = 15.

Income	Saving	$? = \text{Rs. } \frac{540 \times 100}{15} = \text{Rs. } 3,600.$
100	15	
?	Rs. 540	

Ex. 4. A workman is able to save 12½% of his wages, but if his wages were raised 2s a week and his expenditure were increased by 10%, his annual savings

would be diminished by 17s 4d. What were the man's weekly wages, a year being taken as 52 weeks exactly?

(C.U. 1943)

$$\begin{aligned}\text{If his weekly wages be } 100, \text{ his expenses} &= 100 - 12\frac{1}{2} \\ &= 87\frac{1}{2}\end{aligned}$$

$$\therefore \text{ increase in expenses} = 10\% \text{ of } 87\frac{1}{2} = 8\frac{3}{4}$$

$$\therefore \text{ decrease in savings} = 8\frac{3}{4}.$$

$$\text{But decrease in weekly savings} = \frac{17s \ 4d}{52} = 4d.$$

$$\therefore \text{ the actual decrease in savings} = 4d + 2s = 2\frac{1}{2}s.$$

Weekly wages	Decrease in savings
100	$8\frac{3}{4}$
?	$2\frac{1}{2}s$

$$? = \frac{100 \times 2\frac{1}{2}}{8\frac{3}{4}} s = £1 \ 6s \ 8d.$$

Ex. 5. A man embarks his whole capital in 4 successive ventures. In the first he obtains 100% and in each of the others he loses 20%. Show what percentage he has gained on his original capital. (C.U. 1944)

Let original capital	=	100	
Add profit 100%	=	100	
Cap. after 1st. venture	=	200	
Less loss 20%	=	40	$\therefore$ gain per cent
Cap. after 2nd. venture	=	160	= 24.
Less loss. 20%	=	32	
Cap. after 3rd. venture	=	128	
Less loss 20%	=	25.6	
Cap. after 4th. venture	=	102.4	
Less capital	=	100	
Gain	=	2.4	

Ex. 6. Pure milk contains 89% of water. A sample of milk on being analysed is found to contain 90% of water. What is the adulteration in 22 seers of milk?

Let quantity of pure milk = 100 seers

Then quantity of water = 89 seers, other substances present = 11 seers

The given sample contains 90% of water.

Other substances present = 10 seers in every 100 seers.

When water is added, other substances remain unchanged.

Other substances	Pure milk
11	100
10	?

$$\therefore ? = \frac{10 \times 100}{11} = 90\frac{10}{11} \text{ seers.}$$

Hence 100 seers of mixed milk contain $90\frac{10}{11}$ seers of pure milk.

$$\therefore \text{quantity of extra water} = 100 - 90\frac{10}{11} \text{ or } 9\frac{1}{11} \text{ seers.}$$

Mixed milk	Adulteration	?
100	$9\frac{1}{11}$	$= \frac{22 \times 9\frac{1}{11}}{100} \text{ seers}$
22	?	$= 2 \text{ seers.}$

Alternative Method.

Let x seers of water added to 100 seers of pure milk produce the impure sample.

Quantity of impure milk = $100 + x$ seers

“ “ water = $89 + x$ “

$$\therefore \frac{100 + x}{89 + x} = \frac{100}{90}$$

$$\text{or } 900 + 9x = 890 + 10x \quad \text{or } x = 10.$$

$\therefore$ 110 seers of mixed milk contain 100 seers of pure milk etc.

Ex. 7. If the price of coal is raised by 10 p. c., find by how much per cent a man must reduce his consump-

tion of that article so as not to increase his expenditure ?
(P.U. 1952)

Let original consumption = 100 units

„ price of each unit = Rs. 100

Then total original cost = Rs. 100×100 = Rs. 10,000

Increased price of each unit = Rs. 110

If the total cost remains the same, number of units
that can be purchased at the increased price = $\frac{10,000}{110} = 90\frac{10}{11}$

$$\therefore \text{ necessary reduction in consumption} = 100 - 90\frac{10}{11} \\ = 9\frac{1}{11} \%$$

Alternative Method

Original price	Increased price	
100	110	- ? = $\frac{100 \times 100}{110} = 90\frac{10}{11}$
?	100	

Hence reduction in consumption = $9\frac{1}{11} \%$

Ex. 8. In an examination 55% of the candidates failed in English, 40% in Mathematics and 25% in both. Find the percentage of those who passed in both the subjects.

Let total number of candidates = 100

Then number of candidates failed in English = 55

„ „ „ „ Math. = 40

„ „ „ „ both = 25

$\therefore$ „ „ „ „ English only = $55 - 25 = 30$

„ „ „ „ Math. only = $40 - 25 = 15$

„ „ „ „ in one or more subjects
= $30 + 15 + 25 = 70$

$\therefore$ total number of passes = $100 - 70 = 30$

$\therefore$ required percentage of pass = 30.

7. Step and Slab basis

Under the Step System, the entire amount is subjected to a *single uniform rate* which increases as the amount increases.

For instance, under the Indian Income tax Act before 1939, the rates of income tax were fixed on a step system in a graded ascending scale. A person having a large income had to pay tax at a uniformly higher rate on his whole income than another with a lower income.

Under the Slab System, different slabs or portions are subjected to different rates, the bottom slabs having lower and the upper slabs higher rates.

Since 1939 the basis of income taxation has been replaced by slab system.

Ex. 9. Take the *hypothetical case* (Step System)

Tax on income upto Rs. 5000	nil
„ „ „ above Rs 5000 and upto Rs 15,000	5%
„ „ „ „ Rs 15,000 and upto Rs 30,000	15%
„ „ „ „ Rs 30,000 and upto Rs 50,000	30%

(i) An assessee with income of Rs. 10,000 is to pay tax at 5% on his whole income of Rs. 10,000

∴ his tax = 5% of Rs. 10,000 = Rs. 500

(ii) An assessee with income of Rs. 20,000 is to pay tax at 15% on his whole income of Rs. 20,000

∴ his tax = 15% of Rs. 20,000 = Rs. 3,000

Ex. 2. Schedule of rates of income-tax for the fiscal year 1951-52. on total annual income of individuals

First	Rs. 1,500	nil
On next	„ 3,500	9 pies per rupee
On next	„ 5,000	21 „ „ „
On next	„ 5,000	36 „ „ „
On balance of income		48 „ „ „

Calculate income-tax on the annual income of Rs. 15,800.

The total annual income is divided into the following slabs—

	1,500 +	3,500 +	5,000 +	5,000 +	800.
Tax rate	nil	9 pies	21 pies	36 pies	48 pies
		per Re.	per Re.	per Re.	per Re.
∴ total tax on Rs. 1,500					= 0
+Rs. 3,500 × 9 p.					= Rs. 164 1a
+Rs. 5,000 × 21 p.					= Rs. 546 14as
+Rs. 5,000 × 36 p.					= Rs. 937 8as
+Rs. 800 × 48 p.					= Rs. 200
					Rs. 1,848 7as

8. **Effective Rate.** In the above example the annual income is Rs. 15,800 while the tax paid is Rs. 1,848 7 as.

$$\begin{aligned}\text{The effective rate} &= \text{Rs. } \frac{1,848 \text{ 7as}}{15,800} \\ &= 22.46 \text{ pies per rupee}\end{aligned}$$

The effective rate may also be expressed as a percentage.
The effective rate in the previous example

$$= \frac{\text{Rs. } 1,848 \text{ 7as}}{\text{Rs. } 15,800} \times 100 = \frac{1848.44}{158} = 11.7\%$$

Exercise 8

✓ 1. After paying $62\frac{1}{2}\%$ of a bill, Rs. 525 is still due. What is the $12\frac{1}{2}\%$ of the bill?

✓ 2. A factory consumes 3650 tons of coal in a year. Assuming $7\frac{1}{2}\%$ of the consignment is due to wastage, find the actual requirement.

✓ 3. 15% of the inhabitants of a town having died of plague a panic set in during which 25% of the remaining inhabitants left the town. If the population was reduced to 33,660, what was it originally? (P.U. 1948)

✓ 4. 65 per cent of the workers in a factory are men getting an average wage of Rs. 20 a week each, and the rest are women getting an average wage of Rs. 12 a week

each. Find what per cent of the total wages earned is paid to (i) men (ii) women.

5. The population of a city increases at the rate of 25 per thousand per year. If the present population is 1,50,460, what was it a year ago? What will it be 2 years hence?

6. At a census it was found that the minors formed 40%, 42% and 45% of the populations of 3 towns respectively. The total populations of the towns are as 7 : 5 : 2. What percentage of the combined population is minor?

(P.U. 1949)

7. By what per cent would the cost of a square carpet be above or below that of a second carpet if the side of the former be 8% greater than that of latter, but the cost per sq. yard is 2% less than that of latter?

8. A landlord after paying 5 per cent for collection and 2as in the rupee as incometax gets Rs. 13,100 as net rental. Find his gross rental.

9. A landlord pays 1% for collection of rent and 7 pies in the rupee as incometax. Had he paid 2% for collection and 5 pies in the rupee as incometax, his net receipt would increase by Rs. 10 2as. Find his gross income.

(G.U. 1957)

10. At an election ⁴⁷³¹~~4231~~ persons voted out of a total of 5,370 electors. ²⁹⁴⁷~~2947~~ voted for X and the remaining voted for Y. What percentage of the electors did not vote and what percentage of the total votes did X receive?

11. A dealer mixes 90 gallons of wine containing 10% of water with 60 gallons containing 20% of water. What is the percentage of water in the mixture?

(P.U. 1943)

12. (i) Pure milk contains 88.75% of water and a sample of milk is found to contain 90% of water. What is the extent of adulteration in 18 seers of this sample?

χ (ii) Genuine milk contains 88.75% by bulk of water, 2.75% of fat and the remainder non-fatty solids. A purchaser buys 7 gallons of milk at 3d a quart; the milk on being analysed is found to contain 90.84% of water, 2.24% of fat, the residue being non-fatty solids. Find whether anything besides water has been added to it, and find the sum of which the purchaser has been defrauded. (C.U. 1941)

(Hints. Show that the percentages of constituents other than water are unaltered.)

13. A workman saves 15% of his wages, but if his wages were raised Rs. 3 a week and his expenses were increased by $12\frac{1}{2}\%$, his annual savings would be diminished by Rs. 65. What are the man's weekly wages?

14. The fare of a bus journey has been increased by 25 p.c., and there is consequently a fall of 40 per cent in the passenger traffic. Find the percentage decrease in revenue. (G.C. 1947)

15. The price of petrol is increased by 15%. Find by how much per cent a man must reduce his consumption so as not to increase his expenditure?

16. Keeping the expenditure fixed, if the price per unit of any article is decreased by r per cent, show that the quantity that can be had would be increased by $\frac{100r}{100-r}\%$.

17. In 1953 a manufacturer purchased 392 tons of coal at a contract price of Rs. 22 6s a ton. In 1954 the coal merchant increased his price by 15%. If the manufacturer's outlay for the coal was same as before, how many tons could he buy? If he would purchase 339 tons at the increased price, by what per cent would his outlay on coal be increased or decreased? (C.U. 1954)

18. An increase of 50% in the import duty resulted in an increase of 20% in the price of a motor car. What percentage of the old price represented duty? (G.C.)

• 19. In an examination 20% of the candidates fail in English, 25% in Mathematics, and 10% in both. Find the percentage of those who pass in both subjects. (P.U. 1950)

20. The following is the schedule of rates of Super tax and Surcharge for a certain category of assessee

Income of assessee	Super tax-rate per Re	Surcharge
On first Rs. 25,000	nil	nil
On next ,, 15,000	3as $\frac{1}{8}$ of the rate in previous column	,,
„ „ 15,000	4as	„
„ „ 15,000	6as	„
„ „ 15,000	7as	„
„ „ 15,000	7½as	„
„ „ 50,000	8as	„
On balance	8½as	„

Find the Super-tax and Surcharge payable when the total income is (i) Rs. 42,000 (ii) Rs. 1,20,000.

Calculate effective rate in each case.

Chapter IX

Elements of Manufacturing Cost

1. The manufacture of any product involves various elements of expenditure which go to make up its total cost.

The **total cost** of any product may be divided into two principal classes of expenses, direct and indirect.

Direct Cost consists of those expenses which are actually incurred in and wholly identified with a particular product or job.

Indirect Cost includes all other expenses incurred for the benefit of production as a whole and can not be identified with any particular unit of cost.

2. The main divisions into which total cost is divided are (1) **Material** (2) **Labour or Wages** (3) **Expenses**.

Raw Material. It is one of the most important elements of cost. There are two kinds of material entering into factory cost.

That portion of the material wholly used directly for production is known as **Direct Material**. Direct material is the first element of cost. Another portion of material is used for the undertaking as a whole and can not be identified with and allocated to specific work. It is called **Indirect Material**. For instance, oil and cottonwaste used in running machinery form indirect material. Materials of small value such as nails and glue in the manufacture of toys are sometimes treated as a direct charge to production.

Labour. There are also two classes of labour—direct and indirect. Labour which is wholly identified with production of a unit is **Direct** or **Productive**. It is the second element of cost. The wages of a labourer directly engaged in production are direct. That portion of labour associated with

general services and cannot be charged directly to any particular job is called **Indirect** or **Un-productive**. The labour of foremen, time-keepers, supervisors is indirect and forms part of the overhead expenses of a factory.

Expenses. Besides material and labour, other expenses are also incurred in production and those may be charged to administration, selling and distribution. All items of expenses which may be charged direct to the running of the factory such as the cost of administration are termed **Direct Expenses**. Other expenses such as selling and distribution expenses, with which the factory organisation is not concerned are called **Indirect Expenses**.

3. Costing Terminology.

Oncost—Overhead Expenses—Burden—Indirect Expenses.

Direct Expenses—Chargeable Expenses

Works Oncost.—Factory Oncost—Works Expenses. It comprises all indirect expenses incurred in running the factory,

Office Oncost—Office Overhead. It consists of all other indirect expenses such as administrative, selling and distribution expenses incurred upto sale.

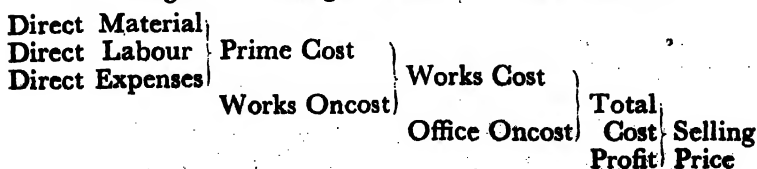
Prime Cost—Basic Cost—Flat Cost. It comprises direct material, direct labour and expenses directly chargeable to production.

Factory Cost—Manufacturing Cost—Cost of Production—Works Cost. It comprises prime cost and works oncost.

Gross Cost—Total Cost—Selling Cost. It comprises works cost and office oncost.

Selling Price—It comprises total cost and profit.

4. Diagram Showing Division of Cost Elements.



5. Items commonly included in

(1) **Works Oncost**—Factory Rent and Rates, Factory Lighting and Heating, Fuel and Power, Repair and Renewals to Plant and Factory, Depreciation of Plant and Machinery, Salary of Works Manager, Works Technical staff, Wages of time keepers & Foremen, Works Telephones, Stationery etc.

(2) **Office Oncost**—Office Rent, Office Lighting and Heating, Repairs of Office Buildings, Stationery, Depreciation of Office Buildings, Furnitures & Fittings, Salaries of Office staff, Insurance of Premises, Printing, Directors' and Audit fees, Warehouse upkeep, Discount, Interest and bad debts, Salaries and Commission of Salesmen and Travellers, Advertisement, Price Lists, Showroom rent and Insurance, Salaries and wages of Showroom staff, Legal Charges, Carriage.

6. Allocation of Oncost

Some expenses can always be charged directly to sales or works. But there are expenses whose classification and apportionment present difficulties. The simple method of allocation of oncost is to charge each unit as a percentage of one or other of the elements of direct expenditure. In many cases such simple subdivision is sufficient but in some others it is most inaccurate.

Generally,

Works Oncost is allocated as a percentage of direct wages. It is particularly suitable where the labour is uniform and rates are fairly constant.

Overtime is charged to jobs concerned

Office Oncost is allocated as a percentage on works cost of each individual job.

Selling Oncost is allocated as a percentage on the selling price of each article.

Depreciation in plant and machinery is generally provided on the diminishing balance system on their values.

Lighting and Heating expenses are allocated on the basis of floor space.

Power is charged on each department on the basis of consumption or Machine Hour or Direct Labour Hour.

7. There are several methods of charging oncost to production

- (1) As a percentage on material
- (2) As a percentage on direct wages
- (3) As a percentage on prime cost
- (4) On machine hour rate
- (5) On direct Labour hour rate.

8. Costing involving fixed and Variable items

Many overheads or indirect expenses such as rent, rates, works manager's salary are fixed in character as they do not vary substantially with the volume of output. Some expenses, however, change with the volume of production. Repairs of Plants and buildings, commissions to salesmen etc are variable overheads. Some expenses are semi-variable. They move with the level of activity but not proportionately. A telephone bill contains a fixed charge for rental and a variable charge for calls and is semi-variable.

Where the items of expenses are partly fixed and partly variable the cost per article decreases as production increases. But there is a certain limit, beyond which the cost does not fall without adverse effects.

9. Office Oncost is purely trading and not manufacturing. It is usually subdivided into

(1) **Administrative Expenses**—expenses incurred in management and administration

(2) **Selling and Distribution Expenses**—expenses incurred in disposing of manufactured goods.

10. Cost Summary

	Rs.
Direct Cost	
Direct Materials	10,000
Direct Wages	3,000
Chargeable Expenses	1,500
Prime Cost	=14,500

Works Expenses

Indirect Materials	500
Indirect Wages	1500
Factory Rent & Rates	500
Repairs	200
Factory Lighting and Heating	600
Depreciation of Plant	800
Insurance	600
Power	1000

Works Cost 20,200

Office & Administrative Expenses

Office Rent & Rates	800
Office Salaries & Wages	1,400
Stationery	400
Repairs	200
Depreciation	500

Cost of Production 23,500

Selling and Distribution Expenses

Rent of Showroom	1000
Lighting	400
Advertising	500
Salaries and commission	800
Travellers' Expenses	1200
Discount	200
Bad debts	400

Selling or Total Cost 28,000

Net profit 4,800

Sales **32,800**

11. Costing. The fundamental object of costing is to analyse expenses and allocate them to individual units of cost. Costing helps to trace wastage and check it. It helps to make forecasts and adjust prices to meet market conditions.

12. Standard Cost and Cost Budgeting

To obtain maximum benefits, production is planned in advance. Standard cost means 'the determination of costs of standard products, produced in standard qualities under standard conditions which may be normal or otherwise.' Standard costing does not only reveal the cost of each unit but at the same time explains why such cost differs from the forecast estimates.

Budgetary control means the systematic control of all factors affecting production by standards prepared well in advance by minute details, thus providing with a standard by which efficiency may be measured. The differences between actual expenditure and predicted standards are thus exposed at every stage. Effective steps can be taken to control expenditure and remedy the position.

The difference between actual expenditure and predicted standard is known as *variance* or *variation* and may arise out of

- (1) variation in quantity consumed, called usage
- (2) variation in price of raw material
- (3) variation in the time taken for production
- (4) variation in efficiency
- (5) variation in wages rate
- (6) variation in volume of production
- (7) variation in working hours.

Efficiency Percentage. The ratio of standard cost of a unit to its actual current cost is called Efficiency Percentage.

Ex. 1. A factory executes two work orders. Materials cost Rs. 80,000, and Rs. 60,000 ; Direct wages Rs. 15,000 and

Rs. 12,000 ; Direct Expenses Rs. 1,500 and Rs. 900 respectively. Factory overhead is 80% of Direct wages, office overhead 20% of Works cost. Find the cost of production of each work order.

	Work Order No 1	Work Order No 2
	Rs.	Rs.
Direct Material	80,000	60,000
Direct Wages	15,000	12,000
Direct Expenses	<u>1,500</u>	<u>900</u>
Prime Cost	96,500	72,900
Factory overhead (80% of Direct Wages)	<u>12,000</u>	<u>9,600</u>
Works Cost	108,500	82,500
Office Overhead (20% of Works Cost)	<u>21,700</u>	<u>16,500</u>
Cost of Production	130,200	99,000

Ex. 2. The cost of production of an article is made up as follows : Materials Rs. 800, Direct Wages Rs. 400, Direct Charges Rs. 75, Works Oncost 75% of Direct Wages, Office Oncost 30% on Works Cost, Selling Oncost Rs. 212. If the net profit is Rs. 205·50, find its normal selling price. Also express profit as percentage of (i) sales, (2) cost of production and (3) prime cost.

	Rs.	
Materials	800	
Direct Wages	400	
Direct Charges	<u>75</u>	
Prime Cost	1,275	(1) Profit on Sales.
Works Oncost (75% of Direct Wages)	<u>300</u>	$= \frac{205\cdot50}{2,465} \times 100 = 8\cdot34\%$
Works Cost	1,575	
Office Oncost (30% of Works Cost)	<u>472·50</u>	(2) Profit on Cost of Production
Cost of Production	2,047·50	$= \frac{205\cdot50}{2,047\cdot50} \times 100 = 10\cdot04\%$
Selling Oncost	<u>212</u>	(3) Profit on Prime Cost
Cost of Sales	2,259·50	$= \frac{205\cdot50}{1,275} \times 100 = 16\cdot12\%$
Net Profit	<u>205·50</u>	
Selling Price	2,465	

Ex. 3. The Cost Accounts reveal the following information in relation to the production of a standard article.

Materials consumed	Rs. 140,000
Direct Labour	„ 80,000
Direct Charges	„ 10,000
Factory Overhead	„ 20,000
Office Overhead	„ 50,000

Prepare (a) Percentage of Factory Overhead to Direct Labour
 (b) Percentage of Office Overhead to Works Cost
 (c) Quotation price for the manufacture of an article consuming materials costing Rs. 2000 and labour Rs. 800 so that a profit of 30% is made on production cost.

Direct Materials	Rs. 140,000
Direct Labour	„ 80,000
Direct Charges	„ 10,000

Prime Cost	<u>230,000</u>
Factory Overhead	20,000

Works Cost	<u>250,000</u>
Office Overhead	50,000

Cost of Production	<u>300,000</u>
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(a) Factory Overhead % to Direct Labour

$$= \frac{20,000}{80,000} \times 100 = 25$$

(b) Office Overhead % to Works Cost

$$= \frac{50,000}{250,000} \times 100 = 20$$

(c) Material	Rs. 2000
Labour	Rs. 800

Prime Cost	<u>Rs. 2800</u>
Factory Overhead (25% of Labour)	Rs. 200

Works Cost	<u>Rs. 3,000</u>
Office Overhead (20% of Works Cost)	Rs. 600

Cost of Production	<u>Rs. 3,600</u>
Profit (30% of Production Cost)	Rs. 1,080

Selling Price	<u>Rs. 4,680</u>
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Exercise 9

1. Tabulate the Elements of Cost, showing the usual items of expenditure relating to each.

2. A cost analysis of an Engineering Co. shows the following particulars.

	Rs.
Materials used	20,50,000
Direct Wages	8,00,000
Works Oncost	1,50,000
Office Oncost	40,000

Show the Works Cost, Total Cost, percentage of Works Oncost to Direct Wages and Office Oncost as percentage of Works Cost.

3. In a factory direct material consumed in 1960 amounted to Rs. 8,000 ; direct wages Rs. 10,000 and total factory overhead Rs. 6,000.

If the factory produced an article whose material cost was Rs. 400 and direct wages Rs. 600, find the factory cost of the article, the factory overhead being calculated on the (1) prime cost basis (2) direct labour cost basis (3) direct material cost basis.

Soln.	Direct Material=Rs. 8,000
	Direct wages =Rs. 10,000
	Prime Cost =Rs. 18,000

Factory overhead

$$= \frac{6,000}{18,000} \times 100 = 33\frac{1}{3}\% \text{ of Prime Cost}$$

$$= \frac{6,000}{10,000} \times 100 = 60\% \text{ of Direct Labour}$$

$$= \frac{6,000}{8,000} \times 100 = 75\% \text{ of Direct Material}$$

	(1) Rs.	(2) Rs.	(3) Rs.
Direct Material	700	700	700
Direct Wages	500	500	500
Prime Cost	1,200	1,200	1,200
Factory Overhead			
33 $\frac{1}{3}$ % of Prime Cost	400	60% Direct Wages 300	75% Direct Material 525
Factory Cost	1,600	1,500	1,725

4. An analysis showed that the cost of production of an article is made up as follows : Materials 40%, Wages 30%, Overhead Charges 30%. If wages rise by 10% and cost of materials fall by 5%, find the reduction necessary in the overhead charges in order that the cost of production may remain unaltered. (P.U. 1944)

5. The prime cost of an article is 3 times the value of the raw materials used. The cost of raw materials increase in the ratio of 5 : 12 and productive wages in the ratio of 3 : 5. Find the prime cost of an article which was used to be made for Rs. 4, 3as, 6p. (G.C., P.U. 1945)

6. The standard cost analysis gives the following percentages on the total cost ; Prime Cost—40 ; Factory Oncost—38 ; Office Oncost—22 $\frac{1}{2}$. The actual analysis showed Prime Cost to have gone down by 5% ; Factory Oncost and Office Oncost gone up by 5% and 2 $\frac{1}{2}$ % respectively. If the selling price was marked 10% above standard cost, how would the rate of profit be affected? P.U. 1944 (S)

7. A Cost Analysis gives the following percentages

Selling price	...	...	100
Prime cost	...	...	32
Factory Oncost	...	...	28
Office Oncost	...	...	29
Profit	...	...	11

A sudden jump in the price of raw materials increases prime cost by 25%, which is sought to be set off by reductions in the estimated profit and Office Oncost in the ratio 1 : 3 ; other things remaining unaltered. Find the adjusted percentages.

8. A factory executes 3 work orders. Materials Cost Rs. 30,000 ; Rs. 40,000 and Rs. 75,000, Direct Labour Rs. 12,000 ; Rs. 18,000 Rs. 50,000 and Direct Charges Rs. 1000, Rs. 1500, Rs. 3,000 respectively.

Factory Overhead is 70% of Direct Labour, Office Oncost is 30% of Works Cost. Find the total cost of each work order.

9. A manufacturing concern made up its Accounts and produced the following data for the year ended 30th June, 1946.

Material used	£ 300,000
Productive Wages	£ 250,000
Factory and General Expenses	£ 50,000
Administrative and other Expenses	£ 30,000

The Factory and General Expenses are applicable to Factory only.

You are required to prepare the following Cost Statement :

- (a) Factory Cost Statement
- (b) Percentage of Factory and General Expenses to Productive Wages
- (c) Percentage of Administrative and other Expenses to Factory Cost
- (d) The price the Company should quote for the production of a Machine requiring material valued £ 1,000 and Wages £ 500 so that the price will yield a profit of 25% on total cost.

(I.C.A.I. 1947)

Chapter X

Price-Fixing and Quotations

1. We have already seen in Chapter IX how cost price of manufactured goods is related to direct and indirect charges incurred in the process of manufacture.

In trading concerns, certain expenses are incurred in presenting goods for sale, some of which have the effect of raising cost price, while others reduce the margin of profit. The usual practice is to add a certain percentage to the cost.

Generally the profit so calculated is a percentage of the total cost. It may also be calculated as a percentage on sale or turnover.

Thus if the cost is Rs. 800 and profit is 20% on cost, the selling price is fixed at Rs. 800 + Rs. 160 or Rs. 960. If, however, the profit is 20% on sale, the selling price is Rs. 1000.

C.P	S.P	
80	100	
Rs. 800	X	$X = \text{Rs. } \frac{800 \times 100}{80} = \text{Rs. } 1,000.$

2. **Catalogue price.** For certain business a different procedure of price-fixing is adopted. Manufacturers or wholesalers usually issue catalogues or price lists of goods sold by them. The price quoted in the catalogues is called **Marked Price, List Price, Catalogue Price or Scheduled Price.** These are abbreviated as **M.P.**

3. **Trade Discount.** When goods are sold to trade, i.e., by a wholesale dealer to a retailer engaged in the same trade, an allowance is made off the list price. This allowance is called *Trade Discount*. It enables the latter to sell the goods at the Marked Price and yet leaves him a margin of profit after meeting all necessary expenses.

Trade discount is always calculated as a percentage on Marked price and deducted from marked price giving selling price

Thus, $M.P - \text{Trade discount} = \text{Selling price}.$

4. Cash Discount. It is an allowance made by the seller to the purchaser for prompt payment. When both trade and cash discounts are allowed, trade discount is allowed *first* and cash discount is allowed *next* on the *reduced amount*.

5. Successive Discounts. When two or more discounts are allowed, each is calculated on the preceding reduced amount.

Equivalent Discount. It is a single discount which has the same effect as several successive discounts.

Thus two successive discounts of 20% and 40% are equivalent to a single discount of 52% and not 60%.

M.P	= 100	Equivalent discount = 100 - 48 = 52%
Less 1st. discount 20%	= 20	
First reduced balance	= 80	
Less 2nd. discount 40%	= 32	
Second reduced balance	= 48	

If d_1, d_2 be two successive discounts, their equivalent discount

$$= 100 - \frac{(100 - d_1)(100 - d_2)}{100}$$

Ex. 1. A trader is allowed a cash discount of 5% in addition to a trade discount of 10% and 10%. Find the net amount of the bill for Rs. 240. (P.U. 1948)

Bill value	= Rs. 240
Less discount 10%	= " 24
Reduced amount	= " 216
Less discount 10%	= 21.60
Reduced amount	= " 194.40
Less discount 5%	= 9.72
Net amount	= Rs. 184.68

6. Method of Proportional Numbers. The above can also be worked out by proportional numbers and simple proportion.

Bill value	= 100
Less discount 10%	= 10
	= 90
Less discount 10%	= 9
	= 81
Less discount 5%	= 4.05
Net amount	= 76.95

Bill value	Net amount	? = Rs. $\frac{240 \times 76.95}{100}$
100	76.95	
Rs. 240	?	= Rs. 184.68

Ex. 2. An article is marked at Rs. 120 and sold at a discount of $12\frac{1}{2}\%$. Find the cost of the article so that a profit of 20% may be made on cost.

M.P	= 120	C.P	= 100
Less discount	= 15	Add profit	= 20
S.P	= 105	S.P	= 120

C.P	S.P	? = Rs. $\frac{105 \times 100}{120}$ = Rs. 87.50
100	120	
?	Rs. 105	

Ex. 3. What price should be marked on an article which costs £ 2-6s-8d so that a profit of 10% may be made after allowing a discount of $12\frac{1}{2}\%$ for cash?

(C.U. 1942)

C.P	S.P	? = £ $\frac{2\frac{1}{2} \times 110}{100}$ = £ $\frac{77}{30}$
100	110	
£ 2½	?	
S.P	M.P	? = £ $\frac{\frac{77}{30} \times 100}{87\frac{1}{2}}$ = £ $\frac{44}{15}$ = £ 2-18s-8d.
87½	100	
£ $\frac{77}{30}$	?	

Note. M.P. can also be found from the formula

$$\text{M.P.} = \frac{100 + \text{profit}\%}{100 - \text{discount}\%} \times \text{C.P.}$$

Ex. 4. A trader is offered a discount of $13\frac{3}{4}\%$ by one wholesaler and 10% and 5% by another. Find which is cheaper. If he purchases at cheaper rate and sells at 5% below list, what profit does he make? (C.U. 1944)

Let the list price of each wholesaler = 100

First wholesaler

M.P. = 100

Less discount $13\frac{3}{4}\%$ = $13\frac{3}{4}$

S.P. = $86\frac{1}{4}$

Second wholesaler

M.P. = 100

Less discount 10% = 10

Invoice Price = 90

Less discount 5% = $4\frac{1}{2}$

S.P. = $85\frac{1}{2}$

The second offer costs less and is cheaper.

Trader's C.P. = $85\frac{1}{2}$

" S.P. = 95

Profit = $95 - 85\frac{1}{2} = 9\frac{1}{2}$

$\therefore$ Profit on cost = $\frac{9\frac{1}{2}}{85\frac{1}{2}} \times 100 = 11\frac{1}{9}\%$

Ex. 5. A manufacturing firm is accustomed to allow to trade customers a discount of 15% on the prices in its published list. This has given it a profit of 19% on the cost of manufacture. The cost goes up 12% and the firm issues a new price list with all the prices put up 10%. If the firm continues to allow its customers the same rate of discount, what percentage profit will it now make on the cost of manufacture? (C.U. 1935)

Let original M.P. = 100

Then " S.P. = 85

" C.P. = $\frac{85 \times 100}{119}$

S.P.

119

85

C.P.

100

?

New M.P. = $100 + 10 = 110$

" S.P. = $85 + 8.5 = 93.5$

" C.P. = $\frac{85 \times 100}{119} \times \frac{112}{100} = 80$

New Profit = $93.5 - 80 = 13.5$

Profit% = $\frac{13.5}{80} \times 100 = 16\frac{7}{8}\%$

Ex. 6. During a sale a shopkeeper reduced his goods 50% below marked price which had originally been fixed to allow 25% profit on selling price after deducting 10% discount for cash. What per cent does he gain or lose ?

(P.U. 1952)

Let original M. P. = 100
Less discount 10% = 10

S. P. = 90
C. P. = 67.5

S. P. C. P. ? = $\frac{90 \times 75}{100}$

100 75

90 ? = 67.5

Reduced M. P. = 50

Less discount 10% = 5

S. P. = 45

$\therefore$ loss = $67.5 - 45 = 22.5$

$\therefore$ percentage loss

on cost = $\frac{22.5}{67.5} \times 100$
= $33\frac{1}{3}$

Note. If no discount is allowed after reduction in M. P., then S. P. is the reduced M. P.

$\therefore$ loss = $67.5 - 50 = 17.5$

$\therefore$ profit % on cost = $\frac{27.5}{67.5} \times 100 = 40\frac{20}{27}$

Ex. 7. A trader altered his trade discount from 15% to 10%. By what percentage was his selling price altered, the percentage of cash discount remaining the same as before ?

Let M. P. = 100

Less discount = 15

S. P. = 85

Cash discount being unaltered did not effect S. P.

$\therefore$ S. P. in second case

= $100 - 10 = 90$

$\therefore$ increase in S. P. = $90 - 85 = 5$.

Percentage increase in S. P. = $\frac{5}{85} \times 100 = 5\frac{15}{17}$

Exercise 10

1. What single discount is equivalent to 3 successive discounts of 20%, 10% and 5% ?

2. A trader is offered a discount of 15% by one wholesaler while he is offered a discount of 12% and a further cash discount of 3% by another. Find which is cheaper, if at all.

If he purchases at the cheaper rate and sells out at 5% below list, what rate per cent profit does he earn (i) on his cost price and (ii) on his selling price ? (*C.U. 1958*)

3. A shopkeeper marks his goods with a price for which he can deduct 15% for ready money and still have $10\frac{1}{2}\%$ profit. What is the marked price on an article which cost him £3.19s. 2d ? (*P.U. 1948*)

4. The list price of an article is 25% above the selling price and the cost price is 40% below the list price. Find the rate of discount and profit. (*P.U. 1947*)

5. If the cost of manufacture of a certain article is increased by 16% and the catalogue price by 15%, what profit per cent (to nearest tenth) on the cost of manufacture does a firm gain after allowing a trade discount of $12\frac{1}{2}\%$? The profit per cent on the former cost of manufacture was 20%. (*C.U. 1941*)

6. During a sale a shopkeeper reduced his goods 40% below marked price, which was originally fixed to allow 20% profit on cost after deducting 10% discount for cash. What per cent does he gain or lose ?

7. The retailer buying from the manufacturer is allowed 25% trade discount on the catalogue price of an article. He pays cash and gets a further rebate of 6d in the £. What does he actually pay for an article listed at £ 30, and what is the catalogue price of an article for which he pays £ 8 15s 6d ? (*C.U. 1948*)

8. The catalogue price of an article has been raised by 12 per cent but the customer is still given discount of 5 per cent on the increased price. Find the original catalogue price of an article which costs the customer now Rs. 345.

(C.U. 1947)

9. A bicycle agent allows 25% discount on his advertised prices and then makes a profit of 20% on his outlay. What is the advertised price of a bicycle on which he gains £ 3?

(P.U. 1952)

10. One firm *A*, offers to pay carriage on all goods bought at cash price, while another *B*, in the same place, gives a discount of $\frac{1}{2}$ d in the shilling but pays no carriage. If I reside 20 miles away and desire to purchase 3 cwt. of a certain article at $4\frac{1}{2}$ d per lb, which firm ought I to go to, the carrier's charges being at the rate of 1s 8d per cwt for a distance of 20 miles?

(C.U. 1942)

[**Hints :** cost of goods from *A* = $336 \times 4\frac{1}{2}$ s = 126s = £ 6 6s.

“ “ “ *B* = $336 \times 4\frac{1}{2}$ s — discount + carriage
= 126s — $126\frac{1}{2}$ d + 3×1 s 8d = £ 6 5s 9d]

11. A consignment of 200 similar rugs entered the show-room of a general stores at the inclusive cost of Rs. 2,400. It was then charged to a share of overhead charges at 15% on the value, and then marked to clear $12\frac{1}{2}$ % after allowing discount of 5%. Find the listed price of a rug.

(P.U. 1943)

12. How much per cent must the cost price of an article be greater than the cost of manufacture so that by selling the same at a trade discount of $12\frac{1}{2}$ % and allowing a cash discount of $2\frac{1}{2}$ %, the manufacturer can make a profit of 6% on the cost of manufacturing the articles? (C.U. 1961)

13. A discount of $33\frac{1}{3}$ per cent is allowed on the list price of certain make of guns. The importer has to pay duty at 20 per cent on the net invoice price, and his expenses are estimated at 10 per cent on the same. If the guns

are sold on the list price plus duty calculated on the price, what is the profit made? (P.U.)

[Hint: M.P.=100. Net invoice price= $100 - 33\frac{1}{3} = 66\frac{2}{3}$, C.P.= $66\frac{2}{3} + \text{duty} + \text{expenses} = 66\frac{2}{3} + 13\frac{1}{3} + 6\frac{2}{3} = 86\frac{2}{3}$; S.P.= $100 + 20 = 120$ etc.]

14. A hosiery goods manufacturer sells to the retailer at 20% off the list price which includes a profit of 30% over the cost of production. The retailer allows his customers 5% discount off the list price. Calculate the percentage of profit the retailer makes on his return. Calculate also the production cost of an article for which his customer pays Rs. 2/47 nP. (C.U. 1960)

15. New Book Publishers make up the following account of the publication of 1,200 copies of a note book sold at Rs. 2.75 nP. a copy.

Paper—120 reams at Rs. 4.25 nP per ream; Printing Rs. 550/-. Binding Rs. 45/- per 100 copies and advertisement, etc, Rs. 150/.

100 copies were distributed free and the rest were all sold. Discount at the rate of $12\frac{1}{2}\%$ was allowed on the sale of 850 copies through other book sellers; and a royalty of 50 nP per copy was paid to author on all sold copies.

How much did the publisher gain or lose? (C.U. 1959)

Chapter XI

Gains and Losses

1. Cost Price and Selling Price. The cost price (*C.P.*) of an article is the total cost incurred in presenting it for sale. Cost includes both direct and indirect charges. The price at which an article is sold is called its selling price (*S.P.*). Profit is the difference between *S.P.* and *C.P.*

$$S.P. - C.P. = \text{gain} \quad C.P. - S.P. = \text{loss.}$$

Profit and loss are usually calculated as percentages on cost unless otherwise stated.

$$\text{Gain \% on cost} = \frac{\text{gain}}{C.P.} \times 100. \quad \text{Loss \% on cost} = \frac{\text{loss}}{C.P.} \times 100$$

$$\text{Gain \% on sale} = \frac{\text{gain}}{S.P.} \times 100, \quad \text{Loss \% on sale} = \frac{\text{loss}}{S.P.} \times 100$$

2. Equivalent terms.

C.P. is also termed outlay or capital

S.P. is also termed turnover, returns or sales.

3. Relation between gain % on cost and gain % on sale.

Ex. 1. If *T* per cent be the rate of profit on turnover and *P* per cent that on cost of production, show that

$$\frac{1}{T} - \frac{1}{P} = \frac{1}{100}$$

Let *C.P.* = 100, then *S.P.* = 100 + *P*.

$$\therefore \text{profit \% on } S.P. = \frac{P}{100 + P} \times 100.$$

$$\text{Hence } T = \frac{100 P}{100 + P} \quad \text{or} \quad \frac{1}{T} = \frac{100 + P}{100 P} = \frac{1}{P} + \frac{1}{100}$$

$$\therefore \frac{1}{T} - \frac{1}{P} = \frac{1}{100}$$

Ex. 2. A retailer bought 13 maunds 14 seers and 3 chattaks of ghee at Rs. 172-8-0 per md, and sold it at Rs. 5-8-0 per seer. Find the percentage gain based (1) on cost price and (2) on selling price. (C.U. 1951)

$$\text{S.P. of 1 md} = \text{Rs. } 5-8-0 \times 40 = \text{Rs. } 220$$

$$\text{C.P. of 1 md} = \text{Rs. } 172.50$$

$$\therefore \text{Gain by selling 1 md} = \text{Rs. } 47.50$$

$$\therefore \text{Percentage gain on cost} = \frac{47.50 \times 100}{172.50} = 27.5$$

$$\text{Percentage gain on sale} = \frac{47.50 \times 100}{220} = 21.6$$

Note. When a quantity of goods is bought at one uniform rate and sold at another uniform rate, the percentage gain or loss does not depend on the quantity so bought and sold.

Ex. 3. A dealer bought a quantity of sugar at a uniform rate. He sold one-third of the quantity at a profit of 40 per cent on cost, one-fourth, at a profit of 25 per cent on cost and the remainder at a loss of 20 per cent. What percentage of the total cost was the total gain?

$$\text{Let total cost} = 1. \text{ Remainder} = 1 - \left(\frac{1}{3} + \frac{1}{4}\right) = 1 - \frac{7}{12} = \frac{5}{12}$$

$$\begin{aligned} \text{Then percentage gain on total cost} &= \frac{\frac{1}{3} \times 40 + \frac{1}{4} \times 25 - \frac{5}{12} \times 20}{1} \\ &= \frac{40}{3} + \frac{25}{4} - \frac{25}{3} = 11\frac{1}{4} \end{aligned}$$

Note. If $c_1, c_2, \dots$ be the cost prices of articles; $r_1, r_2, \dots$ be the percentage gains on cost prices, then

$$\text{percentage gain on total cost} = \frac{c_1 r_1 + c_2 r_2 + \dots}{c_1 + c_2 + \dots}$$

If some articles are sold at a loss say $r_1\%$ then

$$\text{percentage gain on total cost} = \frac{-c_1 r_1 + c_2 r_2 + \dots}{c_1 + c_2 + \dots}$$

A negative percentage indicates loss.

Ex. 4. If the manufacturer makes a profit of 25%, the wholesale dealer a profit of 40% and the retailer a

profit of 75%, what was the cost of the manufacture for a pen that is sold by a retailer for Rs. 25 ? (C.U. 1950)

Let cost of manufacture = 100

Then wholesaler's cost = $100 \times \frac{125}{100}$

retailer's cost = $100 \times \frac{125}{100} \times \frac{140}{100} \times \frac{175}{100} = \frac{1225}{4}$

Retailer's S.P. Cost of manufacture

$\frac{1225}{4}$	100	? = Rs. $\frac{25 \times 100}{\frac{1225}{4}}$
Rs. 25	?	= Rs. $\frac{400}{49}$
		= Rs. 8.16

Ex. 5. A shop keeper can buy his goods at a price which gives him a gross profit of 20% on the turnover. If the price at which he has to buy rises by 10%, while he increases the prices to his customers by 5% only, what will be then his percentage of gross profit on turnover ?

(C.U. 1942)

Old prices	New prices
Let S.P. = 100	S.P. = 100 + 5 = 105
Then C.P. = 80	C.P. = 80 + 8 = 88
∴ profit = 105 - 88 = 17.	
∴ percentage on sale = $\frac{17}{105} \times 100 = \frac{340}{21} = 16\frac{4}{21}$	

Ex. 6. If 5 per cent of the sale price of an article is equal to 6 per cent of its cost, and 8 per cent of the sale price exceeds 9 per cent of its cost by Rs. 3, what were the sale and cost prices ? (C.U. 1945)

Let C.P. = Rs. x , S.P. = Rs. y , then

$$5\% \text{ of S.P.} = \frac{5}{100} \times y = \frac{y}{20}, \quad 6\% \text{ of C.P.} = \frac{6}{100} \times x = \frac{3x}{50}$$

$$\therefore \frac{y}{20} = \frac{3x}{50} \dots (1)$$

$$\text{Again, } 8\% \text{ of S.P.} = \frac{8}{100} \times y = \frac{2y}{25}, \quad 9\% \text{ of C.P.} = \frac{9}{100} \times x = \frac{9x}{100}$$

$$\therefore \frac{2y}{25} - \frac{9x}{100} = 3 \dots (2)$$

Solving (1) & (2), we have $x = \text{Rs. } 500$, $y = \text{Rs. } 600$.

Alternative Method

Let S.P. = 100, then 5% of S.P. = 5

$$\therefore 6\% \text{ of C.P.} = 5 \quad \therefore \text{C.P.} = \frac{500}{6} = \frac{250}{3}$$

$$\text{Again, } 8\% \text{ of S.P.} - 9\% \text{ of C.P.} = 8 - \frac{9}{100} \times \frac{250}{3} = 8 - \frac{15}{2} = \frac{1}{2}$$

Difference	C.P
$\frac{1}{2}$	$\frac{250}{3}$

Rs. 3	?
-------	---

$$\begin{aligned} \text{C.P.} &= \text{Rs. } \frac{3 \times 250}{3} \times 2 \\ &= \text{Rs. } 500 \end{aligned}$$

Difference	S.P
$\frac{1}{2}$	100

Rs. 3	?
-------	---

$$\begin{aligned} \text{S.P.} &= \text{Rs. } 3 \times 100 \times 2 \\ &= \text{Rs. } 600. \end{aligned}$$

Ex. 7. A retailer bought two grosses of lamp shades at Rs. 6/- each. He found some of them damaged which he sold at Rs. 4/50 nP. each and in consequence his profit on total cost price was $20\frac{5}{8}\%$ instead of 25% as expected. How many lamp shades were damaged? (C.U. 1958)

Cost of 2 grosses of lampshades = $2 \times 144 \times \text{Rs. } 6 = \text{Rs. } 1728$

Decrease in expected profit = $(25 - 20\frac{5}{8})\%$ of Rs. 1728

$$= \frac{4\frac{1}{8}}{100} \times \text{Rs. } 1728 = \text{Rs. } 72.$$

Decrease in expected S.P. of each damaged shade

$$= \text{Rs. } 6 \times \frac{125}{100} - \text{Rs. } 4.50 = \text{Rs. } 3.$$

$$\therefore \text{ number of damaged shades} = \frac{72}{3} = 24.$$

Ex. 8. It was decided by a teagarden that its development expenses would be distributed over 20 years. The balance of this on 31st Dec. 1952 was Rs. 30,000. The cost of raising 1 cwt of tea was Rs. 84, while overhead charges were Rs. 18,100 for the year and the total raising

was 350 cwt. What would be the sale price per lb if it is desired that a profit of 20% on turnover is to be earned?

(C.U. 1953)

Development expenses for the year on 20 years' basis

$$= \text{Rs. } \frac{30,000}{20} = \text{Rs. } 1,500.$$

Cost of raising 350 cwt = Rs. 84×350 = Rs. 29,400

Overhead charges = Rs. 18,100

Cost of 350 cwt = Rs. 49,000

$$\therefore \text{cost of 1 lb} = \text{Rs. } \frac{49,000}{350 \times 28 \times 4} = \text{Rs. } 1.25.$$

$$\therefore \text{S.P of 1 lb} = \text{Rs. } 1.25 \times \frac{100}{80} = \text{Rs. } 1.56$$

Ex. 9. A petroleum dealer so constructs his syphon that $\frac{1}{4}$ th. of a gallon of petroleum is left over in the hose every time petrol is delivered. If the total daily sale is 680 gallons to 200 customers on the average, work out an estimate of profit represented by this fraudulent practice, if petroleum is selling at Rs. 1-5-0 a gallon. (P.U. 1944)

On an average daily delivery is 200 times.

$$\therefore \text{quantity left} = 200 \times \frac{1}{4} \text{ gallons.}$$

$$\therefore \text{profit by fraudulent means} = 200 \times \frac{1}{4} \times \text{Rs. } 1 \text{ } 5 \text{ as} \\ = \text{Rs. } 21-14 \text{ as}$$

Ex. 10. A dealer in coke gains 2 cwt per ton by fraudulent means at the time of purchase and defrauds his customers to the extent of 2 cwt per ton. If he buys at Rs. 20 per ton and sells at Rs. 24 per ton, find the percentage of his profit on outlay.

Cost of 1 ton = Rs. 20. He gains 2 cwt per ton.

$$\therefore \text{his cost of 22 cwt} = \text{Rs. } 20$$

$$\therefore \text{ " " " 1 " } = \text{Rs. } \frac{20}{22} = \text{Rs. } \frac{10}{11}$$

Selling price of 1 ton = Rs. 24. He gains 2 cwt per ton.

$$\therefore \text{his selling price of 18 cwt} = \text{Rs. } 24$$

$$\therefore \quad \quad \quad 1 \text{ cwt} = \text{Rs. } \frac{24}{18} = \text{Rs. } \frac{4}{3}$$

$$\text{Hence his gain} = \text{Rs. } \left(\frac{4}{3} - \frac{10}{11} \right) = \text{Rs. } \frac{14}{33}$$

$$\therefore \text{ actual gain} = \frac{14}{33} \times 100 \div \frac{10}{11} = 46\frac{2}{3}\%$$

Exercise 11

1. A trader gains 20% on sales. How much does it represent on cost?

2. A merchant loses 10% on cost. How much does lose on sale?

3. By selling 150 mangoes, a fruit-seller gains the selling price of 30 mangoes. Find his gain per cent. (P.U. 1950)

4. By selling an article at Rs. 12, a trader loses 4 per cent. At what price should it have been sold to gain 20% on sale?

5. Selling some articles at Rs. 5 each, a dealer makes a certain percentage of profit. On increasing the price to Rs. 6 each his percentage of profit is increased by 50%. What profit was he making at the latter price? (C.U. 1945)
[Hints : 50% of C.P = Rs. 6 - Rs. 5 = Re. 1 $\therefore$ C.P = Rs. 2.]

6. An article is sold for Rs. 150 at a gain. Had it been sold for Rs. 135 there would have been a loss of 50% on the original gain. Find the cost price of the article. (P.U. 1952)

7. A difference of 15as in the cost price of an article would mean a loss of 5% to the dealer instead of a gain of 7½%. What is the original cost of the article.

8. An auctioneer charges 4½% on the sale proceeds and spends on the average ⅓% on advertisement and other incidental charges. What is his net profit on a sale which realised Rs 5,784? (P.U. 1945)

9. If T per cent be the rate of profit on turnover and P per cent that on cost of production, show that

$$T = \frac{100P}{100+P}$$

Hence show that a profit of 25% on the cost of production is equivalent to a profit of 20% on the turnover.

10. A man embarks his whole capital in four successive ventures ; in the first he gains 100%, and in each of others he loses 20%. Show what percentage he has gained on his original capital.

11. A manufacturer sells to a wholesaler at a profit of 15% on the actual cost of production. The wholesaler sells to a retailer so that his profit represents 15% on the wholesale price. The retailer's profit is 15% on the actual selling price to a customer who pays £ 4 5s 0d. How much did the article cost to manufacture ?
(L.C.C. 1950)

12. A manufacturer makes a profit of 20%, the wholesale merchant a profit of 25%, and the retail merchant a profit of 40%. What was the cost of the manufacture of an article which was purchased from the retail merchant at 17s 6d ?
(P.U. 1946)

13. A manufacturer sold an article to a merchant thereby gaining 7%. The latter sold it to a retailer thereby gaining $12\frac{1}{2}\%$ and the retailer sold it to a customer thereby gaining 22%. These percentages were based on cost price in each case. Find how much per cent the customer's price was greater than the cost price to the manufacturer. If the customer's price was Rs. 70, what was the cost of production ?
(C.U. 1954)

14. A shop-keeper can buy his goods at a price which gives him a gross profit of 25 per cent on his turnover. If the price at which he buys rises by 20 per cent while he increases the price to his customers by 15% only, what will be his percentage of gross profit on turnover ?

15. Formerly *A* sold goods at a profit of 15 per cent above cost. As the cost price has recently been reduced by 30%, selling price has been lowered 29%. What is the percentage of profit on cost price to *A*? (C.U. 1956)

16. *A* bought the following lots of goods :

311 tons @ Rs. 113-4-0 per ton, 128 tons @ Rs. 127-9-0 per ton, 73 tons @ Rs. 128-11-0 per ton. Find at what rate the entire stock should be sold to enable *A* to earn 10 @ profit. (C.U. 1945)

17. In 1953, a manufacturer purchased 392 tons of coal at a contract price of Rs. 22 6s per ton. In 1954 the coal merchant increased his price by 15%. If the manufacturer's outlay for the coal was same as before, how many tons could he buy? If he would purchase 339 tons at the increased price, by what per cent would his outlay on coal be increased or decreased? (C.U. 1954)

18. A retailer bought a gross of lampshades at 6s. each. He found some of them damaged which he sold at 3s. 6d. each and in consequence his profit on the total selling price was only $14\frac{2}{3}$ p. c. instead of 20 p. c. as expected. How many were damaged? (L.C.C.)

19. A draper bought 7 dozen pairs of stockings for £23 9s and on selling all of them, he made a profit of $12\frac{1}{2}\%$ on the total selling price. He sold 4 dozen pairs at 6s 11d per pair ; what was the selling price of the remaining 3 dozen? (L.C.C.)

20. A dealer sold three pianos for Rs. 300, Rs. 450 and Rs. 500 respectively thereby gaining 5% on the total selling price. He lost 4% by the sale of the first and gained $7\frac{1}{2}\%$ by selling the second, the percentages being based on selling prices. What percentage gain did he obtain by selling the third?

21. A bicycle was sold at a loss of 40 p. c. on the cost price ; a second one was bought with the proceeds plus 5.

guinees paid ; this was also sold at a loss of 35% ; a third one was bought with the proceeds plus £ 5. 15s. 3d. The third cycle costs £ 16. What was the cost of the first one ?

22. A retailer sells goods at a price by which he gains 25% on that price. The cash price of his goods is reduced later by 6% and he lowers his selling price by 4%. Calculate the percentage of profit on sales under such altered conditions. (C.U. 1958)

23. A traveller was appointed by a merchant at the rate of Rs. 1800 per annum plus 5% on the value of orders obtained. In the first four years he obtained orders to the value of Rs. 50,000 ; Rs. 74,000 ; Rs. 97,600 and Rs. 1,08,600, the goods being sold at $33\frac{1}{3}\%$ above cost price. To what salary was the traveller entitled each year ? How much profit each year did the merchant make out of the traveller's efforts, if additional expenses attached to the sale of above goods were 2% of selling price ? (C.U. 1954)

24. (a) A draper professes to sell goods at a profit of 10% but he uses a yard which is 1 inch short. Find his actual gain per cent.

(b) A grocer uses a pound-weight which is half-an-ounce short. How much per cent of cost does he gain by his dishonesty ? (C.U. 1959)

25. If 7% of the sale price of an article is equal to 8% of its cost price, and 9% of the sale price exceeds 10% of the cost by half-a-crown, at what price is the article bought and sold ? (C.U. 1943)

Chapter XII

Alligation or Mixture

1. It is a common practice in trade to mix two or more different varieties of a commodity or similar varieties of different commodities either to produce a standard quality or to make undue profit by fraudulent means.

These problems on mixture may be grouped into 2 main classes.

(1) **The direct problems**—to find the average cost price of the mixture of different constituents of given qualities and prices.

(2) **The inverse problems**—to find the ratio or proportion in which two or more constituents of given prices are mixed up to produce a mixture of a given price.

The most usual case of mixture of direct type is that of tea, milk, water and the like.

Ex. 1. A milkman mixes 44 pints of water with 349 pints of milk which he buys for Rs. 123. At what price should he sell the mixture so as to make one-third of his outlay ? (C.U. 1947)

Water can be had free and costs nothing.

Cost of 349 pints of milk = Rs. 123

Cost of 44 pints of water = 0

Cost of 393 pints of mixture = Rs. 123

Add profit one-third = Rs. 41

S. P. of 393 pints of mixture = Rs. 164

S. P. of 1 pint of mixture = $\frac{\text{Rs. } 164}{393} = 42 \text{ nP.}$

2. The average price of mixture may also be found out thus :—

(a) multiply each quantity by its price

(b) add together these products

(c) divide the sum thus obtained by the total of all the quantities.

Ex. 2. A merchant buys tea at Rs. 2-4as, Rs. 2-6as-6p and Rs. 2-11as-4p per lb., and mixes them in the proportion 5 : 4 : 1. At what price must he sell the mixture so as to make a profit of 25% ? (C.U. 1941)

C. P. of 1 lb of mixture

$$= \frac{5 \times \text{Rs. } 2.4\text{as.} + 4 \times \text{Rs. } 2.6\text{as. } 6\text{p} + 1 \times \text{Rs. } 2.11\text{as. } 4\text{p.}}{5+4+1}$$

$$= \frac{\text{Rs. } 11.4\text{as.} + \text{Rs. } 9.10\text{as.} + \text{Rs. } 2.11\text{as. } 4\text{p.}}{10}$$

$$= \frac{\text{Rs. } 23.9\text{as. } 4\text{p.}}{10} = \text{Rs. } 2.5\text{as. } 8.8\text{p.}$$

Add profit 25% ($\frac{1}{4}$ th) = Rs. 0.9as. 5.2p.

S. P. of 1 lb of mixture = Rs. 2.15as. 2p.

Ex. 3. A person sold 1000 pairs of stockings at Rs. 3-3as per pair to a customer from whom he bought in return equal quantities of cloth at Rs. 4-12 per yd, cambric at 15as per yd, and calico at 11as per yard. How many of each kind did he get for the stockings and a further payment of Rs. 109 ?

Cost of 1 yd. of cloth = Rs. 4-12as

" " cambric = 15as

" " calico = 11as

Cost of 3 yds = Rs. 6-6as

$$\begin{aligned} \text{Total cost of cloth cambric, calico} &= 1000 \times \text{Rs. } 3-3\text{as} + \text{Rs. } 109 \\ &= \text{Rs. } 3187.8\text{as} + \text{Rs. } 109 \\ &= \text{Rs. } 3296.8\text{as} \end{aligned}$$

Number of yds. of each kind of cloth, cambric, calico

$$= \frac{\text{Rs. } 3296.8\text{as}}{\text{Rs. } 6.6\text{as}} = \frac{3296\frac{1}{2}}{6\frac{3}{5}}$$

$$= \frac{26372}{51} = 517.1$$

Ex. 4. There are two one-pint decanters, containing liquor and water in the ratio of 11 to 4 and 10 to 5 respectively. The contents of the two decanters are poured into a two-pint decanter and the mixture is sold at Re 1 per pint, thereby gaining a profit of 2 as per pint. Find the cost price of a pint of liquor.

Quantity of liquor in the first decanter = $\frac{11}{15}$ pints and that in the second decanter = $\frac{10}{15}$ pints.

Total quantity of liquor in the 2 pint-decanter
 $= \frac{11}{15} + \frac{10}{15}$ pints = $\frac{7}{3}$ pints.

Now S. P of 2 pints = Rs. 2 and profit = 4as

$\therefore$ cost of 2 pints of mixture = Rs. 1.12as.

As water is free of cost, and 2 pints of mixture contain $\frac{7}{3}$ pints of liquor

$\therefore$ cost of $\frac{7}{3}$ pints of liquor = Rs. 1.12as.

$\therefore$ cost of 1 pint of liquor = 28as $\times \frac{3}{7}$ = 20as = Rs. 1. 4as.

Ex. 5. Equal number of mangoes are bought at the following rates ; Rs. 2 per score, Rs. 1/8 per score, Rs. 1/4 per doz. and Rs. 1/12 per doz. They are sold at Rs. 2/6 per score and the total profit amounts to Rs. 18/- only. Find the number of mangoes bought at each of the above prices.

(C.U. 1955)

1 doz = 12 ; 1 score = 20 L. C. M of 12 and 20 = 60

Let number of each kind be 60

Then cost of 60 at Rs. 2 per score = Rs. 2×3 = Rs. 6-0-0

„ „ Rs. 1/8 „ = Rs. $1/8 \times 3$ = Rs. 4-8-0

„ „ Rs. 1/4 per doz = Rs. $1/4 \times 5$ = Rs. 6-4-0

„ „ Rs. 1/12 „ = Rs. $1/12 \times 5$ = Rs. 8-12-0

Total cost of 60 of each kind = Rs. 25-8-0

S. P. of 1 score = Rs. 2/6/-

$\therefore$ S. P. of 60 $\times 4$ = Rs. $2/6 \times \frac{240}{30}$ = Rs. 28-8as

Total profit = Rs. 28-8as = Rs. 25-8as = Rs. 3

When total profit in Rs. 3, number of each kind 60 .
 „ „ „ Rs. 18, „ „ ?
 $? = \frac{60}{3} \times 18 = 360$

3. The inverse problems are more complicated and varied in nature.

To illustrate the procedure let us first suppose that two different kinds of tea costing Rs. 4 and Rs 2-12 per lb are mixed up in the ratio 2 : 3.

Then, total cost of a mixture of 5 lb

$$= \text{Rs. } 4 \times 2 + \text{Rs. } 2-12 \text{as} \times 3$$

$$= \text{Rs. } 8 + \text{Rs. } 8-4 \text{as} = \text{Rs. } 16-4 \text{as}$$

$\therefore$ average cost of 1 lb of mixture

$$= \frac{\text{Rs. } 16-4 \text{as}}{5} = \text{Rs. } 3-4 \text{as}$$

Conversely, let us now find the ratio in which two kinds of tea costing Rs. 4 and Rs. 2-12 per lb are mixed up to produce mixture costing Rs. 3-4as per lb.

Evidently, if the first kind were sold at the average price of Rs. 3-4as per lb, there would have been a loss of Rs. 4--Rs. 3-4as or 12as per lb.

But if the second kind were sold at the average price, there would have been a gain of Rs. 3-4as--Rs. 2-12as or 8 as per lb.

But as there is neither loss nor gain due to mixing, the loss in one is counterbalanced by gain in another.

Hence the required ratio must be inversely proportional to the difference of the price of each constituent from that of the mixture.

In the present case the ratio = 8as : 12 as = 2 : 3
 Thus we obtain the following rule :

$$\frac{\text{Quantity of 1st kind}}{\text{Quantity of 2nd kind}} = \frac{\text{difference of the price of the 2nd kind from that of mixture}}{\text{difference of the price of the 1st kind from that of mixture}}$$

4. Hence to deal with an inverse problem of a mixture of two kinds.

1. Find the prices of each kind for the same unit.
2. Place the prices of the two kinds in a horizontal line and that of the mixture below them.
3. Put down the differences in prices diagonally opposite.
4. The quantities mixed are in the ratio of these two differences.

$$\therefore \text{1st kind} : \text{2nd kind} = 8\text{as} : 12\text{as} = 2 : 3.$$

Ex. 6. A dealer mixes oranges bought by him at two different rates, one at 1s. a dozen and the other at 8d. a dozen. He sells the lot at 15 per shilling and makes a profit of 5% on his outlay. Find the ratio of the number of oranges of the two kinds.

Cost price of 1 doz. of 1st kind = 12d

" " " 2nd " = 8d

S.P. of 15 of mixture = 1s

" 12 " = $\frac{1}{15} \times 12\text{d} = \frac{4}{5}\text{d}$

The S.P. being at 5% profit,

the C.P. of 1 doz of mixture = $\frac{100}{105} \times \frac{4}{5}\text{d} = \frac{8}{7}\text{d}$.

Arranging in the following form

	1st kind	2nd kind
Cost price	12d	8d
	↘	↗
Cost price of mixture	$\frac{64}{7}\text{d}$	
	↙	↘
	$\frac{8}{7}\text{d}$	$\frac{20}{7}\text{d}$

$$\therefore \text{1st kind} : \text{2nd kind} = \frac{8}{7}\text{d} : \frac{20}{7}\text{d} = 2 : 5.$$

Ex. 7. The price of pure milk is Re 1 per seer. A milkman adulterates it by mixing water in such a way that by selling 1 maund of the adulterated quality of milk at 12 annas a seer he makes a profit of 25% on cost. Find the proportion of milk and water in the adulterated milk. (C.U. 1953)

Cost price of 1 md. of pure milk = Rs. 40.

Cost price of 1 seer of pure milk = Re. 1.

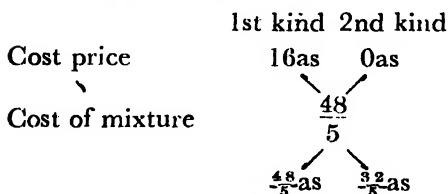
S.P. of 1 seer of mixture = 12as.

As there is a profit of 25% on cost,

C.P. of 1 seer of mixture = $\frac{100}{125} \times 12\text{as} = \frac{48}{5}\text{as}$.

Water can be had free.

Arranging in proper form



∴ Milk : water = $\frac{48}{5} : \frac{32}{5} = 3 : 2$.

Ex. 8. In mixing tea 1 lb in every 100 lbs is wasted. In what proportion must a dealer mix teas which cost him 2s and 1s. 6d. per lb respectively, so as to gain 15 per cent by selling the mixture at 1s 11d per lb? (C.U. 1353)

Take 100 lbs from the two kinds. Then 1 lb is wasted in the process of mixing leaving a mixture of 99 lbs only.

S.P. of 99 lbs of mixture = $99 \times 1\text{s. } 11\text{d} = 99 \times \frac{23}{12}\text{s.}$

C.P. " " " = $\frac{100}{110} \times 99 \times \frac{23}{12}\text{s} = 165\text{s.}$

That is, C.P. of 100 lbs of two kinds = 165s.

" 1 lb " " = $\frac{165}{100} = \frac{33}{20}\text{s.}$

Hence

$$\frac{\text{1st kind}}{\text{2nd kind}} = \frac{\text{diff. in prices per lb of 2nd. kind and mixture}}{\text{diff. in prices per lb of 1st kind and mixture}}$$

$$= \frac{\frac{33}{20}\text{s} - 1\frac{1}{2}\text{s}}{2\text{s} - \frac{33}{20}\text{s}} = \frac{\frac{7}{20}}{\frac{7}{20}} = 1$$

i.e., 1st. kind : 2nd. kind = 7 : 3

Ex. 9. The price of gold is Rs. 80 per tola. An alloy of gold and silver weighing 10 seers is worth Rs. 38,880; but if the weight of gold and silver in the alloy were interchanged it would be worth Rs. 26,320 only. Find the proportion of gold and silver in the alloy and the price of silver per tola. (C.U. 1953)

In the two alloys, the quantity of gold altogether is 10 seers, and that of silver is also 10 seers.

$$\therefore \text{cost of 10 seers of gold} + \text{cost of 10 seers of silver} \\ = \text{Rs. } 38,880 + \text{Rs. } 26,320 = \text{Rs. } 65,200.$$

$$\therefore \text{cost of 1 seer of gold} + \text{cost of 1 seer of silver} = \text{Rs. } 6520.$$

$$\text{or cost of 80 tolas of gold} + \text{cost of 80 tolas of silver} = \text{Rs. } 6520.$$

$$\therefore \text{cost of 1 tola of gold} + \text{cost of 1 tola} \\ \text{of silver} = \text{Rs. } \frac{6520}{80} = \text{Rs. } 81.50$$

$$\therefore \text{Rs. } 80 + \text{cost of 1 tola of silver} = \text{Rs. } 81.50$$

$$\therefore \text{cost of 1 tola of silver} = \text{Rs. } 81.50 - \text{Rs. } 80 = \text{Rs. } 1.50,$$

$$\text{cost of 1 tola of alloy} = \text{Rs. } \frac{38880}{80} = \text{Rs. } 486$$

Hence $\frac{\text{quantity of gold}}{\text{quantity of silver}}$

$$= \frac{\text{diff. in prices per tola of silver and alloy}}{\text{diff. in prices per tola of gold and alloy}}$$

$$= \frac{\text{Rs. } 486 - \text{Rs. } 81.50}{\text{Rs. } 80 - \text{Rs. } 81.50} = \frac{404.5}{-1.5} = 3 : 2$$

Thus proportion of gold to silver = 3 : 2

5. Problems involving three or more kinds are treated in a similar way. Two kinds are to be selected at a time, one having a higher price than the average and the other lower. Each kind must be taken at least once but in so doing one kind may be taken more than once.

It should be noted that such problems admit of infinite number of solutions.

Ex. 10. In what proportion may three kinds of tea worth 2s ; 3s. and 4s. 6d per lb be mixed so as to produce a mixture worth 3s. 3d per lb ?

	1st kind	2nd kind	3rd kind
Cost price (d)	24	36	54
	15	15	3
		39	
Average price of mixture (d)	15		15
	15	15	18

Ratio of 1st kind and 3rd kind = 15 : 15

Ratio of 2nd kind and 3rd kind = 15 : 3

∴ the proportion of 1st, 2nd and 3rd kinds = 15 : 15 : 15 + 3

$$= 15 : 15 : 18$$

$$= 5 : 5 : 6$$

Also, ratio of 1st kind and 3rd kind = 15 : 15 = 1 : 1

Ratio of 2nd kind and 3rd kind = 15 : 3 = 5 : 1

∴ the proportion of 1st, 2nd and 3rd kinds = 1 : 5 : 1 + 1

$$= 1 : 5 : 2$$

Also, ratio of 1st kind and 3rd kind = 15 : 15

Ratio of 2nd kind and 3rd kind = 15 : 3 = 5 : 1

∴ the proportion of 1st, 2nd and 3rd kinds = 15 : 5 : 15 + 1

$$= 15 : 5 : 16$$

and so on.

Thus there is an infinite number of solutions of the above problem.

Alteration in the constituents of a mixture.

Ex. II. An alloy of 20 oz of gold and silver is 80% fine. How much gold is to be mixed up with this alloy to make it 90% fine?

80% fine means that there are 80 parts by weight of the finer metal (gold) in 100 parts of the alloy.

Quantity of gold in the alloy = 80% of 20 oz

$$= \frac{80}{100} \times 20 \text{ oz} = 16 \text{ oz.}$$

∴ quantity of silver = 20 - 16 oz = 4 oz

Evidently, 4 oz = 100 - 90 or 10% of the new alloy

∴ increased quantity of alloy = 40 oz.

∴ quantity of gold to be added = 40 - 20 or 20 oz.

Ex. A tradesman professes to retail his goods at a profit of 10% but he adulterates them by mixing $\frac{1}{4}$ th of their weight of an inferior article which costs him $\frac{1}{5}$ th of the price of the better. How much per cent does he make? Also in what proportion must he mix the two kinds so as to gain 20%? (C.U.)

First Part

Let quantity of better quality = 4 lbs.

„ cost of 1 lb „ „ = 100 units

Then quantity of inferior quality = 1 lb

Cost of 1 lb „ „ = $\frac{4}{5} \times 100$ or 80 units.

∴ cost of 5 lbs of mixture = 400 + 80 or 480 units.

∴ „ 1 lb „ „ = $\frac{480}{5}$ or 96 units ✓

But S. P. of 1 lb = 110 units

∴ profit = 110 - 96 or 14 units

∴ percentage of profit = $\frac{14}{96} \times 100 = 14\frac{7}{12}\%$

Second Part

Here S. P. of 1 lb of mixture = 110 units and expected profit = 20%

C.P	S.P	
100	120	$? = \frac{110 \times 100}{120} = 91\frac{2}{3}$
?	110	

Hence

$$\begin{aligned}
 & \frac{\text{quantity of better quality}}{\text{quantity of inferior quality}} \\
 &= \frac{\text{price of 1 lb of mixture} - \text{price of 1 lb of inferior quality}}{\text{price of 1 lb of better quality} - \text{price of 1 lb of mixture}} \\
 &= \frac{91\frac{2}{3} - 80}{100 - 91\frac{2}{3}} = \frac{11\frac{2}{3}}{8\frac{1}{3}} = \frac{7}{5}
 \end{aligned}$$

Exercise 12

1. A dealer mixes 60 gallons of wine containing 10% of water with 90 gallons containing 20% of water. What is the percentage of water in the mixture?

2. A merchant buys teas at Rs. 2 4as, Rs. 2 6as 6p and Rs. 2 11as 4p per lb and mixes them in the proportion 5 : 4 : 1. At what price must he sell the mixture so as to make a profit of 25%? (C.U. 1941)

3. A milk vendor mixes 10 seers of milk at 5 annas per seer with 20 seers at 6 annas per seer and then adds a quantity of water. By selling the mixture at 6 annas per seer he gains 50 per cent profit. Find the quantity of water added.

4. Two vats contain mixtures of spirit and water. In the first vat there are three parts of spirit to one of water and in the second there are five parts of spirit to seven of water. A 20-gallon cask is filled from these vats so as to contain a mixture of equal parts of spirit and water. How many gallons are taken from the first vat? (G.C. 1954)

5. An alloy is made with copper, tin and zinc in the proportion 4 : 7 : 9 by weight respectively. If the price of copper, tin and zinc be Rs. 220, Rs. 180 and Rs. 300 per unit of weight, find the price of $2\frac{1}{2}$ units of this alloy.

6. A dealer sells two qualities of tea at Rs. 1.75 per lb and Rs. 2.50 per lb, thereby making a profit of 12% and 20% respectively. He mixes them in the ratio 4 : 3. At what price must he sell the mixture to gain at least 10% on outlay?

7. A trader mixed 100 lbs of tea at one price with 70 lbs of tea at a dearer price. By selling the mixture at Rs. 2.12 per lb, he gained 20% on his outlay. Find the value of each kind of tea, the difference in prices being 36nP per lb.

8. Two grades of motor oil *A* and *B* are mixed in the proportion 3 : 1 to make 96 gallons of grade *C*. When half of *C* has been sold, a further quantity of *A* is added to increase the proportion of *A* to *B* in the resulting mixture to 7 : 2. Find the quantity of *A* last added (G.C.)

9. The price of gold is Rs. 38 15as per ounce. An alloy of gold and silver weighing 28 lbs is worth Rs. 7246 8as ; but if the weights of gold and silver in the alloy were interchanged, it would be worth only Rs. 1717 8as. Find the proportion of gold and silver in the alloy and the price of silver per ounce. (S.C.C.)

10. Milk contains by bulk 88.75 per cent of water, 2.75 per cent of fat and 8.5 per cent of non-fatty solids. A milkman added water to the milk so that on analysis the percentage of fat was only 2.31. Find the number of gallons of water added to 100 gallons of milk, and the amount by which a person was defrauded who bought 10 gallons of milk at 2s a gallon. (L.C.C.)

11. If 48 gallons of mixed paint containing $1\frac{1}{2}\%$ of oil are to be made into a paste containing paint and oil in the ratio 2 : 1, find the quantity of oil to be added. If 14 more gallons of oil are added, find the ratio of paint and oil in the new mixture. (C.U. 1959)

12. A sample of 20 lbs of mustard oil is found to contain impurities of 10%. How much of pure mustard oil is to be mixed with it to make it 95% pure?

13. Equal number of mangoes are bought at the following rates : Rs. 3 per score, Rs. 2 per dozen and 32 nP per pair. These are mixed up and sold at Rs. 2.24 per dozen and the total profit amounts to Rs. 80. Find the number of each of the above prices.

14. In making bread 2 parts of water are mixed with 5 parts of flour. In baking part of the water is evaporated so that 140 lb dough is made into 128 lb bread. If flour costs Rs. 15/- per maund, and the cost of labour, rent etc. represents 80% of the cost of flour and a lb loaf is sold for 6as, what percentage of the price is profit? (assume water is obtained free and 1 seer=2 lbs. in weight) (C.U. 1955)

15. With tea half-a-crown a pound a dealer mixes an inferior quality worth 18d a pound. In what proportion must he mix them so that by selling the mixture at the higher price he may gain 16 per cent? (C.U. 1940, P.U. 1945)
(Hints—higher price means here 30d)

✓ 16. A milkman adulterates a quantity of milk as follows: He first takes out 9 seers of milk and replaces

it by water. After thorough mixture, he again takes out 9 seers and replace it by water. The proportion of milk to water in the mixture is now 9 : 16. If the adulterated milk is sold at 5as per seer, what is the extent of illicit profit made by the milkman ? (P.S.C)

17. Dress materials of two qualities costing Rs. 1/6/- and Rs. 1/10/- per yard are to be used in making children's frocks to be sold at Rs. 5/7/- per piece. Assuming that on the average one yard and a half of cloth is used in a frock and that the overhead charges and profit are covered by charging Rs. 1/2/- per piece, find how much of the second quality is to be used with every 180 yards of the first.

(C.U. 1955)

18. A certain fabric is made of Egyptian cotton mixed with Broach cotton in the proportion 2 : 5. The price of the Egyptian variety is three times that of the Broach Variety. The overall cost of the fabric is Rs. 1-10as per lb. including average making charges of 4as per lb. of the fabric. Find the cost of the two varieties of cotton per lb.

How is the overall cost altered if the proportion of Egyptian to Broach cotton is changed to 5 : 2, other charges remaining the same ? (C.U. 1952)

[Hints : cost of 1 lb of E. cotton = cost of 3 lbs of B. cotton. $\therefore$ cost of 2 lbs of E cotton = cost of 6 lbs of B. cotton.

$\therefore$ cost of (2+5) or 7 lbs of fabric = cost of (6+5) or 11 lbs of B. cotton.

But cost of 1 lb. of fabric = Rs. 1-10 less 4as = Rs. 1-6as. etc.]

19. A grocer wishes to mix teas at 2s, 3s, 3s 6d and 4s per lb respectively ; how must he mix them (using the first two kinds in the proportion 2 : 3 and the last two in the proportion of 3 : 4) so that by selling the mixture at 3s 4d per lb, $\frac{1}{10}$ of the receipts may be clear profit ?

(P.U. 1945)

Chapter XIII

Commission & Brokerage, Freight, Rebates,

Insurance Premium & Claims

1. **Commission.** The remuneration paid to Agents who act on behalf of others for buying and selling goods is called *Commission*. Commission is generally fixed on percentage basis on the total value of work effected by him.

There are many kinds of agents such as Brokers, Mercantile agents, Commission agents, Del Credere agents.

A Del credere agent receives an extra percentage of commission for his guarantee that the parties involved will fulfil their contracts.

A broker is merely an agent to negotiate sales and purchases of lands, buildings, stocks and shares etc. A broker's remuneration is termed *brokerage*.

The effect of brokerage or commission is to increase cost and decrease sale price.

2. **Freight.** The charge for transport of goods is called *freight*. Transportation may be made by land, water or air.

Freight is charged by weight, volume, or distance (mileage) to yield higher revenue. Heavy articles are charged by weight and bulky goods by measurement. For charging by volume so many C. ft are charged as for 1 ton.

Thus a steamer ton may be 40 cwt at one place but 14 cwt at another. In Indian Railways freight is charged per km per quintal.

Rebate. It is synonymous with discount. It is an allowance made for prompt payment.

3. **Insurance Premium & claims.** Insurance is an indemnity against loss. The amount paid periodically to the Insurance Company or Corporation for covering risk of property

Special risks such as War Risks are also covered by insurance. Premiums are sometimes quoted with discounts allowed.

For fixation of rates assessment i.e. rateable on each holding is first ascertained on the basis of its net rental and the rates are charged as so many naye paise per Re or so much per cent in India. In England it is charged at so many shillings and pence per £.

Ex. 1. A traveller was appointed by a merchant on a salary of Rs 1800 per annum plus 5% commission on the value of sale effected by him. If in a year he effected sales to the extent of 1,08,000, find his commission and total remuneration.

Annual salary	= Rs 1,800
Commission 5% of Rs 1,08,000	= Rs 5,400
Total remuneration	= Rs 7,200

The charge for 59½ C. ft is the same as that for 60 C ft.

$$\therefore \quad \therefore 60 \quad \therefore = \frac{60}{50} \times 31s \ 4d = 37s \ 7'2d$$

Ex. 3. A property changed hands three times, each agent who sold it charging 1 per cent as commission. If each time it was sold for the net amount obtained at the previous sale,

calculate the original value, if the third sale realised Rs 48514'35.

Let original value	= 1000	
Less com. 1%	= 10	
Net sale proceeds	= 990	
Less com 1%	= 9'90	
Net sale proceeds	= 980'10	
Less com. 1%	= 9'801	
Net sale proceeds	= 970'299	
Net sale	Original value	
970'299	1000	$? = \text{Rs. } \frac{48514'95 \times 1000}{970'299}$
Rs 48514'95	?	
		= Rs 50,000

Ex 4. A broker gets 8% and then 2% commission on the sale value of a car sold by him. If he sells a car for Rs 20,000, find his commission.

Sale value = Rs 20,000	
Less 8%	1,600
	Rs 18,400
Less 2%	368
Net sale price	= Rs 18,032

∴ broker's total commission = Rs 1,600 + Rs 368 = Rs 1,968
(or = Rs 20,000 - Rs 18,032)

Ex 5. What premium at 3 p. c. will be paid for insuring a ship worth Rs 12,000? What sum must be insured for, so that in case of loss the value of the ship and the premium be recovered? (C. U. 1923)

Premium at 3% = $\frac{3}{100} \times \text{Rs } 12,000$	= Rs 360
Let the sum insured = 100	
Less premium paid = 3	
	= 97

Hence to recover loss and premium, the ship valued at Rs 97 must be insured for 100.

Value	Sum insured	
97	100	$? = \text{Rs. } \frac{12,000 \times 100}{97}$
Rs 12,000	?	
		= Rs 12371 (app)

Ex. 6. Calculate the marine insurance premium on a shipment

of goods valued at Rs 12,000, if the rate of premium is 5% less 20% plus transhipment risk $\frac{1}{4}\%$.

5% Less 20% plus $\frac{1}{4}\%$	value	premium
Less 20%	100	$4\frac{1}{4}$
	Rs 12,600	?
	? = Rs $\frac{12,600 \times 4\frac{1}{4}}{100}$	
Plus $\frac{1}{4}\%$		
Net premium = $4\frac{1}{4}\%$		= Rs 535'50

Ex. A Corporation requires a revenue of Rs 5,48,320. If total revenue of the city is Rs 72,37,537 find to the nearest pie the rate per rupee to be levied. If all the rates are paid in full, what surplus would accrue to the Corporation? (I. M. C 1952)

The required rate = Rs $\frac{5,48,320}{72,37,537} = 1 \text{ a } 2'5 \text{ p} = 1 \text{ a } 3 \text{ p}$ to the nearest pie.

Note. Generally no *even rate* can raise the desired amount. The rate must be such as to raise *at least* this amount.

Total amount realised at 1 a 3 p per Re

$$= \text{Rs } 72,37,537 \times 1 \text{ a } 3 \text{ p} = \text{Rs } 565432 \text{ 9 as } 3 \text{ p}$$

$$\therefore \text{ Surplus} = \text{Rs } 565432 \text{ 9 as } 3 \text{ p} - \text{Rs } 5,48,320$$

$$= \text{Rs } 17,112 \text{ 9 a } 3 \text{ p}$$

Exercise 13

1. An agent receives 2% commission on the total annual sales and an extra bonus of 1% on the excess of sales over Rs 20,000. In 1960 his total remuneration was Rs 910, find his sales for the year.

2. A consignee is allowed 10 per cent commission on sales, plus $2\frac{1}{2}$ per cent *del credere* commission. If his total commission amounts to Rs 5,625 what is the total amount of sales effected by him? (G. C. 1964)

3. An agent is paid a commission at the rate of $1\frac{1}{2}\%$ on the value of goods bought for his clients. His own expenses average 2s 6d on each £ 1 of commission received. What is his net income on the purchase of goods worth £ 890 ? (P. U. 1950)

4. A property changed hands three times, each agent who sold it charging 25s per cent as commission. If, each time it was sold for the net amount obtained at the previous sale, calculate the original value if the third sale realised £ 4,622 1s 10d. (P. U. 1941)

5. S sinha, who wanted to sell a few horses refused an offer of Rs 525 for each. Later he sold them through a broker. 4 horses at Rs 570 each and 6 at Rs 555 each, but the broker charged 5% as his commission. How much per cent did sinha gain or lose over the previous offer that was refused by him ? (C. U. 1961)

6. The rate of freight from calcutta to Liverpool is 37s 6d per ton (40 C. ft). Find the charges of two packages one measuring 15 C. ft and the other weighing 15 cwt 3 qr. (P. U. 1944)

7. A cargo is valued at Rs 36,850. of the rate of Insurance premium be $7\frac{1}{2}\%$, policy charges $\frac{1}{4}\%$ and commission $\frac{1}{8}\%$, what should be the amount for which the cargo must be insured to cover the cost of insurance and of the cargo ? (C U. 1953)

8. Calculate the marine insurance premium on a shipment of goods valued at Rs. 12,650 if the rate of premium is $\frac{1}{16}\%$ less 5% plus Transshipment Risk $\frac{1}{8}\%$. (P.U. 1948)

9. A ship worth Rs. 18000 is wrecked ; $\frac{1}{3}$ belonged to A, $\frac{1}{4}$ to B and the rest to C. Find what loss each of them sustains if the ship is insured to the extent of $\frac{3}{4}$ of its value.

10. A city municipality decides to meet half of a new recurring expenditure of 20 lakhs out of its ordinary revenue and to raise the other half by increasing the rates. If the rateable

value of the city be $12\frac{1}{2}$ crores, calculate the additional rate that must be levied to raise the necessary fund.

11. A Government grant is made at the rate of 5s per head for all children at elementary schools. If the grant be distributed at the rate of 5s 9d per child in the town and 3s. 3d per child in country schools, what percentage of the total number of children is in each class of schools ?

12. The scheme of street electrification is estimated to cost a municipality Rs. $12\frac{1}{2}$ lakhs per annum. If the rateable value of the town be Rs. $2\frac{1}{2}$ crores, find to the nearest pie the minimum additional rate that will have to be levied to meet this expenditure. If 15% of the additional rate cannot be collected, find whether it will be possible to meet entire expenditure of the scheme from levy. (G.U. 1947)

13. S. Roy builds a mansion containing 20 flats. He invests inland Rs. 75000 only. To build the mansion, he has to pay Rs. 2,15,000 to the contractors and 5% of that as Architect's fee. The building is assessed at Rs. 14,500 and the yearly rates are 22 nP. in the rupee. Allowing 5% of the total initial cost of the mansion each year for repairs and maintenance and one flat on an average to be vacant, calculate the rent to the nearest rupee that Roy should ask for each flat to get a return of 8% on the money he has sunk. (C.U. 1960)

Chapter XIV

Partnership

Partnership : Indian Partnership Act, 1932 defines a Partnership as the relation between persons who have agreed to share the profits of a business carried on by all or any of them acting for all. Such persons are called *partners* and the partners are collectively called a *firm*.

Generally, there is a partnership agreement in writing which sets forth in-detail, the rights and liabilities, duties and powers of the partners. Such a document is called *Articles of Partnership*.

2. In the absence of any partnership agreement, or specific provisions in the agreement the law lays down that

(1) Partners are entitled to share equally in the profits and must contribute equally towards loss irrespective of the amounts of their capitals

(2) Partners are not entitled to interest on capital and are not liable for interest on withdrawals

(3) The liability of each partner is unlimited.

The contract may be varied by consent of all partners.

The Partnership agreement generally provides for withdrawals of a certain amount by each partner in expectation of future profits. Such drawings may or may not be chargeable with interest as per agreement.

3. If a new partner is admitted into a running business, the new-comer is required to compensate the existing partners for the surrender of a part of their profits. Such compensation is called premium for goodwill. It is quite distinct from any capital that he may bring in.

Ex. 1. A, B, C, D and E are in partnership with a total capital

of Rs. 81,000. *A* contributes $\frac{3}{8}$ this of this amount and *B* contributes $\frac{4}{9}$ this. Of the remainder of the amount left *C* and *D* contribute $\frac{1}{2}$ and $\frac{1}{3}$ respectively. The partnership agreement provides for payment of interest on capital contributed by a partner at 5 per cent per annum. Calculate interest so receivable by each of the partners. (C.U. 1951)

Total capital = Rs. 81,000

A's contribution = $\frac{3}{8} \times \text{Rs. } 81,000 = \text{Rs. } 48,600$

remainder = Rs. 32,400

B's contribution = $\frac{4}{9} \times \text{Rs. } 32,400 = \text{Rs. } 14,400$

remainder = Rs. 18,000

C's contribution = $\frac{1}{2} \times \text{Rs. } 18,000$

= Rs. 9,000

D's contribution = $\frac{1}{3} \times \text{Rs. } 18,000$

= Rs. 6,000 = Rs. 15,000

E's contribution = remainder = Rs. 3,000

Interest on *A*'s capital = 5% of Rs. 48,600 = Rs. 2,430

„ *B*'s „ = 5% of Rs. 14,400 = Rs. 720

„ *C*'s „ = 5% of Rs. 9,000 = Rs. 450

„ *D*'s „ = 5% of Rs. 6,000 = Rs. 300

„ *E*'s „ = 5% of Rs. 3,000 = Rs. 150

Ex. 2. *A* and *B* are partners in a trading venture. *A* contributing £ 2,000, *B* £ 3,000. *A*, however, acts as Manager, the understanding being that of the profits *A* shall get 25% for his services as Manager, the remainder to be divided in the ratio of their contributions. *B* gets £ 120 as his share. What would *A* get? (C.U. 1943)

Let annual profit = 100

Less *A*'s salary = 25

balance = 75

Share ratio

= 2000 : 3000 = 2 : 3

∴ *A*'s share of profit = $\frac{2}{5} \times 75 = 30$

B's „ „ „ = $\frac{3}{5} \times 75 = 45$

∴ A gets 25 + 30 or 55, B gets 45

Share	value	
45	£ 120	$£ = \frac{120 \times 55}{45} = £ 146 \text{ } 13\text{s } 4\text{d.}$
55	?	

Ex. 3. A and B start a business in partnership, A contributing a capital of Rs. 6,000. and B Rs. 9,000. It is arranged that after 5% interest on capital has been paid out of the year's profits, A is to receive, should the profits so admit, an extra 1% on his capital and a yearly salary of Rs. 2,000. Any remaining profits are then divided in proportion to the capital put into the business. One year A receives altogether Rs. 3,210. What does B get ?

Interest on B's capital = 5% of Rs. 9,000 = Rs. 450

Interest on A's capital = 5% of Rs. 6,000 = Rs. 300

Extra 1% on capital = Rs. 60

Salary = Rs. 2,000

∴ A's total receipt = Rs. 2,360 ✓

Now A's ,, , = Rs. 3,210

∴ A's share of profit = Rs. 3,210 - Rs. 2,360 = Rs. 850

Profit sharing ratio = 2 : 3. ✓

	Share	Value	
(A)	2	Rs. 850	$? = \text{Rs. } \frac{\text{Rs. } 850 \times 3}{2} = \text{Rs. } 1275.$
(B)	3	?	

∴ B gets interest on capital + share of profit

= Rs. 450 + Rs. 1275 = Rs. 1725

Ex. 4. A starts a business with capital of £150 and after 6 months B becomes a partner investing £650. After 4 months more C joins and brings a capital of £150. What should each receive out of profit of £1025 made by the time. A has been in business, reckoning that each receives 10% interest per annum on his capital prior to the division of profits in proportion to capital.

Since the investment periods are different the capitals must be reduced to equivalent sums

A's Capital = £450 for 1 year = £450 × 12 for 1 month

B's Capital = £650 for 6 months = £650 × 6 for 1 month

C's Capital = £950 for 2 months = £950 × 2 for 1 month.

∴ profit sharing ratios = 450 × 12 : 650 × 6 : 950 × 2
= 5400 : 3900 : 1900 = 54 : 39 : 19

Total shares = 54 + 39 + 19 = 112

Interest on A's capital = £ 5400 × $\frac{1}{12}$ × $\frac{1}{10}$ = £ 45

" B's " = £ 3900 × $\frac{1}{2}$ × $\frac{1}{10}$ = £ 32½

" C's " = £ 1900 × $\frac{1}{6}$ × $\frac{1}{10}$ = £ 15½

Total = £ 93½

∴ divisible profits = £ 1025 - £ 93½ = £ 931½

A's share of profit = $\frac{54}{112} \times £ 931½$ = £ 449 3s 11d

B's " " = $\frac{39}{112} \times £ 931½$ = £ 324 8s 5d :

C's " " = $\frac{19}{112} \times £ 931½$ = £ 158 1s

A's total receipts = £ 45 + £ 449 3s 11d = £ 494 3s 11d

B's " " = £ 32½ + £ 324 8s 5d = £ 356 18s 5d

C's " " = £ 15½ + £ 158 1s = £ 173 17s 8d

Ex. 5. The gross profits of a business are divided as follows :—

15% for depreciation, 10% for reserve, 5% as interest on capital invested and the remainder is divided among the three partners in the proportion 3 parts to A, 2 parts to B and 1½ to C. Suppose the gross profits to be £ 25,000 and the capital invested by A, B and C respectively to be £ 40,000, £ 35,000 and £ 25,000 ; how much does each of the partners receive ?

(C. U. 1944)

Gross profits = £ 25,000

Depreciation = 15% of " = £ 3,750

Reserve = 10% of " = £ 2,500

Interest = 5% of " = £ 1,250

$$\therefore \text{net profits} = \text{£ } 25,000 - \text{£ } 7,500 = \text{£ } 17,500$$

$$\text{Profit-sharing ratios} = 3 : 2 : \frac{3}{2} = 6 : 4 : 3$$

$$\therefore A^s \text{ share of profit} = \frac{6}{13} \times \text{£ } 17,500 = \text{£ } 8,076 \quad 19s$$

$$B^s \quad \quad \quad \quad \quad = \frac{4}{13} \times \quad \quad \quad = \text{£ } 5,384 \quad 12s$$

$$C^s \quad \quad \quad \quad \quad = \frac{3}{13} \times \quad \quad \quad = \text{£ } 4,038 \quad 9s$$

$$\text{Interest sharing ratios} = 40,000 : 35,000 : 25,000 = 8:7:5$$

$$\therefore A^s \text{ share of interest} = \frac{8}{20} \times \text{£ } 1250 = \text{£ } 500.$$

$$\therefore A^s \text{ total receipts} = \text{£ } 8,076 \quad 19s + \text{£ } 500 = \text{£ } 8,576 \quad 19s$$

$$B^s \quad \quad \quad \quad \quad = \text{£ } 5,384 \quad 12s + \frac{7}{20} \times \text{£ } 1,250 = \text{£ } 5,822 \quad 2s$$

$$C^s \quad \quad \quad \quad \quad = \text{£ } 4,038 \quad 9s + \frac{5}{20} \times \text{£ } 1,250 = \text{£ } 4,350 \quad 19s$$

Ex. 6. In a partnership between *A*, *B* and *C* it is agreed *A* is to be paid a salary of Rs. 250 a month and *B* a commission of 5% on gross sales. The balance of profit and loss is to be divided between them in the proportion of 3:4:9 respectively. In two consecutive years the profits are Rs. 3,000, Rs. 15,000 before paying out *A* and *B*; and the gross sales are Rs. 30,000 and Rs. 1,00,000 respectively. Divide the total of the two years' profit between the partners who agree not to divide the first year's profit. (C. U. 1952)

$$A^s \text{ salary for 1st year} = \text{Rs } 250 \times 12 = \text{Rs } 3,000$$

$$B^s \text{ commission } \quad \quad = 5\% \text{ of Rs } 30,000 = \text{Rs } 1,500$$

$$\text{Rs } 4,500$$

$$A^s \text{ salary for 2nd year} = \text{Rs } 250 \times 12 = \text{Rs } 3,000$$

$$B^s \text{ commission } \quad \quad = 5\% \text{ of Rs } 1,00,000 = \text{Rs } 5,000$$

$$\text{Rs } 8,000$$

$$\text{Total liability for 2 years} \quad \quad \quad \text{Rs } 12,500$$

On the basis of 2nd years' profit, balance for distribution

$$= \text{Rs } 15,000 - \text{Rs } 12,500 = \text{Rs } 2,500 \quad \checkmark$$

Profit sharing ratios = 3 : 4 : 9.

$$\therefore A^s \text{ share of profit} = \frac{3}{16} \times \text{Rs } 2,500 = \text{Rs } 468 \frac{75}{16}$$

$$B^s \quad \quad \quad \quad \quad = \frac{4}{16} \times \text{Rs } 2,500 = \text{Rs } 625$$

$$C^s \quad \quad \quad \quad \quad = \frac{9}{16} \times \text{Rs } 2,500 = \text{Rs } 1406 \frac{25}{16}$$

$$\therefore \text{Amount received by } A = \text{Rs } 3,000 + \text{Rs } 3,000 + \text{Rs } 468.75 \\ = \text{Rs } 6,468.75$$

$$\text{,, ,, } B = \text{Rs } 1,500 + \text{Rs } 5,000 + \text{Rs } 625 \\ = \text{Rs } 7,125$$

$$\text{,, ,, } C = \text{Rs } 1406.25.$$

Ex. 7. *A* and *B* were partners in a manufacturing concern. On 1st January, 1955 *C* was admitted as a partner when *A* agreed to give up $\frac{2}{19}$ ths of his share and *B* $\frac{3}{19}$ ths of his share. When the profits for the year 1955 were divided, *C* got Rs 1693. How much did *A* and *B* receive? (C. U. 1956)

A and *B* have equal shares. Let each have $\frac{1}{2}$ shares. Then total share = $\frac{1}{2} + \frac{1}{2} = 1$.

$$C's \text{ share} = \frac{2}{19} \text{ of } \frac{1}{2} + \frac{3}{19} \text{ of } \frac{1}{2} = \frac{1}{19} + \frac{3}{38} = \frac{5}{38}$$

$$\text{After admission of } C, A's \text{ share} = \frac{1}{2} - \frac{1}{19} = \frac{17}{38}$$

$$B's \text{ share} = \frac{1}{2} - \frac{3}{38} = \frac{15}{38}$$

$$C's \text{ share} = \text{Rs } 1690 \text{ which is } \frac{5}{38} \text{th of the total profits}$$

$$\text{Hence total profit} = \text{Rs } 1690 \times \frac{38}{5} = \text{Rs } 338 \times 38$$

$$\therefore A \text{ received } \frac{17}{38} \times \text{Rs } 338 \times 38 \text{ or Rs } 5,746$$

$$B \text{ received } \frac{15}{38} \times \text{Rs } 338 \times 38 \text{ or Rs } 5,408$$

Ex. 8. Three persons contribute sums £250, £500, of £750 respectively towards a venture on the understanding that the profits shall be divided in such a way that the rate of interest which each receives shall be in proportion to the amount of his contribution. If the profits for a year amount to £245, how much will each receive?

$$\text{The rates of interest} = 250 : 500 : 750 = 1 : 2 : 3.$$

Let the interest on *A*'s capital be 1%, then on *B*'s capital is 2%, that on *C*'s capital is 3%.

$$\therefore \text{Interest on } A's \text{ capital} = 1\% \text{ of } £250 = £2\frac{1}{2}$$

$$\text{,, ,, } B's \text{ ,, } = 2\% \text{ of } £500 = £10$$

$$\text{,, ,, } C's \text{ ,, } = 3\% \text{ of } £750 = £22\frac{1}{2}$$

$$\text{Total } £35$$

Hence the actual rate of interest on A 's capital = $\frac{245}{35}$ or 7%

" " B 's " = 2×7 or 14%

" " C 's " = 3×7 or 21%

Hence amount A receives = 7% of £250 = £17½

" B " = 14% of £500 = £70

" C " = 21% of £750 = £157½

Exercise 14

1. A and B enter into a partnership. A furnishes 25 men for 4 days in addition to 18 men for 5 days; B furnishes 15 men for 8 days and 18 men for 10 days. A receives Rs 162. What should B receive?

2. A , B , C enter into a partnership with capitals of Rs 15,000, Rs 21,000 and Rs 36,000. A , as manager, is to receive $6\frac{1}{2}\%$ of the profits and the remainder is to be divided in proportion to capital. How much did B and C receive if A received Rs 7,458 in a year? (*P. C. 1949*)

3. A and B are partners sharing profits in proportion to $\frac{3}{10}$ ths and $\frac{7}{10}$ ths with capitals of Rs 20,000 and Rs 15,000 respectively. They agreed to an interest of 5% on capital and A was to be allowed an annual salary of Rs 1,500. The profits for the year 1961 prior to calculation of interest on capital but after charging A 's salary, amounted to Rs 6,000. Calculate how much each received.

4. Ramesh, Suren and Matilal had agreed to divide the profits of their partnership firm in proportion to their capitals, which were Rs 25,000/- Rs 22,000/- and Rs 18,000/- respectively at the beginning of the year. After 2 months Ramesh added Rs 2000/- capital and Suren after 3 months withdrew Rs 1,000/- capital and after 8 months Matilal added Rs 2,000/- capital. If the profit for the year was Rs 8960/4/-, what amount should each partner receive? (*C. U. 1954*)

5. Aiyer and Bose engage in a business, their capitals being Rs 40,000 and 30,000 respectively. They decide to place to a reserve fund 5% of the trading profit every year. They also agree that Aiyer should receive 10% and Bose should receive 8% of the balance of profits for their services as remuneration.

How much should Bose receive if, for 1960, Aiyer receives Rs 5814 in all as his share. . (C. U. 1961)

6. A and B enter into a partnership with Rs 40,000 and Rs. 50,000 respectively. It is agreed that A will draw a salary of Rs 300/- per month for supervising the business and B is to receive a commission of 2 per cent on the turnover as salesman and the profits will be shared in the proportion of 2 : 3. In 1953, the sales amounted to Rs 1,47,300 and the profits amounted to Rs. 32, 426 before charging salary and commission. Calculate the returns the income of A and B represent on their respective capitals. (C. U. 1955)

7. A and B are partners contributing Rs 10,000 and Rs 12,000 respectively. It is agreed that after 5% interest on their capital, A should get Rs 6,000 annually as salary and an extra 1% on his capital for supervising the business. It is further agreed that the balance of the profits and losses should be divided in in the ratio of their capitals put in by them. For the year 1959, A gets altogether Rs 16,612. How must does B get and what is the total amount of trading profit or loss of the firm in 1959 ? (C. U. 1960)

8. Bose, Roy and Sen had repectively capital of Rs 31,000/- Rs 10,500/- and Rs 3500/- in their firm. On dissolving the partnership, their liabilities including repayment of capital were Rs 51,600/- and the assets realised Rs 47,400/-. What should each receive as repayment of capital (i) if they had agreed to bear $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{3}$ of the lesses respectively and (i) if they had agreed to share losses in proportion to their capitals ? (C.U.1957)

9. A, B, C and D formed a partnership on 1st July 1956 with capitals of Rs 5,000/-, Rs 4,000/- Rs 6,000/- and Rs 9 000/-

respectively and sharing profits and losses equally. A introduced a further capital of Rs 5000/- on 1st November 1956 and B brought in Rs 3,000/- on 1st December 1956. C withdrew Rs 2000/- of his capital on 31st. January 1957 and D withdrew Rs 3,000/- on 1st. March 1957. The annual trading profit for the year ended 30th. June 1957 amounted to Rs 9690. Find the amount each partner should receive on the basis that each partner would receive 5% interest per annum on his capital prior to the division of profits (no interest being charged on withdrawals). (C. U. 1958)

10. Roy, Mitra and Sen are partners of Romsco with capitals of Rs 20,000, Rs 15000, Rs 12,000 respectively on 1st February 1961. Their agreement provides that Sen should receive Rs 6,000 as salary at the end of each financial year and the profits should be divided in proportion to the capitals employed by them during the year. After 3 months Roy withdrew Rs 4,000 capital and after 4 months Mitra contributed Rs 5,000 capital and after 8 months Sen withdrew Rs 2,000 capital. Net Trading profits for the year ended 31st January 1962, amounted to Rs 42,300. Calculate the total amounts due to each partner for 1961-62 profits (C. U. 1962)

11. A, B, C and D having capitals of Rs 12000, Rs 5000, Rs 2000 and Rs 1000 respectively in their firm, agree to pay interest at 5% p. a. on their capital, and to share profits and losses in the ratios of 4 : 3 : 2 : 1

At the end of the year it is found, that the firm has incurred a net trading loss of Rs 550. If the partners decide to start the next year without an adverse balance, find what sums should be paid by or to the firm in the case of each partner? (C. U. 1962)

12. Sen and Roy carrying on business as partners decided that their capital should bear interest at 6% per annum and that any withdrawal during the year should be treated as withdrawal of capital. They also decided to retain at the end of each year half of their share of profits as capital of the business. On 1st July 1957, Sen's capital was Rs 25,000 and Roy's capital was

Rs 22,000. Sen withdrew Rs 650 on 18th August 1957, and Rs 600 on 16th March 1958. Roy withdrew Rs Rs 400 on 20th September 1957 and Rs 700 on 14th April, 1958. If the net profits for the year ended 30th June 1958 be Rs 9,672 after paying interest on capital, find their total incomes for the year 1957-58 and also their capitals on 1st July 1958. (C. U. 1951)

Chapter XV

Profit In Relation To Turnover And Costs

1. In a trading concern all losses and gains are grouped together under one Account called the Trading and Profit and Loss Account.

The function of the Trading Account is to show **Gross Profit**. Gross profit in a given period is the difference between cost of goods actually sold in the period and their sale proceeds. The office and distribution expenses are not deducted from it.

$$\text{Gross Profit} = \text{Sales} - \text{Cost}$$

The difference between gross profit and working expenses such as rent of the shop, salaries of shopmen etc. is **Net Profit**.

$$\text{Net Profit} = \text{Gross Profit} - \text{Working Expenses}$$

2. Having ascertained the gross profit the trader can find the percentage of gross profit on the buying and selling prices of his goods. This gives him two-fold advantages. He may put a check to increasing cost of goods and fix up selling prices of goods as well.

In a trading concern the word 'Turnover' is used to mean (1) the net sales—i.e., Sales less Returns Inwards, (2) quantity of goods bought and sold by a trader during a certain period, e.g. the quantity of material extracted from a mine during a particular period,

Ex. 1.

	Rs.
Total Sales	60,000
Less Returns Inwards	<u>4,000</u>
Net Sales	56,000

	Rs.	
Opening Stock	5,000	
Purchases Rs. 50,000		
Less Returns		
Outwards 7,000	<u>43,000</u>	
	48,000	
Less Closing Stock	<u>3,000</u>	
	45,000	
Add Direct Expenses	<u>5,000</u>	
	50,000	<u>50,000</u>
	Gross Profit	6,000
Add Other Incomes		<u>1,500</u>
Total Income		7,500
Less Other Expenses		<u>2,000</u>
	Net Profit	5,500

- (1) Gross Profit % on Cost of Production

$$= \frac{6,000}{50,000} \times 100 \text{ or } 12\%$$

- (2) Gross Profit % on Turnover

$$= \frac{6,000}{56,000} \times 100 \text{ or } 10.7\%$$

- (3) Net Profit % on Cost of Production

$$= \frac{5,500}{50,000} \times 100 \text{ or } 11\%$$

- (4) Net Profit % on Turnover

$$= \frac{5,500}{56,000} \times 100 \text{ or } 9.8\%$$

3. Operating Expenses fall under the following heads

- (1) Administrative Expenses : Office expenses, taxes etc.
- (2) Selling Expenses : advertising, show-room rent, commission to Agents, stationery.
- (3) Distributive Expenses : packing, carriage outwards etc.
- (4) Finance Expenses : Discount allowed, interest paid, depreciation etc.

Non-Operating Expenses : Provision for claims for damages etc.

Non-Operating Income : Discount, rebates received, dividends on stocks and shares etc.

4. Gross Profit Ratio—It is the ratio of the gross profit to Net sales, but is generally expressed as percentage on turnover.

$$\text{Gross Profit Ratio} = \frac{\text{gross profit}}{\text{turnover}} \times 100$$

Net Profit Ratio. It is the ratio of the net profit to net sales and expressed as percentage on net sales.

$$\therefore \text{Net Profit Ratio} = \frac{\text{Net Profit}}{\text{turnover}} \times 100.$$

Similarly, any other expense is expressed as a percentage on net sales or turnover.

Operating Ratio. The working of any enterprise is controlled by operational relationship between different items in the Profit and Loss Account.

Rapidity of Turnover. The profit of a business is related to the frequency with which the assets are used.

The cost and selling price of commodities remaining the same, the more the working capital rolled, the more is the profit. By rapidity of turnover we mean the quick rolling of stock. The rate of turnover denotes velocity of turnover.

<i>Ex. 2.</i> Stock of commencement (1.1.61)	= Rs 40,000
Closing stock (31.12.61)	= Rs 50,000
Cost of goods sold	= Rs 70,000

$$\begin{aligned} \text{Then average stock} &= \frac{1}{2}(\text{stock at commencement} + \text{closing stock}) \\ &= \frac{1}{2}(\text{Rs. 40,000} + \text{Rs. 50,000}) \\ &= \text{Rs. 45,000} \end{aligned}$$

$$\begin{aligned} \text{Rate of Stock Turnover} &= \frac{\text{Cost of goods sold}}{\text{Value of average Stock}} \\ &= \frac{70,000}{45,000} = \frac{14}{9} = 1.56 \end{aligned}$$

Thus the stock turn over (stock bought and sold) 1.56 times in 1 year and so average period of turnover is $\frac{365}{1.56}$ or 234.6 days.

$$\begin{aligned}\text{Average period of Turnover} &= \frac{\text{average stock}}{\text{annual turnover}} \times 365 \text{ days.} \\ \text{or} &= \frac{\text{average stock}}{\text{annual turnover}} \times 12 \text{ months.}\end{aligned}$$

6. Inter-relation of capital, turnover, working expenses and gross and net profit.

Let C = cost price, S = Sale price (turnover), T = Total cost,

G = gross profit, N = Net profit, P = Capital

$$\text{Then net profit \% on Cost price} = \frac{N}{C} \times 100 = a. \text{ (say)}$$

$$\text{net profit \% on Sales} = \frac{N}{S} \times 100 = b \text{ (say)}$$

$$\text{net profit \% on Total cost} = \frac{N}{T} \times 100 = d \text{ (say)}$$

$$\text{net profit \% on Capital} = \frac{N}{P} \times 100 = h \text{ (say)}$$

From the above, we establish the following relations :—

$$aC = bS = dT = hP = 100 N.$$

For gross profit similar chain of relations may be established

7. Taxation, Dividend, Retained Profits

Net profit is taxable. The tax is either levied on flat or slab basis. The tax is calculated on the net profit and deducted from it to give *Divisible Profit*. Dividend to shareholders is paid out of divisible profit and the balance left if any, is called *Retained Profit*. Generally divisible profit is not exhausted. A part of it is retained either for creating a sinking fund or strengthening the financial position of the company. Sometimes a part of it is kept for maintaining the rate of dividend for some years. Such a fund is known as *Dividend Equalisation Fund*.

The relations between the different items can be shown as below.

$$\begin{array}{lcl}
 \text{Sales} \} & \text{Gross Profit—} & \} \text{Net Profit—} \} \text{Divisible Profit—} \\
 \text{Cost} \} & \text{Working Expenses} \} & \text{Income Tax} \} \text{Dividend} \} \\
 & & \text{balance = Retained Profit}
 \end{array}$$

Ex. 3. Sunrise Ltd made a net profit of Rs. 2,00,000 in 1961, and after providing for income tax and Super tax at 45 nP. in the rupee decided to distribute and retain the profits in the following manner.

(a) To transfer to General Reserve Account Rs. 15,000

(b) To accumulate Rs. 10,000 in the Debenture Redemption Fund and Rs. 5,000 in the Dividend Equatisation Fund.

(c) To declare a 6% dividend on the Preference Share Capital of Rs. 2,00,000 and of $12\frac{1}{2}\%$ on paid-up Equity Share Capital of Rs. 4,00,000.

Calculate Income tax payable, Retained Profits and Divisible Profits. (C.U. 1962)

$$\text{Income tax} = 45 \text{ nP.} \times \text{Rs. } 2,00,000 = \text{Rs. } 90,000$$

$$\begin{aligned} \text{Amount transferred to Funds} &= \text{Rs. } 15,000 + \text{Rs. } 10,000 \\ &+ \text{Rs. } 5,000 = \text{Rs. } 30,000 \end{aligned}$$

$$\therefore \text{divisible profit} = \text{Rs. } 2,00,000 - \text{Rs. } 90,000 = \text{Rs. } 1,10,000.$$

$$\text{Dividend paid} = 6\% \text{ of Rs. } 2,00,000 + 12\frac{1}{2}\% \text{ of}$$

$$\text{Rs. } 4,00,000 = \text{Rs. } 12,000 + \text{Rs. } 50,000 = \text{Rs. } 62,000$$

$$\therefore \text{retained profits} = \text{Rs. } 1,10,000 - \text{Rs. } 62,000 = \text{Rs. } 48,000$$

Chapter XVI

Approximation

1. In practical problems, approximate results are often sufficient. Thus if the price of an article works out to be 24'575 nP, we are not in a position to pay the exact amount. What we do is this : we take as approximate a value as possible.

2. **Error.** In so doing we take an amount either greater or less than the exact one. This difference between the exact and approximate values is called **Absolute Error**. The ratio of the absolute error to the exact value is called **Relative Error**. Relative error multiplied by 100 is called **Percentage Error**.

Ex. 1. If a metre is equal to 39'3701 and an approximate value of 39'37 is taken, then

$$\text{Absolute error} = 39'3701 - 39'37 \text{ in} = '0001 \text{ in}$$

$$\text{Relative error} = \frac{\text{absolute error}}{\text{real value}} = \frac{'0001}{39'3701} = \frac{1}{393701}$$

$$\text{Percentage error} = \frac{\text{absolute error}}{\text{real value}} \times 100 = \frac{100}{393701}$$

3. Significant digits.

2438'927.

This number = 2438'93 correct to 2 decimal places

„ „ = 2438'9 „ 1 „ place

„ „ = 2439 „ nearest integer

Similarly

38357

„ „ = 38360 „ 2 significant figures

„ „ = 38000 „ 4 „ „

The digits 1 to 9 are called significant figures and 0 having two digits on each side is also called significant.

4. Degree of Accuracy. The degree of accuracy determines the number of significant digits to be taken.

Rule for Approximation. Of the figures rejected, if the first figure is 5 or greater than 5, add 1 to the last figure retained, if it be less than 5, simply ignore it.

5. Limits of Error. Correct to 1 place of decimal the approximate value of any number greater than or equal to 2'75 and less than 2'85 is 2'80. The limits of error in this case are 2'80-2'85 and 2'80-2'75 *ie.*, $\pm .05$.

Now the errors $\pm .05 = \pm \frac{5}{100} = \pm \frac{1}{2} \times \frac{1}{10}$. Here we take $\frac{1}{10}$ or one-tenth as our unit of approximation.

Hence limits of error = $\frac{1}{2} \times$ unit of approximation.

Similarly for approximation correct to 2 decimal places, the limits of error are $\pm .005$ etc.

$\therefore$ Approximate value + $\frac{1}{2}$ unit = upper limit of true value

Approximate value - $\frac{1}{2}$ unit = lower limit of true value

6. In a standard form the multiplier or divisor has one significant digit in its integral part.

In multiplication, if the decimal in the multiplier move 1 place to the left, then in the multiplicand it must move 1 place to the right for compensation, *ie.*, the decimals move through the same number of places but in the opposite directions.

Ex. 1. $24'72 \times 82'91 = 247'2 \times 8'291.$

In division the decimal more in the same direction.

Ex. 2. $\frac{289'612}{87'89} = \frac{28'9612}{8'789}$

7. Contracted Method of Multiplication and Division.

Ex. 3. Multiply 43'6741853 by 59'314 correct to 3 decimal places.

Working Rule :

- (1) Reduce the multiplication to the standard form.
- (2) Put down the numbers so that the decimal points be below or above one another.

$$\begin{array}{r}
 436'74185 \\
 \underline{5'9314} \\
 2183'70927 \\
 393'06767 \\
 13'10225 \\
 43674 \\
 17470 \\
 \hline
 2590'49065
 \end{array}$$

Ans. 2590'49

(3) Draw a vertical line after (3+2) or 5 decimal places. Always take 2 more places than required. Add zeroes if required.

(4) Multiply first by 5 (1st fig before the decimal in the lower line) the number in the top line just before the vertical line. But carry, if any, by multiplying the digit next to it after vertical line by 5 and proceed.

(5) Multiply the top line by 9 starting with 8 but carry 5 from $(9 \times 5) = 45$.

(6) Proceed thus, always placing the first figure of multiplication just to the left of the vertical line.

7. Add and approximate as desired.

Ex. 4. Divide 1113'31762 by '0'8252654 correct to 2 places of decimals.

$$\begin{array}{r}
 7'8252654 \overline{) 128'31762} \\
 \underline{78'2527}
 \end{array}$$

Ans. 16'40.

$$\begin{array}{r}
 50'0649 \\
 \underline{46'9516}
 \end{array}$$

$$\begin{array}{r}
 3'1133 \\
 \underline{2'3476}
 \end{array}$$

$$\begin{array}{r}
 7657 \\
 \underline{7043}
 \end{array}$$

$$\begin{array}{r}
 614 \\
 \underline{547}
 \end{array}$$

$$\begin{array}{r}
 67 \\
 \underline{62}
 \end{array}$$

$$\begin{array}{r}
 6 \\
 \underline{5}
 \end{array}$$

(1) Reduce the division to the standard form.

(2) Find, by short calculation, the number of digits in the integral part of the quotient.

Here $\frac{128}{8} = 16$ (two figs.)

(3) The number of significant figs. in the quotient must then be 2+2 or 4.

(4) Retain two more sig. figs. in the divisor than what is required, so that the total number of sig. figs. in the divisor to be retained = 4+2 or 6. Strike off all other figs. in the divisor.

- (5) Perform the first division but carry if any.
- (6) After first subtraction, do not bring any fig. from the dividend but strike off the last fig. from the divisor.
- (7) After each subtraction strike off the last digit from the divisor and proceed until there is one more fig. than the reqd. number in the quotient.

Exercise 15

1. Multiply 246034'1643 by '0081634 correct to the nearest thousand.

Use contracted method to find the value of

2. $463486'30185 \times 296'1486$ correct to 5 significant figures.
3. 246034'1643 by '0081634 correct to the nearest thousands. (P. U. 1944)
4. 925361'237 by 834'5789 correct to nearest lakh. (P. U. 1949)
5. Divide by contracted method
100'3 by 1'0015 correct to 3 decimal places. (P. U. 1946)
6. Use contracted method to find the reciprocal of 4'515063 correct to 5 decimal places. (P. U. 1945)

PART I

Exercise 1

1. $\frac{1}{80}$; 273.
2. $\frac{5}{8}$.
3. (i) 5198. (ii) '614
4. $3 + \sqrt{2}$
5. '2
6. 36.
7. 157 metres
8. '296.
9. 7 yds. 1 ft. 4 in. ; 49.
10. 561, 943.
11. 121 yards.
12. £ 2' 15' 937.
13. 92 n.P.

Exercise 2

1. 30,000 sq. m.
 2. 24855'28 miles.
 3. 3'79 litres.
 4. 89480000 miles
 5. (i) 4'6094 km. (ii) 4609'4 m. (iii) 460940 cm.
 6. 2'71 sq. yds. ; 19'7%
 7. 7318'5 cm.
 8. (i) 765'646 km. (ii) 367'90 sq. m. (iii) 517 seers
- 50 tolas.

Exercise 3

1. 16 : 15 (decrease).
2. 22;15 (decrease), Rs. 333'33
4. 8", 16 : 441.
5. 27 : 12 : 2.
6. 10 : 9 (increase).
7. 40 : 29.
8. £ 790 9d ; £ 6151 1'5d.

23314'28

Exercise 4

1. Rs. 233314'28.
2. Rs. 1181'25.
3. Rs. 562'50
4. 6 $\frac{2}{3}$ hours.
5. Rs. 1,25,000.
6. 6 weeks.
7. Rs. 305.
8. 1000 men.
9. 68 days.
10. 100 days.
11. Rs. 35-15 as.
12. 406 $\frac{1}{2}$ tons.
13. 216.
14. Rs. 331-12 as.
15. Rs. 108.
16. Rs. 5109.

Exercise 5

1. Each boy Rs. 16'50, each scout Rs. 8'25
2. Rs. 106'25, Rs. 901, Rs. 61'25, Rs. 70, Rs. 122'50.
3. Rs. 405, Rs. 540, Rs. 675.
4. Rs. 54 each.
5. Rs. 30,000 ; Rs. 15,000 ; Rs. 2,000
6. Rs. 413'69.
7. £3550, £3124, £2982 ; 1st lose £ 14 ; 2nd gain £ 4,
3rd gain £ 10.
8. Captain Rs. 439'34 ; each Chief Mate Rs. 164'76 ;
each Second Mate Rs. 117'18 ; each Sailor Rs. 194'70.

Exercise 6

1. (a) Rs. 600. (b) $5\frac{2}{3}$ days. 2. $\frac{mnhl_1}{n_1h_1l}$.
3. $27\frac{3}{4}$ sq. in. 4. 3 in.
5. $\sqrt[3]{r^3 + r'^3}$ 6. 1 cu. ft.
7. 40 in. 9. $y = 3x + \frac{4}{x}$.
10. 4. 11. 4 : 3.
12. Rs. 2800. 13. £ 719 3 s. 4 d.
14. Rs. 53,333.33, Rs. 3,400.
15. Rs. 82'58, Rs. 98'71, Rs. 114'81.
16. Rs. 78. 17. 80% on first Rs. 100,
50% on balance.
18. $375\frac{1}{2}$ amperes.

Exercise 7

1. 25'29. 2. Rs. 72. 3. '018 km.
4. 6s. 5d. 5. Rs. 48,000.
6. (i) 10'253 fr. (ii) Rs. 29'33. 7. 37'9 litres.
8. Rs. 3. 9. 8'90 fr.
10. 3 s. $6\frac{3}{4}$ d., 4'5 fr., 0'9%
11. 11 s. 5 d.

Exercise 8

1. Rs. 175.
2. 3376½ tons.
3. 52,800.
4. (i) 75·6% (ii) 24·4%
5. 146790 ; 1,58,077.
6. 41·4%.
7. 14·31% greater.
8. Rs. 15,759 6 as.
9. Rs 24,300.
10. 11·9%, 54·9%.
11. 14%.
12. (i) 2 seers (ii) no, 1s. 3½d.
13. Rs. 40.
14. 25%.
15. 13⅓%.
17. 340⅔ tons, decrease 0·55%.
18. 40%.
19. 65%.
20. (i) Rs. 3312 8 as., Rs. 160 10 as., 1 a. 3·9 p.
- (ii) Rs. 36,406 4 as., Rs. 1820 5 as., 5 as., 1·1625 p.

Exercise 9

2. W. C. Rs. 30,00,000 ; T. C. Rs. 31,40,000 ; 18 7 4·67%
4. 3⅓%
5. Rs. 8 1a
6. Decrease ½%
7. 9% ; 23%
9. P. C. Rs. 5,50,000 ; W. C. Rs. 6,50,000 , Rs 6,30,000.

Exercise 10

1. 31·6%
2. first ; 11⅓% 10⅓%
3. £ 5—2s—11d
4. 20% ; 33⅓%
5. 19%
6. 26⅔%
7. £ 21 18s 9d ; £ 12
8. Rs. 324·25
9. £ 24
10. B.
11. Rs. 16·34
12. 24·25%
13. 38⅓%
14. 15⅓% ; Rs. 2
15. gain Rs. 432·81.

Exercise 11

1. 25%
2. $11\frac{1}{2}\%$
3. 25%
4. Rs. 9'60 ? *15'625*
5. 200%
6. 140
7. Rs. 7 2 as
8. Re. 267'50
10. 2'4%
- 11.
12. 8s *ad*
- 13.
14. $21\frac{1}{2}\frac{1}{8}\%$
15. $16\frac{1}{4}\%$
16. Rs. 130'94
17. $340\frac{2}{3}$ tons ; $\sim 0'55\%$
18. 18.
19. 5s 8*d*
20. $8\frac{3}{8}\%$ on sales
21. £ 17 10s
22. $26\frac{1}{8}\%$
23. Salary :—Rs. 4300. Rs. 5500,
Rs. 6680, Rs. 7230
Profit—Rs. 7200, Rs. 11520,
Rs. 15768, Rs. 17748
24. (a) $13\frac{1}{7}\%$ (b) $3'22\%$
25. £ 43 15s, £ 50.
26. Rs. 900, Rs. 1,30,000

Exercise 12

1. 14%
2. Rs. 2—15 a 2/-
3. $12\frac{1}{2}$ seers
4. 5 gallons
5. Rs. 605
6. Rs. 1'79
7. Rs. 1'62, Rs. 1'98
8. 6 gallons
9. 23 : 4 ; Rs. 2 9as
10. 19 gallons, 3s 2*d*
11. 15 gallons, 21:20
12. 20lbs
13. .960
14. $42\frac{2}{3}\%$
15. 19:10
16. Rs. 4 8as
17. 360 yds
18. Rs. 2—10as, 14as ;
19. 52 : 78 : 51 : 68
- Rs. 2—6as

Exercise 13

- | | | |
|-----------------|-----------------------------------|---------------------|
| 1. Rs. 37,000 | 2. Rs. 45,000 | 3. £ 11—3—7½d |
| 4. £ 48000(app) | 5. gain 1½% | 6. 14s½d, £ 19s 6½d |
| 7. Rs. 40,000 | 8. Rs. 98'43 | 9. Rs. 900, |
| 10. 0'8nP. | 11. Town school Rs. 1800, Rs. 450 | |
| | 70%, Country school 30% | |

Exercise 13

12. 10p ; Deficit Rs. 1,43,229 (app)
 13. Annual rent Rs 2226.

Exercise 14

- | | |
|---|--------------------------|
| 1. Rs 413'72 | 2. B Rs 7910, C Rs 13560 |
| 2. Rs 3775, B = Rs 3,725 | |
| 4. Rs 3588'56, Rs 2859'66, Rs 2512'03 | |
| 5. Rs 5586 | 6. 34'89%, 36'96% |
| 7. Rs 12,614'40, Rs 29,226'40 | |
| 8. Rs 29400, Rs 9100, Rs 2800 ; Rs 28592'31, Rs 9530'75,
Rs 3176'94 | |
| 9. A Rs 2475'62 ; B. Rs 2346'46 ; C Rs 2558'96 ;
D Rs 2508'96. | |
| 10. Rs 13.223'58 ; Rs 14260'71 ; Rs 14815'71. | |
| 11. A paid Rs 220, B Rs 165, C Rs 110, D Rs 55. | |
| 12. Sen income Rs 6191'78, capital Rs 26,895'89.
Roy income Rs 6128'53, capital Rs 23964'27. | |

Exercise 15

- | | |
|----------------------|--------------------------|
| 1. 2000. | 2. 13726×10^4 . |
| 3. 2×10^8 . | 4. 7723 lakh. |
| 5. 100'15. | 6. '22148. |

Commercial Mathematics

Part II

Chapter-I

LAWS OF INDICES

1. Definition

If m is a positive integer, a^m is used to denote the continued product of m factors each being equal to 'a'. That is,

$$a^m = a \times a \times a \times \dots \text{to } m \text{ factors.}$$

Thus, $a^5 = a.a.a.a.a.$, $5^3 = 5.5.5.$, etc.

2. The Index Laws

If m and n are positive integers,

$$(i) \quad a^m \times a^n = a^{m+n}$$

Proof. Since $a^m = a.a.a.\dots$ to m factors,

and $a^n = a.a.a.\dots$ to n factors,

$$\begin{aligned} \therefore a^m \times a^n &= (a.a.a.\dots \text{to } m \text{ factors}) \\ &\quad \times (a.a.a.\dots \text{to } n \text{ factors}) \\ &= a.a.a.a.\dots \text{to } (m+n) \text{ factors} \\ &= a^{m+n}, \text{ by definition.} \end{aligned}$$

This is called the *Fundamental Index Law*.

Thus, $4^5 \times 4^3 = 4^{5+3} = 4^8$.

Extension of the Law $a^m \times a^n \times a^p \times a^q \dots = a^{m+n+p+q\dots}$

Corollaries

$$(ii) \quad a^m \div a^n = a^{m-n}, \quad m > n$$

Proof. If $m > n$, then $m-n$ is a positive integer

$$\therefore a^{m-n} \times a^n = a^{(m-n)+n} = a^m$$

$$\therefore a^m \div a^n = a^{m-n}.$$

Thus, $4^5 \div 4^3 = 4^{5-3} = 4^2$.

$$(iii) \quad (a^m)^n = a^{m.n}.$$

Proof. $(a^m)^n = a^m.a^m.a^m.\dots$ to n factors

$$= a^{m+m+m \dots \text{to } n \text{ terms}} = a^{mn}.$$

Thus, $(3^4)^2 = 3^{4 \times 2} = 3^8$, $(3^2)^4 = 3^{2 \times 4} = 3^8$ etc.

$$(iv) (ab)^m = a^m b^m$$

Proof. $(ab)^m = ab.ab.ab \dots$ to m factors

$$= (a.a.a \dots \text{to } m \text{ factors}) \times (b.b.b \dots \text{to } m \text{ factors})$$

$$= a^m \times b^m$$

$$= a^m b^m$$

Thus $(2x)^3 = 2^3 x^3 = 8x^3$ etc.

3. We assume that the fundamental Index Law is true for all values of m and n .

4. (i) **Meaning of a^0** ($a \neq 0$)

Since $a^m \times a^n = a^{m+n}$ holds for all values of m and n , putting $m=0$, we have

$$a^0 \times a^n = a^{0+n} = a^n$$

Dividing both sides by a^n , $a \neq 0$,

$$a^0 = \frac{a^n}{a^n} = 1.$$

Thus, $1^0 = 2^0 = 3^0 = 10^0 = 100^0 = \dots = 1$.

(ii) **Meaning of a^{-m}** , where m is a positive integer.

Since $a^{-m} \times a^m \times a^n = a^{-m+m+n} = a^n$,

$$\therefore a^{-m} \times a^m = \frac{a^n}{a^n} = 1$$

$$\therefore a^{-m} = \frac{1}{a^m}, \quad a^m = \frac{1}{a^{-m}}.$$

Therefore, a^{-m} is the reciprocal of a^m and a^m is the reciprocal of a^{-m} .

Thus $2^{-3} = \frac{1}{2^3}$ and $2^3 = \frac{1}{2^{-3}}$ etc.

(iii) **Meaning of $a^{p/q}$** , where p and q are any two positive integers.

Since $a^m \times a^n = a^{m+n}$ holds true for all values of m and n , putting $m=n=p/q$, we have,

$$a^{p/q} \times a^{p/q} = a^{p/q+p/q} = a^{2p/q}$$

Similarly, $a^{p/q} \times a^{p/q} \times a^{p/q} \times \dots$ to q factors

$$= a^{(p/q) \times q} = a^p.$$

$$\therefore (a^{p/q})^q = a^p.$$

$$\therefore a^{p/q} = \sqrt[q]{a^p}$$

Therefore, $a^{p/q}$ represents the q th root of the p th power of a . In a like manner, since $a^{p/q} = (\sqrt[q]{a})^p$, $a^{p/q}$ represents the p th power of the q th root of a .

Note. Both these interpretations do not *always* yield the same result. For example, $\sqrt{4^2} = \sqrt{16} = \pm 4$,

$$\text{but } (\sqrt{4})^2 = (\pm 2)^2 = 4.$$

Ex. 1. Evaluate $16^{-\frac{3}{4}} \times 8^{\frac{4}{3}} \div 4^{\frac{1}{2}}$

$$\begin{aligned} \text{The given exp.} &= (2^4)^{-\frac{3}{4}} \times (2^3)^{\frac{4}{3}} \div (2^2)^{\frac{1}{2}} \\ &= 2^{4 \times -\frac{3}{4}} \times 2^{3 \times \frac{4}{3}} \div 2^{2 \times \frac{1}{2}} \\ &= 2^{-3} \times 2^4 \div 2^1 \\ &= 2^{-3+4-1} = 2^0 = \sqrt{2} \end{aligned}$$

Ex. 2. Show that $\left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a} \times \left(\frac{x^a}{x^b}\right)^{a+b} = 1$

$$\begin{aligned} \text{L.H.S.} &= (x^{b-c})^{(b+c)} \times (x^{c-a})^{(c+a)} \times (x^{a-b})^{(a+b)} \\ &= x^{(b-c)(b+c)} \times x^{(c-a)(c+a)} \times x^{(a-b)(a+b)} \\ &= x^{b^2-c^2} \times x^{c^2-a^2} \times x^{a^2-b^2} \\ &= x^{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)} = x^0 = 1 \end{aligned}$$

Ex. 3. Divide $a+b$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

$$\begin{aligned} \text{We have } a+b &= \left(a^{\frac{1}{3}}\right)^3 + \left(b^{\frac{1}{3}}\right)^3 \\ &= \left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right) \left\{ \left(a^{\frac{1}{3}}\right)^2 - a^{\frac{1}{3}}b^{\frac{1}{3}} + \left(b^{\frac{1}{3}}\right)^2 \right\} \\ &\quad [a^3 + b^3 = (a+b)(a^2 - ab + b^2)] \\ &= \left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right) \left(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right) \end{aligned}$$

$\therefore$ The required quotient $= a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

Ex. 4. Show that $\left\{ \frac{(9^{n+\frac{1}{2}})\sqrt{3 \cdot 3^n}}{3\sqrt{3^{-n}}} \right\}^{1/n} = 27$

$$\begin{aligned}
 \text{L. H. S.} &= \left\{ \frac{(3^2)^{n+\frac{1}{2}} \cdot (3^{n+1})^{\frac{1}{2}}}{3 \cdot (3^{-n})^{\frac{1}{2}}} \right\}^{1/n} \\
 &= \left\{ \frac{3^{2n+\frac{1}{2}} \cdot 3^{(n+1)/2}}{3 \cdot 3^{-n/2}} \right\}^{1/n} \\
 &= \left\{ \frac{3^{(5n/2)+1}}{3^{(1-n/2)}} \right\}^{1/n} \\
 &= \left\{ 3^{(5n/2)+1-n/2} \right\}^{1/n} \\
 &= 3^{(3n \times 1/n)} = 3^3 = 27
 \end{aligned}$$

Ex. 5. Simplify

$$\frac{1}{1+x^{m-n}+x^{m-p}} + \frac{1}{1+x^{n-p}+x^{n-m}} + \frac{1}{1+x^{p-m}+x^{p-n}}$$

(C. U. 1942)

The given exp.

$$\begin{aligned}
 &= \frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}} + \frac{x^{-n}}{x^{-n}+x^{-p}+x^{-m}} + \frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}} \\
 &= \frac{x^{-m}+x^{-n}+x^{-p}}{x^{-m}+x^{-n}+x^{-p}} = 1.
 \end{aligned}$$

Ex. 6. If $a^x=b$, $b^y=c$, $c^z=a$, prove that $xyz=1$.

$$\begin{aligned}
 a^{xyz} &= (a^x)^{yz} = b^{yz} = (b^y)^z = c^z = a^1 \\
 \therefore xyz &= 1.
 \end{aligned}$$

Ex. 7. If $x=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$, prove that $3x^3-9x=10$.

Cubing both sides,

$$\begin{aligned}
 x^3 &= (3^{\frac{1}{3}})^3 + (3^{-\frac{1}{3}})^3 + 3 \cdot 3^{\frac{1}{3}} \cdot 3^{-\frac{1}{3}} \cdot x \\
 &\quad [a^3+b^3+3ab(a+b)=(a+b)^3] \\
 &= 3 + 3^{-1} + 3 \cdot 3^0 \cdot x \\
 &= 3 + \frac{1}{3} + 3x \quad [a^0=1, a \neq 0] \\
 \therefore \text{by transposition } 3x^3-9x &= 10.
 \end{aligned}$$

Exercise 1.

1. Simplify (i) $32^{\frac{3}{5}}$ (ii) $125^{-\frac{2}{3}}$

2. Evaluate—

(i) $(1+a^{-1}+a^{-2})(1-a^{-1})$ (ii) $\sqrt{x^{-1}y} \times \sqrt{y^{-1}z} \times \sqrt{z^{-1}x}$

(iii) $2a\sqrt{x^{2a}} \div 4a\sqrt{x^a}$

3. Simplify:—

(i) $\frac{x^{m+2n} \cdot x^{3m-8n}}{x^{5m-6n}}$

(ii) $\frac{5^{2x+3} \cdot 10^{4x+1}}{2^{53x+2} \cdot 16^{x-1/2}}$

(iii) $\frac{7^{n+2} - 35 \cdot 7^{n-1}}{7^n \times 11}$

4. Find the value of:—

(i) $(a^a)^{b-c} \cdot (a^b)^{c-a} \cdot (a^c)^{a-b}$

(ii) $b^c \sqrt{x^{b-c}} \cdot c^a \sqrt{x^{c-a}} \cdot a^b \sqrt{x^{a-b}}$ (R. U. 1953)

(iii) $(x^{b-c})^{b^2+bc+c^2} \cdot (x^{c-a})^{c^2+ca+a^2} \cdot (x^{a-b})^{a^2+ab+b^2}$

(iv) $\sqrt{\frac{bc \cdot x^{b/c}}{x^{c/b}}} \times \sqrt{\frac{ca \cdot x^{c/a}}{x^{a/c}}} \times \sqrt{\frac{ab \cdot x^{a/b}}{x^{b/a}}}$ (C. U. 1938)

5. Show that

$$\frac{1}{1+x^{b-a}+x^{c-a}} + \frac{1}{1+x^{a-b}+x^{c-b}} + \frac{1}{1+x^{a-c}+x^{b-c}} = 1.$$

(C. U. 1940)

6. (i) If $a=b^x$, $b=c^y$ and $c=a^z$ show that $xyz=1$.

(ii) If $a^x=m$, $a^y=n$ and $a^z=(m^y n^x)^z$, prove that $xyz=1$.

7. If $a^x=b^y=c^z$ and $b^2=ac$, prove that

$$\frac{1}{x} + \frac{1}{z} = \frac{2}{y}.$$

(C. U. 1944)

8. If $xy^{p-1}=a$, $xy^{q-1}=b$, $xy^{r-1}=c$, show that

$$a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1.$$

• 9. Evaluate

$$\frac{\left(p + \frac{1}{q}\right)^m \left(p - \frac{1}{q}\right)^m}{\left(q + \frac{1}{p}\right)^m \left(q - \frac{1}{p}\right)^m}$$

• 10. (i) If $x = 2^{\frac{1}{3}} + 2^{-\frac{1}{3}}$, show that $2x^3 - 6x - 5 = 0$.

(ii) If $x^3 + 2 = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, show that $3x^3 + 9x = 8$.

11. If $x^y = y^x$ show that $\left(\frac{x}{y}\right)^{x/y} = x^{(x/y)^{-1}}$,

and if $x = 2^y$, show that $y = 2$.

• 12. Show that $345^3 + 543^3$ is divisible by 888.

• 13. Find x

• (i) $3^{x-3} = 9^{2-x}$

(ii) $9 \cdot 81^x = \frac{1}{27^{x-3}}$ find x .

• 14. Find x and y

• (i) $2^{2x+1} \cdot 2^{3y+1} = 8$

$2^{x+2} \cdot 2^{y+2} = 16$

(ii) $2^{x-1} \cdot 3^{y+1} = \frac{1}{1^{\frac{1}{3}}}$

$2^{3x+1} \cdot 3^{2y+2} = \frac{1}{3^{\frac{1}{8}}}$

Chapter II

Quadratic Equations

1. Definition. Two algebraic expressions connected by the sign of equality ($=$) is called an equation. The two expressions are called the sides or members of the equation.

If in an equation the highest power of the unknown quantity is two, the equation is called an **equation of the second degree or quadratic equation**.

The unknown quantity is usually denoted by x, y, z etc and the other quantities are assumed to be known and denoted by a, b, c etc.

The most *general form* of a quadratic equation is $ax^2+bx+c=0$, where a, b, c are any numbers and $a \neq 0$.

If an equation contains only x^2 and no x , it is called a *pure quadratic equation*.

If an equation contains both x^2 and x , it is called an *adfectad quadratic equation*.

Thus, $ax^2=c$, $2x^2=16$...are pure quadratic equations and $ax^2+bx+c=0$, $4x^2-5x+1=0$, $3x^2+9x=0$, are adfectad quadratic equations.

2. Definition. Any value of the unknown which satisfies the equation is called a *solution* of the equation. A value is said to satisfy an equation if it makes both the sides equal.

Thus, $x=5$ satisfies the equation $x^2=25$. So $x=5$ is a solution of the equation $x^2=25$. A solution is also called a *root* of the equation.

A quadratic equation has *two roots*.

Generally, an equation of the n th degree has n roots.

3. Methods of Solution

There are two methods of solving a quadratic equation :

- (i) Method of factorisation,
- (ii) Method of completing a perfect square.

(i) **Method of factorisation.** This method is used where the quadratic expression can easily be resolved into linear factors.

Ex. 1. Solve $4x^2=25$

We have, by transposition, $4x^2-25=0$

$$\text{or,} \quad (2x)^2-5^2=0$$

$$\text{or,} \quad (2x+5)(2x-5)=0$$

$$\therefore \text{ either } 2x+5=0, \text{ whence } x=-\frac{5}{2}$$

$$\text{or,} \quad 2x-5=0, \text{ whence } x=\frac{5}{2}$$

$$\therefore \text{ The roots are } -\frac{5}{2}, \frac{5}{2}.$$

Ex. 2. Solve $x^2-6x+8=0$.

We have, $x^2-(4+2)x+8=0$

$$\text{or,} \quad x(x-4)-2(x-4)=0$$

$$\text{or,} \quad (x-4)(x-2)=0$$

$$\therefore \text{ either, } x-4=0, \text{ whence } x=4$$

$$\text{or,} \quad x-2=0, \text{ whence } x=2. \therefore x=4, \text{ or } 2.$$

Ex. 3. Solve $\frac{x}{b} + \frac{b}{x} = \frac{a}{b} + \frac{b}{a}$

By transposition, $\frac{x}{b} - \frac{a}{b} = \frac{b}{a} - \frac{b}{x}$

$$\text{or} \quad \frac{x-a}{b} = \frac{b(x-a)}{ax}$$

$$\therefore \text{ either } x-a=0, \text{ whence } x=a$$

$$\text{or} \quad \frac{1}{b} = \frac{b}{ax}, \text{ whence } ax=b^2 \text{ or } x=\frac{b^2}{a}.$$

Thus the roots are $a, \frac{b^2}{a}$.

Note. $x-a$ is a factor common to both sides and gives a solution.

✓ **Ex. 4.** Solve $\frac{9x-2}{3} + \frac{4x^2-7}{4x^2+3} = \frac{6x-1}{2}$.

By transposition,

$$\frac{4x^2-7}{4x^2+3} = \frac{6x-1}{2} - \frac{9x-2}{3} = \frac{1}{6}$$

$$\begin{aligned}\therefore \quad & 6(4x^2 - 7) = 4x^2 + 3 \\ \therefore \quad & 24x^2 - 4x^2 = 3 + 42 \\ \text{or} \quad & 4x^2 = 9 \\ \therefore \quad & x^2 = \frac{9}{4} \\ \therefore \quad & x = \pm \frac{3}{2}\end{aligned}$$

Note. The extraction of square root gives two values equal in magnitude but opposite in sign.

The roots of $x^2 = 4$ are $+2$ and -2 , for these two values satisfy the equation.

(ii) Method of completing square

Ex. 5. Solve $x^2 - 5x + 4 = 0$

we have $x^2 - 2 \cdot x \cdot \frac{5}{2} = -4$

Adding $(\frac{5}{2})^2$ to both sides,

$$x^2 - 2 \cdot x \cdot \frac{5}{2} + (\frac{5}{2})^2 = (\frac{5}{2})^2 - 4$$

$$\text{or} \quad (x - \frac{5}{2})^2 = \frac{9}{4}.$$

Extracting square root of both sides, we have

$$x - \frac{5}{2} = \pm \frac{3}{2};$$

$$\therefore \quad x = \frac{5}{2} \pm \frac{3}{2} = \frac{5+3}{2} \text{ or } \frac{5-3}{2} = 4 \text{ or } 1$$

General method of completing the square

Let the general quadratic equation be $ax^2 + bx + c = 0$.

By transposition, $ax^2 + bx = -c$

Dividing both sides by a , the co-efficient of x^2 , we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $(\frac{b}{2a})^2$ to both sides,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\text{or} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Extracting square root of both sides, we have

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$\therefore$ the two roots of $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The following method of completing the square is due to the great Hindu Mathematician Sridhar Acharyya.

Since $ax^2 + bx + c = 0$, we have, by transposition

$$ax^2 + bx = -c$$

Multiplying both sides by $4a$, i.e., 4 times the coefficient of x^2 , we have

$$4a^2x^2 + 4abx = -4ac$$

Adding b^2 , i.e. square of the coefficient of x , to both sides,

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$\text{or } (2ax + b)^2 = b^2 - 4ac$$

$$\therefore \text{ extracting square root, } 2ax + b = \pm \sqrt{b^2 - 4ac}$$

$$\text{or } 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex. 6. Solve $3x^2 - 10x + 3 = 0$

By transposition, $3x^2 - 10x = -3$

Multiplying both sides by 4×3 , we have

$$36x^2 - 120x = -36$$

Adding 10^2 to both sides

$$36x^2 - 120x + 10^2 = 10^2 - 36$$

$$\text{or } (6x - 10)^2 = 64$$

$\therefore$ extracting square root, $6x - 10 = \pm 8$

$$\text{or } 6x = 10 \pm 8 = 18 \text{ or } 2$$

$$\therefore x = 3 \text{ or } \frac{1}{3}$$

or, applying the formula, $a=3$, $b=-10$, $c=3$.

$$\begin{aligned} \therefore x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \cdot 3 \cdot 3}}{2 \times 3} = \frac{10 \pm \sqrt{100 - 36}}{6} \\ &= \frac{10 \pm 8}{6} = 3 \text{ or } \frac{1}{3} \end{aligned}$$

4. Equations reducible to quadratic forms by suitable substitutions.

(i) *Algebraic equations of higher degree than two.*

Ex. 7. Solve $x^{10} - 33x^5 + 32 = 0$

Putting $y = x^5$, the given equation reduces to

$$y^2 - 33y + 32 = 0$$

or $y^2 - (32+1)y + 32 = 0$

or $y(y-32) - 1(y-32) = 0$

or $(y-32)(y-1) = 0$

$\therefore$ either $y-32=0$, whence $y=32$, i.e. $x^5=32$, whence $x=2$

or $y-1=0$, whence $y=1$, i.e. $x^5=1$, whence $x=1$

Ex. 8. Solve $(x-1)(x-2)(x+3)(x+4)=36$

Multiplying together the 1st and 3rd factors, and the 2nd and 4th. factors of L. H. S, we have

$$(x^2+2x-3)(x^2+2x-8)=36$$

Putting $y = x^2 + 2x$, we have

$$(y-3)(y-8)=36$$

or $y^2 - 11y - 12 = 0$

or $(y-12)(y+1) = 0 \quad \therefore y = -1 \text{ or } 12$

(i) Taking $y = -1$, $x^2 + 2x = -1$

or $(x+1)^2 = 0$, whence $x = -1, -1$

(ii) Taking $y = 12$, $x^2 + 2x = 12$

or $x = \frac{-2 \pm \sqrt{4+48}}{2} = -1 \pm \sqrt{13}$

(ii) *Algebraic equations of fractional index.*

Ex. 9. Solve $x^{\frac{3}{5}} - 5x^{\frac{1}{5}} + 6 = 0$

Putting $y = x^{\frac{1}{5}}$ the equation reduces to—

$$y^3 - 5y + 6 = 0$$

or $y^3 - (3+2)y + 6 = 0$

or $y(y-3) - 2(y-3) = 0$

or $(y-3)(y-2) = 0$

∴ either $y-3=0$, i.e. $x^{\frac{1}{5}}=3$, whence $x=3^5=243$

or $y-2=0$, i.e. $x^{\frac{1}{5}}=2$, whence $x=2^5=32$.

(iii) *Irrational equations reducible to quadratics*

Ex. 10. Solve $\sqrt{2x+1} + \sqrt{3x+4} = 7$

By transposition of one radical,

$$\sqrt{3x+4} = 7 - \sqrt{2x+1}$$

Squaring both sides,

$$2x+1 = 49 + (3x+4) - 14\sqrt{3x+4}$$

or $x+52 = 14\sqrt{3x+4}$

Squaring, $x^2 + 104x + 2704 = 196(3x+4)$

or $x^2 - 484x + 1920 = 0$

or $x^2 - 4x - 480x + 1920 = 0$

or $x(x-4) - 480(x-4) = 0$

or $(x-4)(x-480) = 0$.

∴ either $x=4$, or $x=480$

The root 4 is found to satisfy the given equation, the other root 480 does not satisfy it.

∴ $x=4$ is a root of the given equation, but $x=480$ is not a root.

Note 1. The value $x=480$, which does not satisfy the original equation and, as such, is not a root of the equation is called an *extraneous root*.

Note 2. It is necessary to verify if the solutions thus obtained are really the roots of the equation.

Note 3. In extracting the square root, take only the *positive value* of the square root *while verifying*.

Ex. 11. Solve $\sqrt{2x+1} - \sqrt{3x+4} = 7$

Proceeding as before, $2x+1=49+(3x+4)+14\sqrt{3x+4}$

or $x+52=-14\sqrt{3x+4}$

Squaring, $x^2+104x+2704=196(3x+4)$

whence, as before, $x=4$ or 480 .

None of them satisfies the given equation.

$\therefore$ The equation has no root.

Ex. 12. Solve $\sqrt{2x^2-3x+16} + \sqrt{2x^2-3x-4} = 10 \dots (1)$

we have *identically*,

$$(2x^2-3x+16) - (2x^2-3x-4) = 20 \dots \dots (2)$$

Dividing (2) by (1) and noting that

$a^2-b^2=(a+b)(a-b)$, we have

$$\sqrt{2x^2-3x+16} - \sqrt{2x^2-3x-4} = 2 \dots \dots (3)$$

Adding (1) and (3),

$$2\sqrt{2x^2-3x+16} = 12$$

$$\therefore \sqrt{2x^2-3x+16} = 6$$

Squaring, $2x^2-3x+16=36$

or $2x^2-3x-20=0$

or $2x^2-8x+5x-20=0$

or $2x(x-4)+5(x-4)=0$

or $(x-4)(2x+5)=0$. Hence, $x=4$ or $-\frac{5}{2}$.

5. Reciprocal equations

Ex. 13. Solve $x^4-2x^3-6x^2-2x+1=0$

Rearranging terms, we have

$$(x^4+1)-2(x^3+x)-6x^2=0$$

Dividing both sides by x^2 ,

$$\left(x^2 + \frac{1}{x^2}\right) - 2\left(x + \frac{1}{x}\right) - 6 = 0$$

$$\text{or} \quad \left\{\left(x + \frac{1}{x}\right)^2 - 2\right\} - 2\left(x + \frac{1}{x}\right) - 6 = 0$$

$$\text{or} \quad \left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) - 8 = 0$$

Putting y for $x + \frac{1}{x}$, we have

$$y^2 - 2y - 8 = 0, \text{ or } (y - 4)(y + 2) = 0$$

$$\therefore y = 4 \text{ or } -2;$$

$$(i) \text{ When } y = 4, x + \frac{1}{x} = 4, \text{ or } x^2 - 4x + 1 = 0$$

$$\therefore x = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm 2\sqrt{3}}{2}, = 2 \pm \sqrt{3}.$$

$$(ii) \text{ When } y = -2, x + \frac{1}{x} = -2, \text{ or } x^2 + 2x + 1 = 0,$$

$$\text{or} \quad (x + 1)^2 = 0$$

$$\therefore x = -1, -1.$$

$$\therefore \text{The roots are } 2 \pm \sqrt{3}, -1, -1.$$

6. Exponential equations

Ex. 14. Solve $3^{2x} - 4 \cdot 3^{x+1} + 27 = 0$.

Putting y for 3^x , we have

$$y^2 - 12y + 27 = 0 \quad [3^{2x} = (3^x)^2 = y^2, 3^{x+1} = 3^x \cdot 3 = 3y]$$

$$\text{or} \quad (y - 3)(y - 9) = 0$$

$$\therefore y = 3 \text{ or } 9.$$

$$\therefore 3^x = 3 \text{ or } 3^x = 9 = 3^2.$$

$$\therefore x = 1 \text{ or } 2.$$

Exercise 2.

Solve

$$1. \quad (i) \quad 6\frac{1}{3} - \frac{x-7}{3} = \frac{4x-2}{5} \quad (ii) \quad \frac{3}{x-2} + \frac{5}{x-6} = \frac{8}{x+3}$$

$$(iii) \quad 4x - \frac{x-1}{3} = x + \frac{2(x-1)}{5} + 3$$

$$\checkmark (iv) \quad \frac{a-x}{a} + \frac{2a-x}{2a} = \frac{3a-x}{3a}$$

$$(v) \quad \frac{x-bc}{b+c} + \frac{x-ca}{c+a} + \frac{x-ab}{a+b} = a+b+c$$

$$2. \quad (i) \quad 4x^2=9. \quad (ii) \quad \frac{x}{x+2} = \frac{x+3}{5(x+11)}$$

$$\checkmark (iii) \quad \frac{2(45+2x^2)}{x^2+9} + \frac{3(x^2-9)}{x^2+3} = 7$$

$$3. \quad (i) \quad x^2-10x+9=0 \quad (ii) \quad 3x^2-14x+11=0$$

$$\checkmark (iii) \quad x^2-(a+b)x+ab=0 \quad (iv) \quad \frac{x}{3} + \frac{3}{x} = \frac{10}{3}$$

$$\checkmark (v) \quad \frac{x}{a} + \frac{a}{x} = \frac{a}{b} + \frac{b}{a}$$

$$4. \quad \checkmark (i) \quad 2x^2+3x=6. \quad \checkmark (ii) \quad 3x^2-7x+3=0.$$

$$5. \quad (i) \quad x^2-2\sqrt{3}x+1=0,$$

$$(ii) \quad x^2-(\sqrt{3}+3)x+\sqrt{3}+2=0$$

$$6. \quad (i) \quad \frac{x+2}{x-2} + \frac{x+3}{x-3} = \frac{x-2}{x+2} + \frac{x-3}{x+3}$$

$$(ii) \quad \frac{1}{x+6} + \frac{1}{x+12} + \frac{1}{x+18} = \frac{3}{x}$$

$$(iii) \quad \frac{x-a^2-b^2}{c^2} + \frac{c^2}{x-a^2-b^2} = 2$$

$$7. \quad (i) \quad x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 15$$

$$(ii) \quad x^{\frac{2}{3}} + 12 = 7x^{\frac{1}{3}}$$

$$(iii) \quad x^{\frac{1}{3}} - 9x^{\frac{1}{3}} + 18 = 0$$

8. $x^{-4} + 4 = 5x^{-2}$
9. $6x^{\frac{3}{4}} + 3x^{-\frac{1}{4}} = 11x^{\frac{1}{4}}$
10. (i) $(7-x)(5-x)(6+x)(4+x) = 504$
 (ii) $(x-1)(x+5)(x+8)(x+2) = 880$ (P. U. 1946)
11. $(x^2 - 3x)^2 - 8(x^2 - 3x) - 20 = 0$
12. (i) $\sqrt{(6x-5)} - \sqrt{(3x-2)} = 2$ (C. U. 1954)
 (ii) $\sqrt{(3x+10)} + \sqrt{(9x+7)} = 9$ (C. U. 1950)
13. $2\sqrt{(x+5)} - \sqrt{(2x+8)} = 2$
14. $\sqrt{(x+6)} - \sqrt{(x+3)} = \sqrt{(2x+5)}$
15. $\sqrt{(x+5)} + \sqrt{(x+21)} = \sqrt{(6x+40)}$
16. $x^2 + 2\sqrt{x^2 + 16x} = 24 - 16x$ (C. U. 1953)
17. $3x^2 - 18 + \sqrt{3x^2 - 4x - 6} = 4x$
18. (i) $10x^4 + 63x^3 + 52x^2 - 63x + 10 = 0$
 (ii) $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$ (P. U. 1954)
19. (i) $\left(x - \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 12 = 0$
 (ii) $\left(x - \frac{1}{x}\right)^2 - 10\left(x - \frac{1}{x}\right) + 24 = 0$
20. $5^x + 5^{-x} = 5\frac{1}{5}$
21. $3^{2x} + 3^2 = 10 \cdot 3^x$
22. $2^{2x+3} = 57 + 65(2^x - 1)$
23. $9^{1+x} + 9^{1-x} = 82$

Chapter III

Simultaneous Quadratic Equations

(Two unknowns)

7. One linear equation, another quadratic

The general method of solution consists in

- (i) expressing one unknown, say x , in terms of another, say y , from the linear equation.
- (ii) substituting the value of y in the quadratic equation and obtaining values of y , and then
- (iii) finding corresponding values of x .

Ex. 15. Solve $x^2 + y^2 = 100$(1)

$$x + y = 14$$
.....(2)

From (2), $x = 14 - y$(3)

Substituting this value of x in (1), we have

$$(14 - y)^2 + y^2 = 100$$

$$\text{or } y^2 - 14y + 48 = 0, \text{ or } (y - 8)(y - 6) = 0,$$

$$\text{whence } y = 8 \text{ or } 6.$$

$$\therefore \text{ from (3) } x = 6 \text{ or } 8.$$

$$\therefore \text{ the roots are } x = 6, y = 8 \text{ or } x = 8, y = 6.$$

Ex. 16. Solve $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}$ (1)

$$x + y = 10$$
..... (2) (C.U. 1958)

On simplification, (1) reduces to

$$\frac{x + y}{\sqrt{xy}} = \frac{5}{2}$$

$$\text{or } \frac{10}{\sqrt{xy}} = \frac{5}{2} \text{ from (2), } \therefore \sqrt{xy} = 4 \text{ and } xy = 16$$
.....(3)

From (2) $x=10-y$...(4)

Substituting this value of x in (3), we have

$$(10-y)y=16$$

$$\text{or } y^2-10y+16=0, \text{ or } (y-8)(y-2)=0,$$

$$\text{whence } y=8 \text{ or } 2.$$

$$\therefore \text{ from (4), } x=2 \text{ or } 8.$$

$$\therefore \text{ the roots are } x=2, y=8 \text{ or } x=8, y=2.$$

Ex. 17. Solve $\frac{1}{x^2} + \frac{1}{y^2} = 13$

$$\frac{1}{x} + \frac{1}{y} = 5$$

Put $u = \frac{1}{x}, v = \frac{1}{y}$ then the equations reduce to

$$u^2 + v^2 = 13 \dots (1)$$

$$u + v = 5 \dots (2)$$

From (2), $u = 5 - v \dots (3)$

Substituting this value of u in (1), we have

$$(5-v)^2 + v^2 = 13$$

$$\text{or } v^2 - 5v + 6 = 0, \text{ or } (v-2)(v-3) = 0, \text{ whence } v=2 \text{ or } 3$$

$$\therefore \text{ from (3) } u=3 \text{ or } 2.$$

$$\therefore u=3, v=2 \text{ or } u=2, v=3.$$

$$\therefore \text{ the roots are } x=\frac{1}{3}, y=\frac{1}{2} \text{ or } x=\frac{1}{2}, y=\frac{1}{3}.$$

8. Both equations quadratic

(a) If the equations are *homogeneous* and of the second degree, i.e., if the sum of the indices of x and y in each term is 2, they may be solved by putting $y=mx$ as follows :—

Ex. 18. Solve $x^2 + xy + y^2 = 19$.

$$3xy + 2y^2 = 36.$$

Putting $y=mx$ in the equations, we have

$$x^2(1+m+m^2) = 19 \dots (1)$$

$$x^2(3m+2m^2) = 36 \dots (2)$$

Dividing (1) by (2), $\frac{1+m+n^2}{3m+2m^2} = \frac{19}{36}$

$$\therefore 57m + 38m^2 - 36 + 36m + 36m^2$$

$$\text{or } 2m^2 + 21m - 36 = 0.$$

$$\text{or } (m+12)(2m-3) = 0 \quad \therefore m = -12 \text{ or } \frac{3}{2}$$

(i) taking $m = -12$, from (1)

$$x^2 \times 133 = 19 \quad \therefore x = \pm \frac{1}{\sqrt{7}} = \pm \frac{1}{7} \sqrt{7}$$

$$\therefore y = -12 \times \pm \frac{1}{\sqrt{7}} = \mp \frac{12}{\sqrt{7}} = \mp \frac{12}{7} \sqrt{7}$$

(ii) taking $m = \frac{3}{2}$, from (1)

$$x^2 \times \frac{19}{4} = 19 \quad \therefore x = \pm 2$$

$$\therefore y = \frac{3}{2} \times \pm 2 = \pm 3$$

Thus the roots are

$$\left. \begin{array}{l} x = \frac{1}{7} \sqrt{7} \\ y = -\frac{12}{7} \sqrt{7} \end{array} \right\} \quad \left. \begin{array}{l} x = -\frac{1}{7} \sqrt{7} \\ y = \frac{12}{7} \sqrt{7} \end{array} \right\} \quad \left. \begin{array}{l} x = 2 \\ y = 3 \end{array} \right\} \quad \left. \begin{array}{l} x = -2 \\ y = -3 \end{array} \right\}$$

(b) If the equations are *symmetrical* with respect to x and y , i.e., if the equations remain the same if x is put for y and y is put for x , they may be solved by putting $x = u + v$ and $y = u - v$.

Ex. 19. Solve $\frac{x}{y} + \frac{y}{x} = 3 \dots (1)$

$$x + y = 4 \dots (2)$$

Put $x = u + v$ and $y = u - v$.

$$\therefore \text{ from (2), } (u+v) + (u-v) = 4 \text{ or } u = 2.$$

$$\text{From (1), } \frac{(u+v)^2 + (u-v)^2}{u^2 - v^2} = 10$$

$$\text{or } 3(2u^2 + 2v^2) = 10(u^2 - v^2)$$

$$16v^2 = 4u^2 \quad \text{or } 4v^2 = u^2 = 4$$

$$\therefore v = \pm 1.$$

$$\therefore x = u + v = 2 \pm 1 = 3 \text{ or } 1$$

$$y = u - v = 2 \mp 1 = 1 \text{ or } 3.$$

9. Exponential equation

The method of solution consists of

(i) reducing both members of the equations to the same base

and (ii) equating their indices.

Ex. 20. Solve $9^x = 3^y$... (1)

$$4^{y+1} = 8^{xy} \dots (2)$$

$$\text{From (1), } (3^2)^x = 3^y \text{ or } 3^{2x} = 3^y \quad \therefore 2x = y \quad \dots (3)$$

$$\text{From (2) } (2^2)^{(y+1)} = (2^3)^{xy} \text{ or } 2^{2y+2} = 2^{3xy} \quad \therefore 2y+2 = 3xy \dots (4)$$

$$\text{From (3) and (4) } 2.2x+2 = 3x.2x$$

$$\text{or } 3x^2 - 2x - 1 = 0$$

$$\text{or } (x-1)(3x+1) = 0, \text{ whence } x = 1 \text{ or } -\frac{1}{3}.$$

$$\therefore \text{ from (3) } y = 2 \times 1 = 2 \text{ or } 2 \times -\frac{1}{3} = -\frac{2}{3}.$$

$$\therefore \text{ the roots are } \left. \begin{array}{l} x = 1 \\ y = 2 \end{array} \right\} \left. \begin{array}{l} x = -\frac{1}{3} \\ y = -\frac{2}{3} \end{array} \right\}$$

Three Unknowns

Excepting the case of simultaneous linear equations, no general rule can be laid down for the solution of quadratic equations involving three variables.

10. Method of cross multiplication

Let $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ be three linear equations in x , y and z .

Then multiplying the 1st. equation by a_2 and 2nd. equation by a_1 , we have

$$a_1a_2x + b_1a_2y + c_1a_2z = 0$$

$$a_2a_1x + b_2a_1y + c_2a_1z = 0$$

$$\text{By subtraction, } (b_1a_2 - b_2a_1)y + (c_1a_2 - c_2a_1)z = 0$$

$$\therefore \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \quad \dots (1)$$

Similarly, multiplying the 1st. equation by b_2 and the 2nd. equation by b_1 , we have by subtraction,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{z}{a_1b_2 - a_2b_1} \quad \dots (2)$$

$\therefore$ from (1) and (2),

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

This is called *the Rule of cross multiplication*.

Let each ratio be denoted by k , then

$$x = k(b_1c_2 - b_2c_1), \quad y = k(c_1a_2 - c_2a_1), \quad z = k(a_1b_2 - a_2b_1).$$

Substituting these values of x , y and z in the third equation, we can find the value of k and, therefore, those of x , y and z .

II. If in the above two equations we put $z=1$, then

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

which give solution of the equations

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0$$

Ex. 21. Solve $3x + 4y = 18$, $2x - y = 1$

By transposition, $3x + 4y - 18 = 0$

$$2x - y - 1 = 0$$

By the method of cross multiplication,

$$4.(-1) - (-1).(-18) = (-18).2 - (-1).3 = 3(-1) - 2.4$$

$$\text{or} \quad \frac{x}{-4 - 18} = \frac{y}{-36 + 3} = \frac{1}{-3 - 8}$$

$$\text{or} \quad \frac{x}{2} = \frac{y}{3} = 1$$

$$\therefore \quad x=2, \quad y=3.$$

Ex. 22. Solve $3x - 4y - z = 0$... (1)

$$2x - 5y + 4z = 0 \quad \dots (2)$$

$$x^2 + y^2 + z^2 + z = 15 \quad \dots (3)$$

From (1) and (2) by cross multiplication,

$$\frac{x}{-16-5} = \frac{y}{-2-12} = \frac{z}{-15+8}$$

or $\frac{x}{-21} = \frac{y}{-14} = \frac{z}{-7}$

or $\frac{x}{3} = \frac{y}{2} = \frac{z}{1} = k$. (say)

$\therefore x=3k, y=2k, z=k$.

Substituting in (3), we have

$$9k^2 + 4k^2 + k^2 + k = 15$$

or $14k^2 + k = 15$ or $(k-1)(14k+15) = 0$

$\therefore k=1$ or $-\frac{15}{14}$.

$\therefore x=3, y=2, z=1$

or $x=-\frac{45}{14}, y=-\frac{15}{7}, z=-\frac{15}{14}$.

Ex. 23. Solve $x+y+3z=5$... (1)

$y+z-3x=-11$... (2)

$z+x-3y=-3$... (3)

Adding together, $-(x+y+z)=-9 \therefore x+y+z=9$ (4)

Subtracting (4) from (1), $-4z=-4 \therefore z=1$

Subtracting (4) from (2), $-4x=-20 \therefore x=5$

Subtracting (4) from (3), $-4y=-12 \therefore y=3$.

Ex. 24. Solve $xy+2x+2y=52$

$yz+2y+2z=38$

$zx+2z+2x=44$

Adding 4 to both sides,

$(y+2)(y+2)=56$... (1)

$(y+2)(z+2)=42$... (2)

$(z+2)(x+2)=48$... (3)

Multiplying together (1), (2) and (3)

$$(x+2)^2(y+2)^2(z+2)^2=56 \times 42 \times 48 = (6 \times 7 \times 8)^2$$

$\therefore (x+2)(y+2)(z+2) = \pm 6 \times 7 \times 8$... (4)

Dividing (4) by (1), (2) and (3) in succession, we have

$z+2 = \pm 6 \therefore z = -2 \pm 6 = 4 \text{ or } -8$

$x+2 = \pm 8 \therefore x = -2 \pm 8 = 6 \text{ or } -10$

$y+2 = \pm 7 \therefore y = -2 \pm 7 = 5 \text{ or } -9$

Ex. 25. Solve $x+y=z^{-1}$
 $y+z=x^{-1}$
 $z+x=y^{-2}$

Since $z^{-1}=1/z$, $x^{-1}=1/x$, $y^{-1}=1/y$, the equations can be put in the form

$$z(x+y)=1 \quad \text{or} \quad zx+zy=1 \dots (1)$$

$$x(y+z)=1 \quad \text{or} \quad xy+xz=1 \dots (2)$$

$$y(z+x)=1 \quad \text{or} \quad yz+yx=1 \dots (3)$$

Adding (1), (2) and (3), and dividing by 2,

$$xy+yz+zx=\frac{3}{2} \dots (4)$$

Subtracting (1), (2), (3) from (4) in succession,

$$xy=\frac{1}{2}, \quad yz=\frac{1}{2}, \quad zx=\frac{1}{2}$$

Multiplying these last equations together and extracting square root, we have

$$xyz=\pm \frac{1}{2\sqrt{2}}$$

Hence by division, $z=\pm \frac{1}{\sqrt{2}}$, $x=\pm \frac{1}{\sqrt{2}}$, $y=\pm \frac{1}{\sqrt{2}}$.

$\therefore$ the roots are

$$x=y=z=\frac{1}{\sqrt{2}} \quad \text{or} \quad x=y=z=-\frac{1}{\sqrt{2}}$$

Exercise 3

Solve

1. (i) $x^2+y^2=25$, $x+y=7$ (ii) $x^2+y^2=185$, $x-y=3$,

(iii) $3x^2-7xy+4y^2=0$, $4x+3y=5$

2. (i) $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, $x+y=5$ (ii) $\frac{1}{x}+\frac{1}{y}=\frac{7}{12}$, $xy=12$

3. (i) $\frac{x}{y}+\frac{y}{x}=5\frac{1}{5}$, $x+y=6$

(ii) $\frac{x}{2}+\frac{y}{5}=5$, $\frac{2}{x}+\frac{5}{y}=\frac{5}{6}$ (C.U. 1953)

4. (i) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}, \quad x+y=5$
 (ii) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{13}{6}, \quad xy=36$
 (iii) $\sqrt{x} - \sqrt{y} = \sqrt{\frac{8}{3}}, \quad \sqrt{xy}=3 \quad (\text{C.U. 1955})$
5. $x^2+xy+y^2=84, \quad x+y+\sqrt{xy}=14.$
6. $\frac{4}{x+y} + \frac{3}{x-y} = 1, \quad \frac{24}{x+y} + \frac{12}{x-y} = 5$
7. $x+3y=2, \quad x^2+2y^2+3xy=0 \quad (\text{R.U. 1954})$
8. $x+\frac{4}{y}=1, \quad y+\frac{4}{x}=25 \quad (\text{C.U. 1951, 1954})$
9. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4}, \quad \frac{1}{x} + \frac{1}{y} = \frac{3}{2} \quad (\text{C.U. 1959})$
10. $x+y+xy=27, \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{2} \quad (\text{P.U. 1949})$
- 11(i) $x+y=a+b, \quad \frac{a}{x} + \frac{b}{y} = 2 \quad (\text{ii}) \quad \frac{a}{x} + \frac{b}{y} = \frac{a^2}{x^2} + \frac{b^2}{y^2} = 2$
- 12(i) $3y^2+xy+x^2=15, \quad 31xy-3y^2-5x^2=45$
 (ii) $\frac{x^2}{y} + \frac{y^2}{x} = 18, \quad x+v=12.$

Solve the following equations by the rule of cross multiplication.

13. (i) $3x+4y=11, \quad 4x+3y=10$
 (ii) $7x-11y=17, \quad 2x-5y=3$
 (iii) $\frac{1}{3}(x+2y) + \frac{1}{4}(2x-y) = 7\frac{1}{2}$
 $\frac{1}{4}(x-2y) + \frac{1}{3}(2x+y) = 4\frac{1}{2}$
14. Solve (i) $y^x=4, \quad y^2=2^x.$
 (ii) $3^x-2^y=1, \quad 3^{x+1}+2^{y-1}=31.$
 (iii) $2^x+3^y=17, \quad 2^{x+2}-3^{y+1}=5.$

Solve the following equations

15. $3x+4y-5z=0$
 $6x-y+z=0$
 $x+y+z=14.$
16. $9x+y-8z=0$
 $4x-8y+7z=0$
 $yz+zx+xy=47.$
 (R.U. 1951)

17. $3x+y-5z=0$

$7x-3y-9z=0$

$x^2+2y^2+3z^2=23.$

(C. U. 1955)

18. $x+y+z=9$

$x^2+y^2+z^2=29$

$yz+zx=20.$

19. $x^2+y^2+z^2=84$

$x+y+z=14$

$y^2=zx.$

20. $x^2-(y-z)^2=3$

$y^2-(z-x)^2=-9$

$z^2-(x-y)^2=-3$

21. (i) $xy=72, yz=60, zx=30$

(ii) $xy+x+y=23, yz+y+z=27, zx+z+x=41$

C

(C. U. 1945)

22. $x(y+z)=2, y(z+x)=2, z(x+y)=2.$

23. $x(x+y+z)=a^2, y(x+y+z)=b^2, z(x+y+z)=c^2.$

(C. U. 1948)

24. $x^2-yz=5, y^2-zx=3, z^2-xy=-1$

(C. U. 1946)

Equational Problems

Ans P. 224

1. A boy bought a number of oranges for Rs 4. Had he bought one-third more for the same price, he would have paid 8 pies less for each. How many oranges did he buy?

2. A man walked from A to B at the rate of 4 miles per hour. Had he walked at the rate of $3\frac{2}{3}$ miles per hour, he would have taken 3 hours more to reach B. What is the distance between A and B?

3. A labourer is engaged for 30 days on condition that he receives 2s. 6d for each working day, and loses 1s. for each day he is idle. He receives £2. 7s in all. For how many days is he idle?

4. A man walks one-third of the distance from A to B at a m.p.h. and the remainder at $2b$ m.p.h. and travelling back at $3c$ m.p.h. takes the same time.

Prove that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$.

5. A train running from A to B meets an accident 50 miles from A, after which it moves with $\frac{3}{5}$ th. of its original speed and arrives at B three hours late. Had the accident happened 50 miles further on, it would have been only two hours late. Find the original speed of the train and distance from A to B. (C.U.1958)

6. 15 per cent of the sale price of an article is equal to 25 per cent of its cost, and 10 per cent of the sale price exceeds 15 per cent of the cost by Rs 10. Find the cost and sale prices.

7. A man travelled 60 miles in 3 hours partly by train and partly by car. If he had gone all the way by rail he would have arrived 1 hour sooner and saved $\frac{2}{5}$ ths of the time he was in the car. How far did he travel by car? (C. U. 1937)

8. A trader bought an equal number of oranges at 3 for 40 nP and 4 for 50 nP. By selling them at 5 for 70 nP he gained Rs 2·60. Find the total number of oranges.

9. Find two numbers, whose sum multiplied by the difference of their squares is 648, and whose difference multiplied by the sum of their squares is 328.

(Hints : Take $x+y$ and $x-y$ as the two numbers.)

10. Divide the number 26 into three parts such that their squares may have equal differences, and that the sum of those squares may be 300.

11. From a jar of water containing x gallons, y gallons are drawn off and the jar is filled with wine. This process has been repeated four times. Find the quantity of water remaining in the jar.

12. To complete a certain work A requires ' a ' times as long as B and C take together ; B requires ' b ' times as long as A and C together and C requires ' c ' times as long as A and B together. Show that the times taken by them to complete the work separately are in proportion $a+1 : b+1 : c+1$ and

that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$

Chapter IV

Progressions

1. A *series* is a succession of terms formed according to some definite law.

Examples

- (1) 4, 6, 8, 10,.....each term is formed by adding 2 to the preceding term
- (2) 4, 8, 16, 32,.....each term is formed by doubling the preceding term
- (3) 20, 18, 16,.....each term is formed by subtracting 2 from preceding term.

2. A series is said to be in *Arithmetical Progression* (A.P) if the difference between any term and the preceding term is always the same.

The algebraic difference between any term and the preceding term is called the *common difference* (c.d).

Examples

- (i) the series 4, 6, 8, 10,.....are in A.P, the c.d. being 2
- (ii) the series 20, 18, 16,.....are in A.P, the c.d. being -2 .

3. The n th term of an A.P.

Let a be the first term and d the common difference.
Then 1st term $= a$

$$2\text{nd term} = a + d$$

$$3\text{rd term} = (a + d) + d = a + 2d$$

.....

$$\therefore \text{nth term} = \underline{a + (n-1)d}.$$

If there are n terms in the series, the last term

$$l = a + (n-1)d.$$

Ex 1. Find the 5th term of the A.P. 7, 4, 1,.....

Here c.d. $= 4 - 7 = -3$.

$$\therefore 5\text{th term} = 7 + (5-1) \times -3 = 7 - 12 = -5.$$

Ex. 2. The 10th term of an A.P is 40 and the 15th term is 50. Find the series.

Taking a and d as the first term and common difference, we have

$$40 = a + (10-1)d = a + 9d$$

$$50 = a + (15-1)d = a + 14d$$

By subtraction, $-10 = -5d \quad \therefore d = 2.$

$$\therefore 40 = a + 9 \times 2 \quad \text{or} \quad a = 40 - 18 = 22$$

$\therefore$ the series is 22, 24, 26, 28.....

4. Sum of an A.P.

Let a be the first term, d the common difference, n the number of terms and l the last term of an A.P.

Let S denote the sum to n terms, then

$$S = a + (a+d) + \dots + (l-d) + l$$

Writing the series in the reverse order, we have

$$S = l + (l-d) + \dots + (a+d) + a$$

Adding together,

$$\begin{aligned} 2S &= (a+l) + (a+l) + \dots + (a+l) + (a+l) \\ &= (a+l) + (a+l) + \dots \text{to } n \text{ terms} \\ &= n(a+l) \end{aligned}$$

$$\therefore S = \frac{n}{2}(a+l)$$

Putting $l = a + (n-1)d$,

$$S = \frac{n}{2}\{2a + (n-1)d\}$$

Ex. 3. Find the sum of 20 terms of the A.P. 15, 17, 19,...

$$\begin{aligned} S &= \frac{n}{2}\{2a + (n-1)d\} = \frac{20}{2}\{2 \times 15 + 19 \times 2\} \quad d=2. \\ &= 10(30 + 38) \\ &= 680. \end{aligned}$$

or, 20th term $= a + (n-1)d = 15 + (20-1) \times 2 = 53$:

$$S = 15 + 17 + \dots + 51 + 53$$

$$\text{and} \quad S = 53 + 51 + \dots + 17 + 15$$

Adding together,

$$2S = 68 + 68 + \dots \text{to 20 terms} \\ = 68 \times 20$$

$$\therefore S = \frac{68 \times 20}{2} = 680.$$

5. Arithmetic Means

When three quantities are in A.P., the middle one is called *arithmetic mean* (A.M).

If a, b, c , are in A.P., we have

$$b - a = c - b \quad \text{or} \quad 2b = a + c \quad \text{or} \quad b = \frac{a + c}{2}.$$

Hence *arithmetic mean of two numbers is half their sum.*
The *arithmetic mean of n numbers $x_1, x_2, \dots, x_n$ is the sum of the numbers divided by n .*

$$\text{i.e., A. M. of } x_1, x_2, \dots, x_n = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\Sigma x}{n}$$

Find the arithmetic mean of

$$16, 25, 4.7, .62$$

$$\checkmark \text{ Reqd. A.M} = \frac{16 + 25 + 4.7 + .62}{4} = \frac{46.32}{4} = 11.58.$$

Ex. 4. Insert 4 arithmetic means between 28 and 13

The first term is 28 and 6th term is 13.

Let d be the common difference

$$\text{Then } 13 = a + 5d = 28 + 5d$$

$$\therefore 5d = 13 - 28 = -15 \quad \text{or} \quad d = -3.$$

$$\therefore \text{the 2nd term} = 28 + d = 28 - 3 = 25$$

$$\text{3rd term} = 25 + d = 25 - 3 = 22$$

4th term = $22 + d = 22 - 3 = 19$, which are the three* required means.

6. (i) Find the sum of first n natural numbers

Let S denote the sum, then

$$S = 1 + 2 + 3 + \dots + n = \frac{n}{2}(1 + n) = \frac{n(n+1)}{2}$$

(ii) Find the sum of first n odd natural numbers

Let S denote the sum, then

$$S = 1 + 3 + 5 + \dots \text{to } n \text{ terms}$$

$$= \frac{n}{2} \{ 2 \times 1 + (n-1)2 \} \quad d=2$$

$$= \frac{n}{2} (2 + 2n - 2)$$

$$= \frac{n}{2} \times 2n = n^2.$$

(iii) Find the sum of the squares of first n natural numbers

Let S denote the sum, then

$$S = 1^2 + 2^2 + 3^2 + \dots + n^2$$

$$\text{we have } n^3 - (n-1)^3 = 3n^2 - 3n + 1.$$

Putting $n=1, 2, 3$, etc. in succession, we have

$$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$$

$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$$

$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$$

$$\dots \quad \dots \quad \dots$$

$$\dots \quad \dots \quad \dots$$

$$n^3 - (n-1)^3 = 3 \cdot n^2 - 3 \cdot n + 1$$

By addition,

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \dots + n^2) - 3(1 + 2 + 3 + \dots + n) + n$$

$$= 3 \cdot S - 3 \cdot \frac{n(n+1)}{2} + n$$

$$\therefore 3S = n^3 + \frac{3n(n+1)}{2} - n = \frac{n}{2} (2n^2 + 3n + 3 - 2)$$

$$= \frac{n}{2} (2n^2 + 3n + 1) = \frac{n(n+1)(2n+1)}{2}$$

$$\therefore S = \frac{n(n+1)(2n+1)}{6}$$

Ex. 5. Find the 20th term of the arithmetical progression 15, 13, 11..... Calculate the number of terms required to make the sum equal to zero.

$$\begin{aligned} 20\text{th term} &= 2 \times 15 + (20-1) \times -2 \quad d = -2 \\ &= 10(30-38) = -80 \end{aligned}$$

In the second case, $\text{sum} = 0$, to find n ,

$$\therefore 0 = n/2 \{2 \times 15 + (n-1) \times -2\} = n(16-n)$$

$$\therefore \text{either } n=0, \text{ or } n=16.$$

Hence the only admissible value is 16.

Ex. 6. If a, b, c are the $p^{\text{th}}, q^{\text{th}}$ and r^{th} terms of an A.P. prove that $a(q-r) + b(r-p) + c(p-q) = 0$.

Let x be the first term and d the common difference of the A.P. Then

$$a = x + (p-1)d \dots (1)$$

$$b = x + (q-1)d \dots (2)$$

$$c = x + (r-1)d \dots (3)$$

Subtracting (2) from (1) and (3) from (2), we have

$$a-b = (p-q)d \dots (4)$$

$$b-c = (q-r)d \dots (5)$$

$\therefore$ dividing (4) by (5)

$$\frac{a-b}{b-c} = \frac{p-q}{q-r}$$

$$\text{or } a(q-r) - b(q-r) = b(p-q) - c(p-q)$$

$$\text{or } a(q-r) + b(r-p) + c(p-q) = 0.$$

Ex. 7. A man is employed to count Rs 10710/- He counts at the rate of Rs. 180/- per minute for half an hour. After this he counts at the rate of Rs. 3/- less every minute than the preceding minute. Find the time taken by him to count the entire amount.

Amount counted in half an hour

$$= \text{Rs. } 180 \times 30 = \text{Rs. } 5400/-$$

$$\therefore \text{balance to be counted} = \text{Rs. } 10710/- - \text{Rs. } 5400/- \\ = \text{Rs. } 5310/-$$

In one minute just after half-an hour, the amount counted is Rs. 177/-.

Let n minutes be taken to count Rs. 5310/-, then

$$5310 = \frac{n}{2} \{2 \times 177 + (n-1) \times -3\} = \frac{n}{2} (357 - 3n)$$

$$\text{or } 3n^2 - 357n + 10620 = 0$$

$$\text{or } n^2 - 119n + 3540 = 0$$

$$\text{or } (n-59)(n-60) = 0$$

$$\therefore \text{ either } n=59 \quad \text{or } 60$$

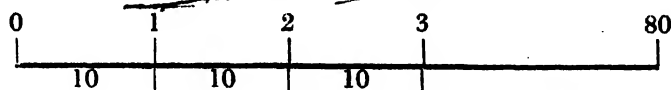
Now in 59 minutes after half an hour, the amount counted $= \frac{59}{2} \{2 \times 177 - 58 \times 3\} = \frac{59}{2} (354 - 174) = 5310$ ✓

and nothing is counted in the 60th minute.

$$(a + 59d = 177 - 59 \times 3 = 0)$$

$$\therefore \text{ the required time } = 30 + 59 = 89 \text{ min.}$$

Ex. 8. 80 coins are placed in a straight line on the ground. The distance between any two consecutive coins is 10 ft. How far must a person travel to bring them, one by one, to a basket placed 10 ft. behind the first coin?



Let 1, 2, 3, ... 80 represent the positions of the coins and 0 that of the basket.

The distance covered in bringing the first coin $= 10 + 10$ or 20ft.

" " " 2nd " $= 20 + 20$ or 40ft.

" " " 3rd " $= 30 + 30$ or 60ft.

etc.

$\therefore$ total distance covered

$$= \frac{n}{2} \{2a + (n-1)d\} = 40(40 + 1580) \text{ ft.} \quad \begin{matrix} n=80 \\ a=20 \\ d=20 \end{matrix}$$

$$= 12\frac{3}{4} \text{ miles.}$$

Exercise 4.

1. Find the n th term and the sum of the first n terms of the following A.P.

(i) 4, 6, 8, ...

$n=10$

(ii) 1, 3, 5, ...

$n=18$

$$(iii) \quad 1, \frac{4}{8}, \frac{9}{8}, \dots \quad n=58$$

$$(iv) \quad 4, 3\frac{3}{4}, 3\frac{1}{2}, \dots \quad n=45$$

$$(v) \quad (1-\frac{1}{2}) + (3-\frac{1}{2}) + (5-\frac{1}{2}) + \dots \quad n=20$$

$$(vi) \quad \begin{matrix} a+b & a+3b & a+5b \\ a+b & a+b & a+b \end{matrix}, \dots \quad n=15$$

$$(vii) \quad (3+4) + (8+9) + (13+14) + \dots \quad n=20$$

2. Find the n th term and sum to n terms of the following A.P.

$$(i) \quad 2, 4, 6, \dots$$

$$(ii) \quad 1, 5, 9, \dots$$

$$(iii) \quad 10, 7, 4, \dots$$

$$(iv) \quad x-y, 3x-2y, 5x-3y, \dots$$

$$(v) \quad a+b, 2a, 3a-b, \dots$$

$$(vi) \quad (x+y)^2, (x^2+y^2), (x-y)^2, \dots$$

$$(xii) \quad \frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \dots$$

$$3. \quad \text{Sum the series (i) } 72+70+68+\dots+40$$

$$(ii) \quad 40+44+48+\dots+1000$$

4. If the 10th term of an A.P. is 15 and the 15th term is 10, find the series.

$$5. \quad (a) \quad \text{If } a_1, a_2, a_3, a_4, \text{ are in A.P., show that}$$

$$a_1 + a_4 = a_2 + a_3$$

(b) If S_1, S_2, S_3 be the sums of n terms of three arithmetic series, the first term of each being 1 and the respective common differences 1, 2, 3, prove that

$$S_1 + S_3 = 2S_2$$

6. Find the 20th term of the A.P. 81, 78, 75,
Calculate the number of terms required to make the sum equal to zero. (R. U. 1953)

$$7. \quad (a) \quad \text{Find the sum of } 6^2 + 7^2 + 8^2 + \dots + 20^2$$

(b) Show that the sum of the first n odd numbers is always a perfect square.

8. ✓ The sum of three numbers in A.P. is 18 and their product is 192. Find them.

[Take $a-\beta$, a , $a+\beta$ as three numbers in A.P. so that $3a=18$ or $a=6$. Also $(a-\beta)a(a+\beta)=192 \therefore a(a^2-\beta^2)=192$, whence $\beta=\pm 2$. The reqd. numbers are 4, 6, 8.]

9. ✓ (i) The sum of four numbers in A.P. is 20 and the sum of their squares is 120. Find the numbers.

[Take the numbers as $a-3\beta$, $a-\beta$, $a+\beta$, $a+3\beta$.]

(ii) The sum of four integers in A.P. is 24 and their product is 945, find them.

10. Show that the sum of the first n natural numbers is equal to $(1+1/n)$ times the sum of the first n odd numbers.
(P. U. 1944)

11. (a) The sum of n terms of an A. P. is $2n^2$. Find the 5th term.

(b) The sum of n terms of an A. P. is $2n^2+3n$. Find the n th term and the series.

✓ 12. The p th term of an A. P. is q and q th term is p . Show that the r th term is $p+q-r$.

13. (i) The sum of p terms of an A.P. is q and the sum of q terms is p . Find the sum of $p+q$ terms.

(ii) If p th term of an A.P. is $1/q$, and q th term is $1/p$, show that the sum of pq terms is $\frac{1}{2}(pq+1)$.

14. Find the sum of the following series without assuming any formula.

(i) $108+102+96+\dots$ to 50 terms

(ii) $8+0-8-\dots$ to 8 terms

(iii) $27+24+21+\dots$ to 19 terms

15. (a) Insert two numbers between 8 and 125. so that the number form 4 terms of an A.P.

(q) Insert n arithmetic means between a and b .

[b is $(n+2)$ th term. $\therefore b=a+(n+1)d$ whence $d=\frac{b-a}{n+1}$]

16. (a) a, b, c are in A.P., show that

$$\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab} \text{ are in A.P.}$$

(b) If a^2, b^2, c^2 are in A.P., prove that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P.}$$

(c) If x, y, z are in A.P., show that

$$\frac{1}{\sqrt{y}+\sqrt{z}}, \frac{1}{\sqrt{z}+\sqrt{x}}, \frac{1}{\sqrt{x}+\sqrt{y}} \text{ are in A.P.}$$

17. If a, b, c are respectively the sums of p, q and r terms of an A.P., prove that

$$(i) \quad a(q-r)/p + b(r-p)/q + c(p-q)/r = 0.$$

$$(ii) \quad (b-c)(p-q) = (a-b)(q-r)$$

18. (a) The n th term of a series is $3n^2 + n$, find the sum to n terms

(b) Sum the series to n terms whose n th term is $n(n+1)$.

19. Sum the following series to n terms

$$(i) \quad \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots,$$

$$(ii) \quad \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$$

$$(iii) \quad \frac{1}{4.7} + \frac{1}{7.10} + \frac{1}{10.13} + \dots,$$

$$(iv) \quad \frac{1}{x(x+y)} + \frac{1}{(x+y)(x+2y)} + \frac{1}{(x+2y)(x+3y)} + \dots,$$

[In (i) 1st. term = $\frac{1}{2} - \frac{1}{3}$, 2nd term = $\frac{1}{3} - \frac{1}{4}$,

$$n\text{th term} = \frac{1}{n+1} - \frac{1}{n+2}$$

In (iii), 1st. term = $\frac{1}{3} (\frac{1}{4} - \frac{1}{7})$, n th term = $\frac{1}{(3n+1)(3n+4)}$

$$= \frac{1}{3} \left\{ \frac{1}{3n+1} - \frac{1}{3n+4} \right\} \text{ etc.]}$$

✓20. Prove that if 1 is added to the sum of any number of terms of the A.P. 3, 5, 7, 9,....., the resulting sum is a perfect square.

[sum to n terms $= \frac{n}{2} \{6 + (n-1)2\} = n^2 + 2n$, which is a perfect square if 1 is added to it.]

✓21. 45 boys contribute to a charity fund. One contributes 5 nP., another 10 nP., the third 15 nP., and so on each contributing 5 nP. more than the former. What is their total contribution ?

✓22. A money lender lends Rs 1000 and charges an overall interest of Rs 140. He recovers the loan and interest by 12 monthly instalments each less by Rs 10 than the preceding. Find the amount of the first instalment.

✓23. A person pays Rs 975 by monthly instalments each less than the former by Rs 5. The first instalment is Rs 100. In what time will the entire amount be paid ?

✓24. 20 stones are placed in a straight line on the ground. The distance between any two consecutive stones is 3 yds. How far must a person travel to bring them, one by one, to a basket placed at the first stone ?

25. To verify cash balances, the auditor of Joy Bank Ltd., employs his assistant to count cash in hand of Rs 4500. At first he counts quickly at the rate of Rs 150 per minute for 10 minutes only but at the end of that time he begins to count at the rate of Rs 2 less every minute than he could count in the previous minute. Ascertain how much time he will take to count this sum of Rs 4,500. (C.U. 1961)

✓26. B arranges to pay off a debt of Rs 9600 in 48 annual instalments which form an arithmetical series. When 40 of these instalments are paid, B becomes insolvent and his creditor finds that Rs 2400 still remains unpaid. Find the value of the first three instalments paid by B. Ignore interest. (C.U. 1957)

Chapter V

Progressions (continued)

7. A series or set of numbers is said to be in *Geometric Progression (G.P.)*, when the ratio of any number of the set to its preceding number is always the same.

This ratio is called *common ratio* and denoted by r .

Examples

(i) the series 4, 8, 16, 32.... are in G.P., the common ratio being 2.

(ii) the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ are in G.P., the common ratio being $\frac{1}{2}$.

(iii) the series 3, -9, 27, ... are in G.P., the common ratio being -3.

8. The n th term of a G.P.

Let a be the first term and r the common ratio of a geometric progression, we have

1st. term = a

2nd. term = ar

3rd. term = $ar.r = ar^2$

4th. term = $ar^2.r = ar^3$

... ..

$\therefore n$ th term = ar^{n-1}

If there are n terms in the series, the last term is $l = ar^{n-1}$

Ex. 9. Find the 15th term of the G.P. 2, 4, 8,

1st. term = $a = 2$, common ratio = $r = \frac{4}{2} = 2$

$\therefore$ 15th term = $2.2^{15-1} = 2.2^{14} = 32768$

Ex. 10. The 10th. term of a G.P. is 16, and the 16th term is 1024. Find the series and the 12th term.

Let a be the first term and r the common ratio.
Then $16 = ar^{10-1} = ar^9 \dots (1)$

$$1024 = ar^{16-1} = ar^{15} \dots (2)$$

By division, $r^6 = \frac{1024}{16} = 64 = 2^6 \quad \therefore r = 2$

$\therefore$ from (i) $16 = a \cdot 2^9 \quad \therefore a = \frac{16}{2^9} = \frac{2^4}{2^9} = \frac{1}{2^5} = \frac{1}{32}$

$\therefore$ the series is $\frac{1}{32}, \frac{1}{32} \times 2, \frac{1}{32} \times 2^2, \dots$

i.e., $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \dots$

$\therefore$ 12th. term $= \frac{1}{32} \cdot 2^{12-1} = \frac{1}{32} \times 2^{11} = 64.$

9. Sum of a G.P.

Let a be the first term, r the common ratio and n the number of terms.

Let S denote the sum to n terms, then

$$S = a + ar + ar^2 + \dots + ar^{n-1}. \quad (1)$$

Multiplying both sides by r ,

$$S \cdot r = ar + ar^2 + \dots + ar^{n-1} + ar^n. \quad (2)$$

$\therefore$ by subtraction

$$S(1-r) = a - ar^n$$

$$\therefore S = \frac{a(1-r^n)}{1-r} \quad r \neq 1 \quad (3)$$

If $r = 1$, $S = a + a \cdot 1 + a \cdot 1 + \dots + a \cdot 1.$

$$= n \cdot a.$$

If l denote the last term i.e., the n th. term, $l = ar^{n-1}$

Then $S = \frac{a-l \cdot r}{1-r}$

If $r < 1$, (3) is conveniently written as

$$S = \frac{a(r^n - 1)}{r - 1}$$

Ex. 11. Sum the series $9+18+36+\dots$ to 12 terms.

First term $=a=9$, common ratio $=r=\frac{18}{9}=2$.

$$\therefore S = \frac{a(r^n - 1)}{r - 1} = \frac{9(2^{12} - 1)}{2 - 1} = 9(4096 - 1) = 36855$$

$$\text{or, 12th. term} = ar^{12-1} = 9 \cdot 2^{11}$$

$$S = 9 + 18 + \dots + 9 \cdot 2^{11}$$

$$2S = 18 + \dots + 9 \cdot 2^{11} + 9 \cdot 2^{12}$$

$\therefore$ by subtraction,

$$-S = 9 - 9 \cdot 2^{12}$$

$$\therefore S = 9(2^{12} - 1) = 36855.$$

10. Geometric means

When three quantities are in G.P., the middle one is called the *Geometric mean* (G.M.)

If x, y, z are in G.P., then

$$\frac{y}{x} = \frac{z}{y} \quad \text{or} \quad y^2 = z \cdot x. \quad \text{or} \quad y = \sqrt{z \cdot x}$$

Hence *geometric mean of two numbers is the square root of their product.*

The geometric mean of n numbers is the nth root of their product.

$$\text{i.e., G.M. of } x_1, x_2, \dots, x_n = (x_1 \cdot x_2 \cdot \dots \cdot x_n)^{1/n}$$

Ex. 12. Find the geometric mean of 12, 18, 72, 108

$$\begin{aligned} \text{Reqd. G.M.} &= (12 \cdot 18 \cdot 72 \cdot 108)^{\frac{1}{4}} = (2^3 \cdot 3^8)^{\frac{1}{4}} \\ &= 2^3 \cdot 3^2 = 36 \end{aligned}$$

Ex. 13. Insert 4 geometric means between 6 and 1458.

The first term is 6 and the 6th. term is 1458.

Let r be the common ratio.

$$\text{Then } 1458 = ar^{6-1} = 6 \cdot r^5.$$

$$\therefore r^5 = \frac{1458}{6} = 243 = 3^5$$

$$\therefore r=3.$$

$\therefore$ the 4 geometric means are

$$6.3, 6.3^2, 6.3^3, 6.3^4$$

$$\text{or } 18, 54, 162, 486$$

11. When three numbers are in G.P. they may be conveniently taken as $\frac{a}{r}, a, ar$, of which the common ratio is r .

When four numbers are in G.P, they may be taken to be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ of which the common ratio is r^2 .

Ex. 14. Three numbers are in G.P. and their sum and product are respectively 63 and 1728. Find them:

Let the numbers be $\frac{a}{r}, a, ar$,

$$\therefore \frac{a}{r} + a + ar = 63 \dots\dots\dots (1)$$

$$\text{and } \frac{a}{r} \cdot a \cdot ar = 1728 \dots\dots\dots (2)$$

$$\text{From (2) } a^3 = 1728 = 2^6 \cdot 3^3. \quad \therefore a = 2^2 \cdot 3 = 12.$$

$$\text{From (1) } \frac{12}{r} + 12 + 12r = 63$$

$$\text{or } 4r^2 - 17r + 4 = 0.$$

$$\text{or } (r-4)(4r-1) = 0.$$

$$\therefore r = 4, \text{ or } \frac{1}{4}.$$

$\therefore$ the reqd. numbers are

$$\frac{12}{4}, 12, 12.4 \quad \text{or } 3, 12, 48$$

Ex. 15. If a, b, c are the p th., q th. and r th. terms of a G.P., prove that $a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$

Let t be the first term and d the common difference of the A.P., then

$$a = t + (p-1)d \quad \therefore a - b = (p-q)d \dots\dots\dots (1)$$

$$b = t + (q-1)d \quad b - c = (q-r)d \dots\dots\dots (2)$$

$$c = t + (r-1)d.$$

Let g be the first term and v the common ratio of the G.P., then

$$a = g.v^{p-1} \quad \therefore \frac{a}{b} = v^{p-q} \dots (3)$$

$$b = g.v^{q-1} \quad \frac{b}{c} = v^{q-r} \dots (4)$$

$$c = g.v^{r-1}$$

$$\text{From (3)} \quad \left(v^{p-q}\right)^{q-r} = \left(\frac{a}{b}\right)^{q-r} = \left(\frac{a}{b}\right)^{(b-c)/d} \quad \text{from (2)}$$

$$\text{From (4)} \quad \left(v^{q-r}\right)^{p-q} = \left(\frac{b}{c}\right)^{p-q} = \left(\frac{b}{c}\right)^{(a-b)/d} \quad \text{from (1)}$$

$$\therefore \left(\frac{a}{b}\right)^{b-c} = \left(\frac{b}{c}\right)^{a-b}$$

$$\text{or } a^{b-c} \cdot b^{-b+c} = \frac{1}{b^{b-a}} \cdot \frac{1}{c^{a-b}}$$

$$\text{or } a^{b-c} \cdot b^{-b+c} \cdot b^{b-a} \cdot c^{a-b} = 1$$

$$\therefore a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$$

12. *The arithmetic mean of two different numbers is greater than their geometric mean.*

Let a and b be the numbers.

Then their A. M. = $\frac{a+b}{2}$ and G.M. = $\sqrt{ab}$.

$$\text{Now } \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a+b-2\sqrt{ab}) = \frac{1}{2}(\sqrt{a}-\sqrt{b})^2$$

which is a positive number.

$$\therefore \frac{a+b}{2} > \sqrt{ab}.$$

13. Infinite Geometric Progression

We have seen that the sum of the G. P.

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

This is a finite G.P.

If the common ratio r is numerically less than 1, r^n decreases as n increases. By taking n indefinitely large, we can make r^n as small as we please.

$\therefore$ if $r < 1$, and n increases without limit

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

$\therefore$ the sum of an infinite geometric series

$$= \frac{\text{first term}}{1 - \text{common ratio}}$$

Ex. 16. Find the sum of the infinite geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

Here first term = 1, common ratio = $\frac{1}{2} (< 1)$

$$\therefore S = \frac{1}{1 - \frac{1}{2}} = 2$$

$$\text{or } S = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + 0$$

$$\frac{1}{2} \cdot S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + 0$$

By subtraction,

$$\frac{1}{2}S = 1 \quad \therefore S = 2.$$

14. Recurring decimals

A recurring decimal is an infinite G.P.

Ex. 17. Express $\cdot\dot{3}\dot{9}$ as an infinite geometric progression and hence find its value.

$$\cdot\dot{3}\dot{9} = \cdot 39 \ 39 \ 39 \dots\dots\dots$$

$$= \cdot 39 + \cdot 0039 + \cdot 000039 + \dots\dots\dots$$

$$= \cdot 39 (1 + \cdot 01 + \cdot 0001 + \dots\dots\dots)$$

Now $1 + \cdot 01 + \cdot 0001 + \dots\dots$ is an infinite G.P. whose first term is 1 and common ratio is $\cdot 01 (< 1)$

$$\therefore \cdot\dot{3}\dot{9} = \cdot 39 \times \frac{1}{1 - \cdot 01} = \cdot 39 \times \frac{1}{\cdot 99} = \frac{39}{99} = \frac{13}{33}$$

Ex. 18. Sum to n terms $4+44+444+\dots$

Let S denote the sum, then

$$\begin{aligned}
 S &= 4 + 44 + 444 + \dots \text{to } n \text{ terms} \\
 &= 4(1 + 11 + 111 + \dots \text{to } n \text{ terms}) \\
 &= \frac{4}{9} \times 9(1 + 11 + 111 + \dots \text{to } n \text{ terms}) \\
 &= \frac{4}{9}(9 + 99 + 999 + \dots \text{to } n \text{ terms}) \\
 &= \frac{4}{9}\{(10-1) + (10^2-1) + (10^3-1) + \dots \text{to } n \text{ terms}\} \\
 &= \frac{4}{9}\{(10 + 10^2 + 10^3 + \dots \text{to } n \text{ terms}) - n\} \\
 &= \frac{4}{9}\left\{\frac{10(10^n-1)}{10-1} - n\right\} = \frac{40}{81}(10^n-1) - \frac{4n}{9}
 \end{aligned}$$

Ex. 19. If a, b, c, d are in G.P., prove that

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2.$$

We have $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} \therefore b^2 = ac, c^2 = bd, bc = ad.$

$$\begin{aligned}
 \therefore L.H.S. &= (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2) + (d^2 - 2db + b^2) \\
 &= 2(b^2 - ac) + 2(c^2 - bd) + a^2 + d^2 - 2bc \\
 &= 2 \times 0 + 2 \times 0 + a^2 + d^2 - 2ad = (a-d)^2.
 \end{aligned}$$

Ex. 20. If the population of a town be 150000 and increases by 5 per cent every year, what will it be in 3 years? What will be the total percentage increase?

Population at the beginning of the first year = 150000

$$\begin{aligned}
 \text{Then } & \text{ " " end " " " } = 150000 \times \frac{21}{20} \\
 & \text{ " " " " 2nd " } = 150000 \times \left(\frac{21}{20}\right)^2 \\
 & \text{ " " " " 3rd " } = 150000 \times \left(\frac{21}{20}\right)^3 \\
 & \quad = \frac{150 \times 9261}{8} \\
 & \quad = 173,644.
 \end{aligned}$$

$$\therefore \text{total increase} = 172,644 - 150,000 = 23,644$$

$$\therefore \text{percentage increase} = \frac{23,644}{150,000} \times 100 = 15.76.$$

Ex. 21. Accumulated at 10% per annum compound interest a sum of money amounts to Rs 8650/- in 4 years. Find the sum.

Let P be the principal, then

$$\text{amount of } P \text{ after 1 year} = P \left(1 + \frac{10}{100}\right) = P \times 1.1$$

$$\text{'' '' '' 2 years} = P \times (1.1)^2$$

$$\text{'' '' '' 3 years} = P \times (1.1)^3$$

$$\text{'' '' '' 4 years} = P \times (1.1)^4.$$

$$\therefore P \times (1.1)^4 = 8650$$

$$\text{or } P = \frac{8650}{(1.1)^4} = \frac{8650}{1.4641} = 5976.37$$

$$\therefore \text{the required principal} = \text{Rs } 5976.37$$

Exercise 5.

1. Find the n th term and sum to n terms of the following series

$$(i) \quad 1 + 2 + 4 + \dots \quad n = 13$$

$$(ii) \quad 1024 + 512 + 256 + \dots \quad n = 15$$

$$(iii) \quad 8 - 16 + 32 - \dots \quad n = 12$$

$$(iv) \quad \frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \dots \quad n = 8$$

$$(v) \quad 1 + \sqrt{3} + 3 + \dots \quad n = 10 \quad (A. U. 1948)$$

$$(vi) \quad \frac{1}{\sqrt{2}} - 1 + \sqrt{2} - \dots \quad n = 10$$

$$(vii) \quad \frac{1}{\sqrt{3}} + 1 + \frac{3}{\sqrt{8}} + \dots \quad n = 18$$

2. Find the sum to n terms.

(i) $1 + 2 + 2^2 + \dots$

(ii) $1 + \frac{1}{3} + \frac{1}{3^2} + \dots$

(iii) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

(iv) $2\frac{1}{2} - 1 + \frac{2}{5} - \dots$

(v) $\frac{x+y}{x-y} + 1 + \frac{x-y}{x+y} + \dots$

3. (a) Find the sum of 9 terms of the G.P. $3 + 3^{\frac{4}{3}} + 3^{\frac{8}{3}} + \dots$ and show that it can not be less than 78.

(b) In the progression $1 + 3 + 3^2 + \dots$, find the least value of n such that $1 + 3 + 3^2 + \dots + 3^n > 500$.

4. (a) Find the sum of 6 terms of a G.P. whose first two terms are 1 and $\frac{3}{8}$.

(b) The first of a set of n numbers in G.P. is a and the last is l . Express the product of the n numbers in terms of a and l .

[last term = n th term = $a.r^{n-1}$, where r = common ratio.

$\therefore ar^{n-1} = l$. The product of terms

$$= a \cdot ar \cdot ar^2 \dots ar^{n-1} = a^n \cdot r^{\frac{1}{2}n(n-1)}$$

$$= a^n \left(\frac{l}{a} \right)^{\frac{1}{2}n} = (al)^{\frac{n}{2}}]$$

5. (i) Find the 8th term of the series $3\frac{1}{2}, 2\frac{1}{4}, 1\frac{1}{2}$.

(ii) The product of first and second terms of a G.P. is 256 and that of second and third terms is 16, find the 5th term.

6. (i) Find the three terms in G.P. whose sum is 19 and the sum of whose squares is 133. (P. U. 1946)

(ii) Find three terms in G.P., whose sum is 19 and whose product is 216.

(iii) The sum of first 3 terms in a G.P. is 31 and their product is 125, find the series.

7. Find the n th. term and sum to n terms in a G.P. whose

(i) first term is 64 and sixth term is 512

(ii) 8th. term is 48 and 14th. term is 384.

8. The first two terms of a G.P. are

(i) 125 and 25, find the 8th. term

(ii) 2187 and 729, find the 10th. term.

9. (i) Insert two numbers between 8 and 125 so that the numbers form four terms of a geometric progression.

(ii) Insert three geometric means between 3 and 9

(iii) Insert n geometric means between a and b .

10. The sum of an infinite G.P. whose common ratio is less than 1 is 32. If the sum of the first two terms is 24, find the series.

11. Sum to infinity

(i) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

(ii) $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \dots$

(iii) $(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \dots$

(iv) $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \dots$

12. Sum to infinity ($x < 1$)

(i) $1 + 2x + 3x^2 + 4x^3 + \dots$

(ii) $1 + 3x + 5x^2 + 7x^3 + \dots$

(iii) $1 + (1+b)x + (1+b+b^2)x^2 + \dots$ where both b, x are proper fractions. (A. U. 1952.)

13. Sum to n terms

(i) $3+33+333+\dots$

(ii) $4+44+444+\dots$

(iii) $\cdot 5+\cdot 55+\cdot 555+\dots$

(iv) $\cdot 9+\cdot 99+999+\dots$

14. Sum to n terms

(i) $1\cdot 03+(1\cdot 03)^2+(1\cdot 03)^3+\dots$

(ii) $\frac{1}{1\cdot 05}+\frac{1}{(1\cdot 05)^2}+\frac{1}{(1\cdot 05)^3}+\dots$

(iii) $\frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\dots+\frac{a}{(1+i)^n}$

15. Show that the sum to infinity of the following series is 25.

$$\frac{1}{1\cdot 04}+\frac{1}{(1\cdot 04)^2}+\frac{1}{(1\cdot 04)^3}+\dots\infty$$

16. Sum to n terms

$$1+3+7+15+\dots$$

17. (i) Three numbers are in A.P. and their sum is 27; if 1, 1, and 6 are added to them respectively, the results are in G.P. Determine the numbers.

(ii) The continued product of three numbers in G.P. is 27, and the sum of their products in pairs is 39. Find the numbers.

18. If a, b, c, d are in G.P., prove that

(i) $a+b, b+c, c+d$ are in G.P.

(ii) $(b+c)(b+d)=(c+a)(c+d)$

(iii) $a^2-b^2, b^2-c^2, c^2-d^2$ are in G.P.

(iv) $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$

19. (i) If a, b, c are in A.P. and x, y, z are in G.P., prove that $x^{a-b}y^{b-c}z^{c-a}=1$

(ii) If R, S, T be the r th. s th. and t th. terms of a G.P., show that $R^{s-t} S^{t-r} T^{r-s} = 1$ (A. U. 1945)

20. If $S_1, S_2, \dots, S_n$ are the sums of infinite geometric series whose first terms are $1, 2, \dots, n$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+1}$ respectively, show that

$$S_1 + S_2 + \dots + S_n = \frac{n(n+3)}{2}$$

21. If $x = y + y^2 + y^3 + \dots \infty$ ($0 < y < \frac{1}{2}$) show that $y = x - x^2 + x^3 - \dots \infty$

22. If the population of a town increases 25 per thousand per year of the population at the beginning of a year and the present population is 26,24,000, what will be the population in three years' time? What was it a year ago?

23. Show that a given sum of money accumulated at 20 per cent per annum more than doubles itself in 4 years at compound interest.

24. A particle moves 40 ft. in the 1st. second, 36 ft. in the 2nd. second, $32\frac{2}{3}$ ft. in the 3rd. second and so on. How far will it move before coming to rest?

25. A person proposes to give alms to a beggar 1 nP. on the first day, 2 nP. on the second day, 4 nP. on the third day, 8 nP. on the fourth day and so on. How much does he need to pay in the month of February, 1962?

26. A pendulum passes over distances in successive swings which form the G.P. 25, 15, 9, ... inches respectively. Find the total distance described by it before coming to rest.

15 A set of numbers is said to be in *harmonic progression* when their reciprocals are in arithmetic progression.

Thus $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ are in harmonic progression, since their reciprocals 2, 3, 4, 5, ... are in A.P.

Ex. 22. Find the Harmonic Mean between two numbers a and b .

Let H be the harmonic mean, then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$$\therefore \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\text{or } \frac{2}{H} = \frac{1}{a} + \frac{1}{b} \quad \therefore H = \frac{2ab}{a+b}$$

Exercise 6

- Find the harmonic mean between 12 and 21.
- If A , G and H be the arithmetic, geometric and harmonic means between two unequal quantities a and b , show that $A > G > H$. If the quantities be equal, show that $A = G = H$.
- The harmonic mean between two numbers is $33\frac{1}{3}$ and their geometric mean is 36, find the numbers.

4. If x, y, z are in H.P., show that

$$\frac{1}{y-x} + \frac{1}{y-z} = \frac{1}{x} + \frac{1}{z}$$

5. a, b, c be in A.P., p, q, r in H.P., ap, bq, cr in G.P., then show that $\frac{p}{r} + \frac{r}{p} = \frac{a}{c} + \frac{c}{a}$ (P. U. 1948)

Chapter VI

Permutations and Combinations

1. Definition

The different *arrangements* of things that can be made by taking some or all of them at a time are called *permutations*.

The different *selections* or *groups* of things that can be made by taking some or all of them at a time are called *combinations*.

In permutation we consider the *order* of things, but in combination the order is immaterial.

For example, the permutations of 3 things a, b, c taken

(i) one at a time are 3 viz a, b, c

(ii) two at a time are 6 viz $\begin{Bmatrix} ab, bc, ca \\ ba, cb, ac \end{Bmatrix}$

(iii) three at a time are 6 viz. $abc, acb, bca, bac, cab, cba$.

The permutations taken 2 at a time viz. ab and ba are formed of the same two things a and b in different orders, so they belong to the same group. Similarly, the permutations ac and ca form a second group and the permutations bc and cb form a third group.

Thus, the number of combinations of 3 things taken 2 at a time is only 3.

In like manner, the 6 permutations of 3 things taken all at a time are formed of the same things but in different orders and so they form one and only one group.

Thus, the number of combinations of 3 things taken all together is only 1.

Permutations

2. Fundamental Principle

If one thing can be done in m ways and when it is done in any of the m ways, a second thing can be done in n ways, then the two things together can be done in $m \times n$ ways.

The above principle can be extended to cases which involve more than two things.

The following examples illustrate the above principle.

Ex. 1. There are four routes for going from a station A to a station B . In how many different ways can a man go from A to B and return, if for returning

- (i) any of the routes is taken
- (ii) the same route is taken
- (iii) the same route is not taken ?

(i) The man can go from A to B in 4 different ways, for he may take any one of the four routes. When he has done so in any of the 4 ways, he may return in 4 different ways.

$\therefore$ corresponding to one way of going there are 4 different ways of returning.

$\therefore$ the total number of different ways $= 4 \times 4 = 16$.

(ii) Corresponding to one way of going there is only one way of returning.

$\therefore$ the total number of different ways $= 4 \times 1 = 4$

(iii) Corresponding to one way of going there are 3 different ways of returning.

$\therefore$ the total number of different ways $= 4 \times 3 = 12$

Ex. 2. Two persons enter a house where there are 5 vacant chairs arranged in a line. In how many different ways can they take seats ?

The first person can take his seat in 5 different ways, for he can take any one of the 5 chairs. When he has taken his seat in any way, the second man can take his seat in 4 ways only, for he can occupy any one of the remaining 4 chairs.

Since each way of occupying the chairs by the first person can be associated with each way of occupying the chairs by the second person, the total number of different ways $= 5 \times 4 = 20$.

Notation

The number of permutations of n different things taken r at a time is usually denoted by the symbol ${}^n P_r$ or ${}_n P_r$.

3. Permutations of n different things

Permutations of n different things taken r at a time ($r \leq n$)
 $i \quad n(n-1)(n-2) \dots (n-r+1)$

The number of permutations of n different things taken r at a time is the same as the number of different ways in which r places in a line can be filled up by n things.

The first place can be filled up in n ways, for any one of the n things can be put in it. $\therefore {}^n P_1 = n$.

When the first place has been filled up in any one of the n ways, the second place can be filled up in $n-1$ different ways, for any one of the remaining $n-1$ things can be put in it. Since each way of filling up the first place can be associated with each way of filling up the second place, the first two places can be filled up in $n(n-1)$ ways.

$$\therefore {}^n P_2 = n(n-1)$$

When the first two places have been filled up in any one of the $n(n-1)$ ways, the third place can be filled up in $n-2$ ways.

∴ the first three places can be filled up in $n(n-1)(n-2)$ ways

$$i. e., \quad {}^nP_3 = n(n-1)(n-2).$$

Proceeding in this way and noticing that the number of factors is equal to the number of places filled up and that each factor is less than the former by 1, we have the total number of ways in which r places can be filled up

$$= n(n-1)(n-2) \dots \text{to } r \text{ factors}$$

$$= n(n-1)(n-2) \dots (n-r+1)$$

$$= n(n-1)(n-2) \dots (n-r+1)$$

$$i. e., \quad {}^nP_r = n(n-1)(n-2) \dots (n-r+1)$$

$$\text{Cor. 1. } {}^nP_n = n(n-1)(n-2) \dots 3.2.1$$

∴ the number of permutations of n different things taken all at a time

$$= n(n-1)(n-2) \dots 3.2.1.$$

$$\text{Cor. 2 } {}^nP_{n-1} = n(n-1)(n-2) \dots 3.2 = {}^nP_n.$$

4. Factorial Notation

The product of first n natural numbers is denoted by the symbol $\lfloor n$ or $n!$ and read as 'factorial n '.

$$\therefore \lfloor 1 = 1, \lfloor 2 = 1.2 = 2, \lfloor 3 = 1.2.3 = 6, \lfloor 4 = 1.2.3.4 = 24$$

$$\lfloor 5 = 120, \lfloor 6 = 720, \lfloor 7 = 5040 \text{ etc.}$$

$$\text{Also } \lfloor 2 = 2.\lfloor 1, \lfloor 3 = 3.\lfloor 2, \lfloor 4 = 4.\lfloor 3$$

$$\dots \lfloor n = n.\lfloor n-1 = n(n-1) \lfloor n-2 \text{ etc.}$$

$$\text{Hence } {}^nP_r = n(n-1)(n-2) \dots (n-r+1)$$

$$= \frac{n(n-1)(n-2) \dots (n-r+1) \lfloor n-r}{\lfloor n-r} = \frac{\lfloor n}{\lfloor n-r}$$

$${}^nP_n = n(n-1)(n-2) \dots 2.1 = \lfloor n.$$

5. **Meaning of $\underline{0}$.** We have ${}^nP_n = \underline{1}n$,

$$\text{Also } {}^nP_n = \frac{\underline{1}n}{n-n} = \underline{1}0$$

$$\therefore \underline{1}n = \underline{1}0$$

$$\text{or } \underline{1}0 = \frac{\underline{1}n}{n} = 1$$

According to the definition, $\underline{1}0$ is meaningless. But used as symbol its value is 1.

Ex. 3. Find the values of ${}^{10}P_3$, 8P_4 and $\frac{\underline{1}8}{\underline{2}.\underline{2}.\underline{2}}$

$${}^{10}P_3 = 10.9.8 = 720, \text{ or } {}^{10}P_3 = \frac{\underline{1}0}{\underline{1}0-3} = \underline{1}7 = 10.9.8 = 720.$$

$${}^8P_4 = 8.7.6.5 = 1680, \text{ or } {}^8P_4 = \frac{\underline{1}8}{8-4} = \underline{1}4 = 8.7.6.5 = 1680.$$

$$\frac{\underline{1}8}{\underline{2}.\underline{2}.\underline{2}} = \frac{\underline{1}8}{2.2.2} = \frac{\underline{1}7.8}{8} = \underline{1}7 = 5040.$$

Ex. 4. If ${}^{m+2n}P_2 = 240$ and ${}^{m-2n}P_2 = 12$, find m and n .

$$\text{We have } {}^{m+2n}P_2 = 16.15 \quad \therefore m+2n = 16 \dots (1)$$

$$\text{and } {}^{m-2n}P_2 = 4.3 \quad \therefore m-2n = 4 \dots (2)$$

Solving (1) and (2), we have $m=10$, $n=3$.

Ex. 5. How many different words can be formed out of the letters of the word FRIDAY? How many of them begin with F? How many begin with F and end with Y?

There are 6 letters, all different.

$$\therefore \text{reqd. number of words} = {}^6P_6 = \underline{1}6 = 720.$$

Keeping aside F, the number of words that can be formed with the remaining 5 letters = $\underline{1}5$. With each of these we associate F at the beginning.

Keeping aside F at the beginning and Y at the end, the number of words that can be formed with the remain-

ing 4 letters=4. With each of these we associate F at the beginning and Y at the end.

6. Permutations in which things are repeated

The number of permutations of n different things taken r at a time in which each thing is repeated r times in any arrangement is n^r .

Evidently, the number of permutations is the same as the number of ways in which r places can be filled up by n different things, each being repeated r times.

The first place can be filled up in n ways, for any one of the n things can be put there. When the first has been filled up in any of the n ways, the second place can also be filled up in n ways, since the same thing can be used again. Thus the first two places can be filled up in $n \times n$ or n^2 ways.

Similarly, the first three places can be filled up in n^3 ways. Proceeding thus and observing that the power of n is the same as the number of places filled up, the required number of ways in filling up r places $= n^r$.

Ex. 6. In how many ways can 6 prizes be given to 5 boys each being entitled to one or more prizes?

Each prize can be given in 5 ways, for it can be given to any one of the 5 boys.

$\therefore$ 6 prizes can be given in $5 \times 5 \times 5 \times 5 \times 5 \times 5$ or 5^6 or 15625 ways.

Note. 5^6 is different from 6^5 .

Ex. 7. In how many ways can 5 letters be posted in 3 letter boxes?

Each letter can be posted in 3 ways, for it can be put in any one of the 3 boxes.

$\therefore$ 5 letters can be posted in $3 \times 3 \times 3 \times 3 \times 3$ or 3^5 or 243 ways.

7. Permutations of things not all different.

The number of permutations of n things of which p things are alike of one kind, q things are alike of a second kind, r things are alike of a third kind and all the rest are different is

$$\frac{n!}{p! \cdot q! \cdot r!}$$

Let the n things be represented by n letters and suppose p of them are each equal to a , q of them are each equal to b , r of them are each equal to c and the rest different.

Let x be the required number of permutations.

If the p letters ' a ' be replaced by p new letters, different from each other and different from the rest, then without changing the position of any other letter, each permutation would produce p permutations.

$\therefore x$ permutations would produce $x \times p$ permutations i.e., the total number of permutations would become $x \times p$.

Again, if the q letters ' b ' be replaced by q new letters different from each other and different from the rest, then the total number of permutations would become $x \times p \times q$.

Again, if the r letters ' c ' be replaced by r new letters different from each other and different from the rest, the total number of permutations would become $x \times p \times q \times r$.

But now the n things are all different and the permutations of n different things taken all at a time is $n!$

$$\therefore x \times p \times q \times r = n!$$

$$\therefore x = \frac{n!}{p! \cdot q! \cdot r!}$$

The above principle can easily be generalised.

Ex. 8. In how many ways can the letters of the word CALCUTTA be arranged?

There are 8 letters of which C occurs twice, A occurs twice and T occurs twice and the rest are all different.

Hence, the required number of words

$$= \frac{8!}{2! \cdot 2! \cdot 2!} = \frac{40320}{8} = 5040.$$

8. Restricted Permutations

(i) The number of permutations of n different things taken r at a time in which p particular things do not occur is ${}^{n-p}P_r$.

Keep aside the p particular things and fill up the r places with the remaining $n-p$ things at disposal.

The number of such ways $= {}^{n-p}P_r$.

(ii) The number of permutations of n different things taken r at a time in which p particular things are present is ${}^{n-p}P_{r-p} \times {}^rP_p$.

Keep aside the p particular things and form the permutations of the remaining $n-p$ things taken $r-p$ at a time. The number of such permutations is ${}^{n-p}P_{r-p}$.

In each of these permutations introduce the p particular things taken aside, one by one.

The first thing can be introduced in $r-p+1$ ways.

The second thing can be introduced in $r-p+2$ ways and the p th thing in $r-p+p$ or r ways.

$\therefore$ the p things can be introduced in each permutation in $(r-p+1)(r-p+2)\dots r$ ways which is clearly equal to rP_p ways.

$\therefore$ the required number of ways $= {}^{n-p}P_{r-p} \times {}^rP_p$.

Ex. 9. In how many different ways can 8 examination papers be arranged in a line so that the best and worst papers are never together?

The total number of arrangements that can be made of 8 papers is $8!$. Now let the best and worst papers be

taken together. This and the remaining 6 totalling 7 papers can be arranged amongst themselves in $7!$ ways. In each of these arrangements the best and the worst papers can be arranged in $2!$ ways.

$\therefore$ the total number of arrangements in which the best and worst papers come together $= 7! \times 2! = 2 \times 7!$.

$\therefore$ the number of arrangements in which the two particular papers are not together

$$= 8! - 2 \times 7! = 40,320 - 10,080 = 30,240.$$

Ex. 10. In how many ways can the letters of the word FAILURE be arranged so that the consonants may occupy only odd positions?

There are 7 letters of which 3 (FLR) are consonants and 4 (AIUE) are vowels.

The 4 positions to be filled up with consonants are indicated below

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (F) & & (L) & & (R) & & () \end{array}$$

The 3 consonants can be placed only in the 3 out of 4 positions marked 1, 3, 5, 7. The total number of ways in which this can be done $= {}^4P_3 = 4 \cdot 3 \cdot 2 = 24$.

One such arrangement is shown above. For this arrangement the 4 vowels can occupy the four remaining positions such as 2, 4, 6, 7 i.e., positions not occupied by vowels.

$\therefore$ total number of arrangement $= 24 \times 24 = 576$.

Ex. 11. How many numbers greater than a million can be formed with the digits 4, 5, 5, 0, 4, 5, 3?

Each number must consist of 7 or more digits.

There are 7 digits in all, of which there are 2 fours, 3 fives and the rest different.

$$\therefore \text{the total number of numbers} = \frac{7!}{2! \cdot 3!} = 420.$$

Of this numbers some begin with 0 and are less than one million. They must be rejected.

$$\text{The number of numbers beginning with 0} = \frac{6!}{2! \cdot 3!} = 60.$$

$$\therefore \text{the required number} = 420 - 60 = 360.$$

9. Circular Permutations

In placing a number of things round a circle, we regard two arrangements as different if they are different with respect to their relative positions, but same if their relative positions are the same.

The three circular arrangements ABC , BCA , CAB are the same, because in the above arrangements the same order of the letters A , B , C is maintained. It does not matter if the order is clockwise or counterclockwise.

The two circular arrangements ABC , ACB where the same order is not maintained are regarded as different.

Now, permutations of three letters A , B , C taken all together are 6 viz. ABC , ACB , BCA , BAC , CAB , CBA . But if the arrangements are circular, three of them viz., ABC , BCA , CAB are the same and regarded as one only. Similarly, three others ACB , BAC , CBA are the same and regarded as one.

$\therefore$ number of circular permutations of three different things taken all together is only 2.

Generally, the number of circular permutations of n different things taken all together is $(n-1)!$.

Ex. 12. In how many ways can 8 persons form a ring?

Here we are concerned with the positions of the persons relative to themselves. So keeping one person fixed we arrange the 7 others with respect to him.

$\therefore$ the required number of ways $= 7! = 5040$.

Note. If no distinction is made between clockwise and counterclockwise arrangements, the number is halved.

Ex. 13. In how many ways can 8 persons be seated at a round-table so that all shall not have the same neighbours in any two arrangements?

The total number of circular arrangements in which one person occupies a fixed position $= 7! = 5040$.

Since corresponding to each clockwise arrangement there is a counter clockwise arrangement and both of them are counted as one, the required number of different arrangements $= \frac{1}{2} \times 5040 = 2520$.

Note : To distinguish from circular arrangement, arrangement in a row or line is called linear. In a circular arrangement there are no ends but in a linear arrangement two ends exist.

Exercise 7

1. There are four routes for going from A to B and five routes for going from B to C . In how many different ways can a man go from A to C via B .

2. There are 8 vacant chairs in a room. In how many ways can 5 persons take their seats?

3. How many different words can be formed of the letters of the word 'failure', the four vowels always coming together.
(C. U., 1940)

4. There are 50 stations on a railway line. How many different kinds of single first class tickets must be printed so as to enable a passenger to go from one station to another?

5. How many numbers between 1,000 and 10,000 can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9? How many of them are odd?

6. How many numbers between 3000 and 4000 can be formed with the digits 1, 2, 3, 4, 5, 6?

7. Of the numbers formed with the digits 2, 3, 4, 6, 8 taking all together but none repeated, how many are even?

8. How many different words can be formed with the letters of the word SUNDAY? How many of the words begin with N? How many begin with N and end in Y?

9. Find how many words can be formed of the letters of the word FAILURE, the four vowels always coming together. (C. U. 1940)

10. In how many ways can 4 prizes, one for recitation, one for sports, one for smartness and one for general proficiency be given away to 8 boys?

11. A telegraph post has five arms and each arm is capable of 4 distinct positions including the position of rest. What is the total number of signals that can be made? (C. U. 1913)

12. (a) In how many ways can 10 letters be posted in 5 letter boxes?

(b) In a cross-word puzzle 20 words are to be guessed, of which 8 words have each an alternative solution also. Find the total number of possible solutions.

13. Find the number of different arrangements that can be made of bars of the seven prismatic colours so that the blue and the green shall never come together? (C. U. 1955)

14. In how many ways 10 examination papers be arranged so that the best and worst papers never come together? (C. U. 1953, Pat. U. 1947)

15. Find the number of ways in which n books can be arranged on a shelf so that two particular books are not together. (C. U. 1956, 1959)

16. In how many ways can 5 letters a, b, c, d, e be arranged so that the vowels may occupy contiguous positions?

17. In how many ways can 3 books on Commercial Mathematics and 5 books on Secretarial Practice be placed on a shelf so that books on the same subject always remain together? (no two books are identical.)

18. A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books and single copies of 8 books. In how many ways can all the books be arranged? (C. U. 1934)

19. How many permutations can be made out of the letters of the following words taken all together?

- | | |
|-------------------|------------------|
| (i) CALCUTTA | (ii) BANARAS |
| (iii) PERMUTATION | (iv) COLLEGE |
| (v) EXAMINATION | (vi) MISSISSIPPI |

20. Find the number of permutations of the letters of the word SIGNAL such that the vowels may occupy only odd positions.

21. There are m men and n mohars ($m < n$). In how many ways each man may become the owner of one mohar?

22. In how many of the permutations of 10 things taken 4 at a time will one particular thing (i) always occur (ii) never occur? (C. U. 1936)

23. Find the number of numbers less than 1000 and divisible by 5 which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occurring more than once in each number. (C. U. 1942)

24. How many numbers greater than a million can be formed with the digits 1, 7, 1, 0, 7, 3, 7?

25. In how many ways can 8 different beads be strung on a necklace?

26. In how many ways can 8 boys form a ring?

27. In how many ways 6 men can sit at a round table, so that all shall not have the same neighbours in any two arrangements?

28. In how many ways can 4 men and 3 ladies be arranged at a round table if the 3 ladies (1) never sit together, (2) always sit together?

29. In how many ways can 7 Englishmen and 6 Indians sit down a round table so that no two Indians are together?

Combinations

10. The number of combinations of n different things taken r at a time is ${}_rC_n = \frac{{}_nP_r}{r!}$ ($r \leq n$)

Let nC_r denote the number of combinations.

Each of these combinations contains r different things.

$\therefore$ if the r different things be arranged among themselves in all possible ways, each combination would produce $r!$ permutations.

$\therefore {}^nC_r$ combinations would produce ${}^nC_r \times r!$ permutations. But this number is clearly equal to the number of permutations of n different things taken r at a time.

Hence, ${}^nC_r \times r! = {}^nP_r$

$$\begin{aligned} \therefore {}^nC_r &= \frac{{}^nP_r}{r!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \\ &= \frac{n(n-1)(n-2)\dots(n-r+1) \cdot (n-r)!}{r! \cdot (n-r)!} \\ &= \frac{{}_nP_r}{r! \cdot (n-r)!} \end{aligned}$$

$$\text{Cor. 1. } {}^nC_1 = n, \quad {}^nC_2 = \frac{n(n-1)}{2}, \dots, {}^nC_n = 1, \quad {}^nC_{n-1} = n$$

$$\text{Cor. 2. } {}^nC_0 = \frac{{}_nP_0}{0! \cdot (n-0)!} = \frac{1}{0! \cdot n!} = \frac{1}{n!} = 1$$

11. Complementary Combinations

The number of combinations of n different things taken r at a time, is equal to the number of combinations of n different things taken $(n-r)$ at a time.

$$\text{i.e., } {}^nC_r = {}^nC_{n-r}.$$

$$\text{we have, } {}^nC_r = \frac{n!}{r! \cdot (n-r)!}$$

$$\text{and } {}^nC_{n-r} = \frac{n!}{(n-r)! \cdot (n-(n-r))!} = \frac{n!}{(n-r)! \cdot r!}$$

$$\therefore {}^nC_r = {}^nC_{n-r}.$$

Cor. 1. If ${}^nC_r = {}^nC_p$ then either $r=p$ or $r+p=n$.
for since ${}^nC_r = {}^nC_{n-r}$ $\therefore n-r=p$ or $r+p=n$.

Cor. 2. If in the formula ${}^nC_{n-r} = {}^nC_r$, we put

$$(i) \ r=n, \text{ then } {}^nC_0 = {}^nC_n = 1$$

$$(ii) \ r=n-1, \text{ then } {}^nC_1 = {}^nC_{n-1} = n, \text{ etc.}$$

12. To prove that the number of combinations of n things taken r at a time plus the number of combinations of n things taken $(r-1)$ at a time is equal to the number of combinations of $(n+1)$ things taken r at a time.

$$\text{i.e., } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$$

$$\begin{aligned} \text{we have, } {}^nC_r + {}^nC_{r-1} &= \frac{n!}{r! \cdot (n-r)!} + \frac{n!}{(r-1)! \cdot (n-r+1)!} \\ &= \frac{n!}{(r-1)! \cdot (n-r)!} \left\{ \frac{1}{r} + \frac{1}{n-r+1} \right\} \\ &= \frac{n!}{(r-1)! \cdot (n-r)!} \cdot \frac{n+1}{r(n-r+1)} \\ &= \frac{(n+1)!}{r! \cdot (n-r+1)!} = {}^{n+1}C_r. \end{aligned}$$

Ex. 14. Find the values of ${}^{10}C_8$, ${}^{40}C_{38}$, ${}^{2n}C_{2n-2}$

we have, ${}^{10}C_8 = {}^{10}C_2 = \frac{10 \times 9}{1 \times 2} = 45$

$${}^{40}C_{38} = {}^{40}C_2 = \frac{40 \times 39 \times 38 \times 37}{1 \times 2 \times 3 \times 4} = 5 \times 13 \times 38 \times 37 = 91390$$

$${}^{2n}C_{2n-2} = {}^{2n}C_{2n-(2n-2)} = {}^{2n}C_2 = \frac{2n \times (2n-1)}{1 \times 2} = n(2n-1)$$

Ex. 15. Prove that $2n \cdot 1.3.5 \dots (2n-1) \cdot 2^n \cdot \underline{n}$

We have $2n \cdot 1.2.3 \cdot 4.5 \cdot 6 \dots (2n-1) \cdot 2n$

$$= 1 \cdot 3.5 \dots (2n-1) \times 2 \cdot 4 \cdot 6 \dots 2n$$

$$= 1.3.5 \dots (2n-1) \times 2^n \cdot 1 \cdot 2.3 \dots n$$

$$= 1.3.5 \dots (2n-1) \times 2^n \cdot \underline{n}$$

Ex. 16. If $5 \cdot {}^{2n}C_3 = 12 \cdot {}^{2n}C_4$, find nC_2

We have ${}^{2n}C_3 = \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)}{1 \times 2 \times 3}$

and ${}^{2n}C_4 = \frac{2n(2n-1)(2n-2)(2n-3)}{1 \times 2 \times 3 \times 4}$

$\therefore$ cancelling like factors, we have

$$2n-4=12 \text{ whence } 2n=16 \text{ or } n=8$$

Hence ${}^nC_2 = {}^8C_2 = \frac{8 \times 7}{1 \times 2} = 28$.

Ex. 17. In an examination paper on Commercial Mathematics 10 questions are set. In how many different ways can an examinee choose 7 questions?

The number of different choice is evidently equal to the number of ways in which 7 places can be filled up by 10 different things.

$\therefore$ the required number of ways

$$= {}^{10}C_7 = {}^{10}C_3 = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120.$$

13. Restricted Combinations

(i) The number of combinations of n things taken r at a time in which p particular things always occur is ${}^{n-p}C_{r-p}$.

(ii) The number of combinations of n things taken r at a time in which p particular things never occur is ${}^{n-p}C_r$.

(i) If the p particular things are set aside, there remain $(n-p)$ things out of which $(r-p)$ things may be chosen in ${}^{n-p}C_{r-p}$ ways. With each of these groups we combine the p particular things so that we get all the combinations in each of which the p particular things will always occur.

$\therefore$ the required number of combinations $= {}^{n-p}C_{r-p}$.

(ii) Let the p particular things be set aside, then there will remain $(n-p)$ things out of which r things may be selected in ${}^{n-p}C_r$ ways.

In none of these groups p particular things will occur. Hence, the required number of combinations $= {}^{n-p}C_r$.

Ex. 18. In a mercantile firm 4 posts fall vacant and 35 candidates apply for the posts. In how many ways can a selection be made

(i) if one particular candidate is always *included*

(ii) if one particular candidate is always *excluded*?

(i) Since a particular person is always to be selected, we must select the remaining 3 candidates out of the remaining 34.

$\therefore$ the required number of selections

$$= {}^{34}C_3 = \frac{34 \times 33 \times 32}{1 \times 2 \times 3} = 5984$$

(ii) Since a particular person is never to be selected, the choice is restricted to 4 candidates out of the remaining 34.

$\therefore$ the required number of selections

$$= {}^{34}C_4 = \frac{34 \times 33 \times 32 \times 31}{1 \times 2 \times 3 \times 4} = 46376$$

Ex. 19. In how many ways can 12 Science students and 9 Arts students sit in a row so that no two Arts students may be together ?

Let the science students be arranged as shown below

$$\times S \times S \times S \times S \times \dots \times S \times S \times$$

In order that the Arts students may not sit together, they can only occupy positions marked $\times$.

The number of such positions is 13, (number of science students being 12).

Hence, we must select 9 positions out of available 13.

$\therefore$ the required number of ways

$$= {}^{13}C_9 = {}^{13}C_4 = \frac{13 \times 12 \times 11 \times 10}{1 \times 2 \times 3 \times 4} = 715$$

Ex. 20. There are n points in a plane no three of which are collinear (lying on the same straight line) with the exception of p points which are collinear. Find the number of

- (i) different straight lines and
- (ii) different triangles formed by joining them.
- (i) Any two points when joined give a straight line.

$\therefore$ the number of possible straight lines formed by joining n points in pairs $= {}^nC_2$

But p of the points lie in the same straight line.

$\therefore$ pC_2 straight lines are lost and instead we get only 1 straight line in which they lie.

$\therefore$ the required number of straight lines $= {}^nC_2 - {}^pC_2 + 1$

$$= \frac{n(n-1)}{2} - \frac{p(p-1)}{2} + 1$$

(ii) Any 3 non-collinear points give a triangle.

$\therefore$ the number of triangles formed by joining n points taken three at a time $= {}^nC_3$.

Since p of the points are collinear, pC_3 triangles are lost.

$\therefore$ the required number of triangles

$$= {}^nC_3 - {}^pC_3 = \frac{n(n-1)(n-2)}{6} - \frac{p(p-1)(p-2)}{6}$$

Ex. 21. A question paper contains ten questions divided into two groups of five questions each. In how many ways can you answer six questions taking at least two questions from each group?

The questions may be answered in the following ways

- | | | | | | | |
|-------|---|--------------------------|---|----|--------------------|---|
| (i) | 2 | questions from 1st group | + | 4 | questions from 2nd | |
| (ii) | 3 | " | " | +3 | " | " |
| (iii) | 4 | " | " | +2 | " | " |

(i) 2 questions can be chosen from 1st. group in 5C_2 ways and 4 questions from 2nd. group in 5C_4 ways.

Since each way of selecting questions from the 1st group can be associated with each way of selecting questions from the 2nd group, the total number of ways of selecting questions from the two groups $= {}^5C_2 \times {}^5C_4$.

(ii) In like manner, 3 questions from the 1st. group and 3 questions from the 2nd group may be selected in ${}^5C_3 \times {}^5C_3$ ways.

(iii) Again, 4 questions from the 1st group and 2 questions from the 2nd group may be selected in ${}^5C_4 \times {}^5C_2$ ways.

$\therefore$ the total number of ways

$$\begin{aligned} &= {}^5C_2 \times {}^5C_4 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_2 \\ &= {}^5C_2 \times {}^5C_1 + {}^5C_3 \times {}^5C_2 + {}^5C_4 \times {}^5C_1 \end{aligned}$$

$$\begin{aligned}
 &= \frac{5 \times 4}{1 \times 2} \times 5 + \frac{5 \times 4}{1 \times 2} \times \frac{5 \times 4}{1 \times 2} + 5 \times \frac{5 \times 4}{1 \times 2} \\
 &- 10 \times 5 + 10 \times 10 + 5 \times 10 = 50 + 100 + 50 = 200
 \end{aligned}$$

14. Total number of combinations of n things, all different

The total number of combinations of n different things taken some or all at a time is $2^n - 1$.

The first thing can be dealt with in 2 ways, for it may either be taken or left. The second thing can also be dealt with in 2 ways.

Since each way of dealing with the first can be associated with each way of dealing with the second, the total number of ways of dealing with the two things $= 2 \times 2 = 2^2$.

Proceeding similarly, the total number of ways $= 2^n$.

But this number includes one case in which none of the things are taken.

$\therefore$ the required number of combinations $= 2^n - 1$.

Cor. ${}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n - 1$.

15. Total number of combinations of n things not all different

The total number of combinations of $(p+q+r+\dots)$ things of which p are alike of one kind, q of second kind, r of third kind, and so on, taken any number at a time is

$$(p+1)(q+1)(r+1)\dots - 1$$

Consider the p like things. The p things can be dealt with in $(p+1)$ ways, for we may take 1, or 2, or 3,..... or p or none in any selection. Similarly, the q like things can be dealt with in $(q+1)$ ways, r things in $(r+1)$ ways etc. Associating each group of selections with the others, the total number of dealing with them is $(p+1)(q+1)(r+1)\dots$

But this number includes one case when all things are left.

$\therefore$ the total number of ways $= (p+1)(q+1)(r+1)\dots - 1$.

Ex. 22. In how many ways can you draw a sum of money from a bag containing a rupee, a half-rupee, a quarter-rupee, a ten nP. and a five nP. pieces?

According to the question we are permitted to draw one or more of the five coins at a time.

Each coin can be dealt with in 2 ways, for it can either be drawn or left.

$\therefore$ the required number of dealing with the coins $= 2^5$.

But this number includes one case in which none is drawn.

$\therefore$ the required number of ways $= 2^5 - 1 = 32 - 1 = 31$.

Ex. 23. A question-paper contains 6 questions, each having an alternative. In how many ways can an examinee answer one or more questions?

The first question can be dealt with in 3 ways, for the question itself may be answered, or its alternative may be answered or none of them may be answered.

Similarly, the second question also can be dealt with in 3 ways. Hence, the first two questions can be dealt with in 3×3 or 3^2 ways. Proceeding thus, all the 6 questions may be dealt with in 3^6 ways.

But this number includes one case in which none of the questions is answered.

$\therefore$ the required number of ways $= 3^6 - 1 = 729 - 1 = 728$

Ex. 24. Out of 10 consonants and 4 vowels, how many words can be formed each containing 6 consonants and 3 vowels?

6 consonants can be chosen out of 10 in $^{10}C_6$ ways and 3 vowels can be chosen out of 4 in 4C_3 ways.

∴ combining each way of selecting the consonants with each way of selecting the vowels, the number of selections having 6 consonants and 3 vowels = $^{10}C_6 \times ^4C_3$.

Each of these selections contains 9 letters which can be arranged among themselves in $\underline{9}$ ways.

$$\begin{aligned}\therefore \text{the total number of words} &= ^{10}C_6 \times ^4C_3 \times \underline{9} \\ &= ^{10}C_4 \times ^4C_1 \times \underline{9} \\ &= \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} \times 4 \times 362880 \\ &= 304819200\end{aligned}$$

16. Group Divisions

The number of ways in which $(m+n)$ things, m and n being unequal, can be divided into two groups containing m and n

things respectively is $\frac{m+n}{m \cdot n}$

We can select m things out of $(m+n)$ things in $^{m+n}C_m$ ways. Each time m things are taken to form a group of m things, n things are left out which form nC_n or 1 group of n things.

$$\text{Hence the required number of ways} = ^{m+n}C_m \times 1 = \frac{m+n}{m \cdot n}$$

Note 1. If $m=n$, the two groups are equal in number, and they can be interchanged without obtaining a new subdivision.

Hence the number of different ways by which $2m$ things can be divided into two equal groups is

$$\frac{2m}{2 \cdot (\underline{m})^2}$$

Note 2. If the $2m$ things be distributed equally between two persons, the number of ways $= \frac{|2m}{(\underline{1}m)^2}$

For example, let a, b, c, d be four things. Evidently when we select a group ab , c and d are left to form a second group cd and when we select cd , a and b are left to form a group ab . But these two subdivisions are the same because they differ only in orders.

But if we divide the four things between two persons, the first person gets ab and the second cd . This is quite different from the case in which the first person gets cd and the second ab .

Similarly, if $(m+n+p)$ things be divided into three groups containing m, n and p things respectively, the first selection of m things can be made in ${}^{m+n+p}C_m$ ways. The next selection of n things out of the remaining $(n+p)$ things can be made in ${}^{n+p}C_n$ ways. The next selection of p things out of the remaining p things can be made in pC_p ways.

$\therefore$ associating the different ways of selection, we have the required number of ways

$$\begin{aligned} &= {}^{m+n+p}C_m \times {}^{n+p}C_n \times {}^pC_p \\ &= \frac{|m+n+p}{\underline{m} \cdot \underline{n+p}} \times \frac{|n+p}{\underline{n} \cdot \underline{p}} \times 1 = \frac{|m+n+p}{\underline{m} \cdot \underline{n} \cdot \underline{p}} \end{aligned}$$

Note 1. If $m=n=p$, the groups are equal. Since three groups can be interchanged in $\underline{3}$ ways, the total number

of ways of subdivision $= \frac{|3m}{\underline{3} \cdot (\underline{1}m)^3}$

Note 2. If the $3m$ things be distributed equally between three persons the number of ways $= \frac{|3m}{(\underline{1}m)^3}$

Ex. 25. In how many ways can a party of 12 men divide themselves into two batches? (C. U. 1960)

The batches may consist of unequal or equal numbers.
 $\therefore$ the required number of ways

$$\begin{aligned}
 &= {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + {}^{12}C_4 + {}^{12}C_5 + \frac{1}{12} \cdot {}^{12}C_6 \\
 &= 12 + \frac{12 \times 11}{1 \times 2} + \frac{12 \times 11 \times 10}{1 \times 2 \times 3} + \frac{12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4} \\
 &\quad + \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} + \frac{1}{2} \cdot \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \\
 &= 12 + 66 + 220 + 495 + 792 + 462 = 2047
 \end{aligned}$$

Exercise 8

- Find the values of 8C_3 , ${}^{21}C_{10}$, ${}^nC_{n-3}$.
- (i) If ${}^{2n}C_3 : {}^{2n}C_2 = 8$, find n .
 (ii) If ${}^nC_3 = {}^nC_5$, find the value of ${}^{2n}C_2$
- Prove that

$$(i) \quad {}^nC_r = \frac{n!}{r!} \quad {}^{n-1}C_{r-1} \quad (ii) \quad {}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}.$$

4. In how many ways can you choose six out of nine questions? In how many of these ways the first question is always excluded? In how many the first and second questions are always included?

5. From a company of 15 men, how many selections of 9 men can be made so as to exclude three particular men? (C. U. 1954)

6. In how many ways can 15 white balls and 13 black balls be arranged in a line so that no two black balls are together?

7. A father takes 6 children, three at a time to the Zoo, as often as he can without taking the same three together more than once. (i) how often will he go? (ii) how often will each child go?

8. A candidate is required to answer 6 out of 10 questions which are divided into two groups each containing 5 questions, and he is not permitted to attempt more than 4 from any group. In how many different ways can he make up his choice? (C. U. 1932)

9. (i) How many triangles can be formed by joining the angular points of a decagon? How many diagonals has it? (C. U. 1911)

(ii) There are 20 points in a plane no three of which are collinear with the exception of 8 points which are collinear. Find the number of straight lines and the number of triangles formed by them.

10. At an election there are five candidates and three members to be elected, and a voter is entitled to vote for any number of candidates not greater than the number to be elected. In how many ways, may a voter choose to vote. (C. U. 1935)

[Hints : ${}^5C_1 + {}^5C_2 + {}^5C_3 = 5 + 10 + 10 = 25$]

11. In a municipal corporation there are 20 councillors and 5 aldermen. How many committees can be formed consisting of 6 councillors and 3 aldermen. (C. U. 1933)

12. From 7 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include *at least* one lady? (C. U. 1957)

13. (a) In a group of 15 boys there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include (i) *exactly* 6 boy-scouts (ii) *at least* 6 boy-scouts? (C. U. 1943)

(b) A council consists of 10 members, 6 belonging to the party A and 4 to the party B. In how many ways can a committee of 5 be selected so that the members of the party A are in a majority? (R. U. 1953)

14. In how many ways can a committee of 2 ladies and 3 gentlemen be appointed from a meeting consisting of 10 ladies and 12 gentlemen? In how many of these Mr. X refuses to serve in a committee if Mrs. Y is a member?

15. A cricket club consists of 16 members of which only 6 can bowl. In how many ways can an eleven be chosen to include *at least* 4 bowlers?

16. An examination paper which is divided into two groups consisting of 3 and 4 questions respectively, carries the note "It is not necessary to answer all the questions. One question must be answered from each group."

In how many ways can an examinee select the questions? (A. U. 1950)

17. In how many ways can you invite eight friends, one or more at a time to dinner?

18. In how many ways can you draw a sum of money from a bag containing a rupee, an eight-anna piece and a-quarter piece?

19. In how many ways it is possible to make selections by taking some or all of 9 fruits, 3 of which are oranges, 2 lemons and the rest apples?

20. Out of 17 consonants and 5 vowels, how many different words can be formed each consisting of 3 consonants and 2 vowels? (C. U. 1939)

21. Find the number of words which can be formed with two different consonants and one vowel, out of 7 different consonants and 3 different vowels, the vowel to lie between two consonants?

[Hints : ${}^7C_2 \times {}^3C_1 \times 2 = 21 \times 3 \times 2 = 126$]

22. In order to pass an examination, a minimum is to be secured in each of seven subjects. In how many ways can a student fail? (A. U. 1952)

23. You are given 6 questions, each having an alternative. In how many ways can you answer one or more questions ?

24. Show that there are 2^{n-1} ways of selecting an odd number of things out of n things. (A. U. 1947)

25. How many combinations can be formed of 8 counters marked 1, 2, 3, 4, 5, 6, 7, 8 taking them 4 at a time, there being at least one odd and one even counter in each combination ? (C. U. 1941)

26. A boat's crew consists of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.

[Hints : 4 men must row on each side. But on one side there are already three men and on the other side two. So one must be placed on the side of three men and two on the other side. This can be done in 3C_1 or 3 ways.

Again, 4 men on *each* side can be arranged among themselves in $\underline{4}$ ways. Hence the required number

$$= 3 \times \underline{4} \times \underline{4} = 3 \times 24 \times 24 = 1728]$$

27. In how many ways can 22 people be divided into two cricket teams to play against each other in a friendly game ? (C. U. 1950)

28. In how many ways can 18 different books be divided equally among three students ? (C. U. 1951)

29. In how many ways can a party of 12 men divide themselves into two batches ? (C. U. 1960)

30. In how many ways can 52 cards be equally divided among 4 players ?

31. Find the number of combinations that can be made by taking 4 letters of the word "combination."

[Hints : There are 11 letters of 8 different kinds, $c, (0, 0), m, b, (i, i), (n, n), a, t$. The groups shall contain (i) 2 alike, 2 different (ii) 2 alike, 2 others alike (iii) all four different.

$$\therefore \text{the required number} = {}^3C_1 \times {}^7C_3 + {}^3C_2 + {}^3C_4 = 136]$$

Chapter VII

Logarithms

1. **Logarithm** is a very powerful mathematical tool for numerical calculations—multiplication, division, raising to powers and extraction of roots. John Napier (1550–1617) invented logarithm and Henry Briggs (1561–1630) simplified its application.

2. **Definition.** The logarithm of a number to a given base is the index of the power to which the base must be raised to produce the number. Thus if $a^n = b$, then n is called the logarithm of 'b' to the base 'a' and briefly written as $\log_a b = n$.

$$\text{Since } 2^6 = 64, \quad \therefore \log_2 64 = 6$$

$$4^3 = 64, \quad \therefore \log_4 64 = 3$$

$$8^2 = 64, \quad \therefore \log_8 64 = 2 \text{ etc.}$$

Note. Logarithms of the same number to different bases are different.

3. Properties of logarithm

Special properties

✓1. *Logarithm of any number to the same base is 1*

If 'a' is a finite quantity not equal zero,
then $a^1 = a \quad \therefore \log_a a = 1$.

2. *Logarithm of 1 to any base is 0*

We have $a^0 = 1$ (a being finite and $\neq 0$) $\therefore \log_a 1 = 0$.

4. Laws of logarithm

$$1. \log_a (m \times n) = \log_a m + \log_a n.$$

Logarithm of the product of two numbers is equal to the sum of their logarithms.

Put $\log_a(m \times n) = p$, $\log_a m = q$ and $\log_a n = r$,
Then from the definition,

$$a^p = m \times n, \quad a^q = m, \quad a^r = n.$$

We have, $a^p = m \times n = a^q \times a^r = a^{q+r}$.

$$\therefore p = q + r$$

$$\therefore \log_a(m \times n) = \log_a m + \log_a n.$$

Cor. $\log_a(m.n.p.....) = \log_a m + \log_a n + \log_a p + \dots$

$$(ii) \quad \log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n.$$

Logarithm of quotient of two numbers is equal to the difference of their logarithms.

Let $\log_a m = x$, $\log_a n = y$

Then by definition, $a^x = m$ and $a^y = n$.

$$\text{Now } \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}.$$

$$\therefore \log_a \left(\frac{m}{n} \right) = x - y = \log_a m - \log_a n$$

$$(iii) \quad \log_a(m)^n = n \cdot \log_a m.$$

Logarithm of the power of a number is equal to the product of the power and the logarithm of the number.

Let $\log_a m = x$, then by definition, $a^x = m$

$$\therefore (m)^n = (a^x)^n = a^{nx}$$

$$\therefore \log_a(m)^n = n \cdot x = n \cdot \log_a m.$$

5. Change of Base

If the logarithm of a number to any base is given, then the logarithm of the same number to any other base can be determined from the following relation

$$\log_a m = \log_b m \times \log_a b.$$

$$\text{or} \quad \log_b m = \frac{\log_a m}{\log_a b}$$

Let $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$,

Then by definition, $a^x = m$, $b^y = m$, and $a^z = b$.

Hence $a^x = b^y = (a^z)^y = a^{yz}$.

$\therefore x = yz$, or $\log_a m = \log_b m \times \log_a b$.

Cor 1. Putting $m = a$, we have

$$\log_a a = \log_b a \times \log_a b.$$

$\therefore \log_b a \times \log_a b = 1$.

This result can also be shown otherwise.

Let $\log_b a = x$ and $\log_a b = y$, then from definition, we have $b^x = a$ and $a^y = b$.

$\therefore b = a^{\frac{1}{x}} = a^y$.

Hence $\frac{1}{x} = y$ or $xy = 1$

i.e. $\log_b a \times \log_a b = 1$.

Hence to change logarithm of a number from base 'b' to base 'a', divide the given logarithm by $\log_a b$.

Cor 2. $\log_b a \times \log_c b \times \log_a c = 1$.

Since $\log_b a = \log_c a / \log_c b$

Left hand side = $\frac{\log_c a}{\log_c b} \times \log_c b \times \log_a c = \log_c a \times \log_a c = 1$

Examples

✓ **Ex. 1.** Find the logarithm of 1728 to the base $2\sqrt{3}$.
We have $1728 = 2^6 \times 3^3 = 2^6 \times (\sqrt{3})^6 = (2\sqrt{3})^6$
 $\therefore \log_{2\sqrt{3}} 1728 = 6$.

Ex. 2. Find the logarithm of $\frac{1}{324}$ to the base $3\sqrt{2}$.

• We have $\frac{1}{324} = \frac{1}{3^4 \cdot 2^2} = \frac{1}{3^4 (\sqrt{2})^4} = (3\sqrt{2})^{-4}$

$\therefore \log_{3\sqrt{2}} \left(\frac{1}{324} \right) = -4$.

✓ **Ex. 3.** Prove that $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1$

Let $u = \text{L.H.S.}$

Then $\log u$

$$\begin{aligned} &= \log (x^{\log y - \log z}) + \log (y^{\log z - \log x}) + \log (z^{\log x - \log y}) \\ &= (\log y - \log z) \log x + (\log z - \log x) \log y \\ &\quad + (\log x - \log y) \log z \\ &= 0 \end{aligned}$$

$\therefore u = a^0 = 1$ (a being any base $\neq 0$)

Ex. 4. Show that

$$\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$$

$$\begin{aligned} \text{L.H.S.} &= \log 2 + 16 \log \frac{2^4}{3 \times 5} + 12 \log \frac{5^2}{2^3 \times 3} + 7 \log \frac{3^4}{2^4 \times 5} \\ &= \log 2 + 16 (\log 2^4 - \log 3 - \log 5) \\ &\quad + 12 (\log 5^2 - \log 2^3 - \log 3) + 7 (\log 3^4 - \log 2^4 - \log 5) \\ &= \log 2 + 16 (4 \log 2 - \log 3 - \log 5) \\ &\quad + 12 (2 \log 5 - 3 \log 2 - \log 3) \\ &\quad + 7 (4 \log 3 - 4 \log 2 - \log 5) \\ &= (1 + 64 - 36 - 28) \log 2 + (-16 - 12 + 28) \log 3 \\ &\quad + (-16 + 24 - 7) \log 5 \\ &= \log 2 + 0 \times \log 3 + \log 5 = \log 2 + \log 5 = \log 10 = 1 \end{aligned}$$

Otherwise,

$$\begin{aligned} \text{L.H.S.} &= \log 2 + \log \left(\frac{16}{15} \right)^{16} + \log \left(\frac{25}{24} \right)^{12} + \log \left(\frac{81}{80} \right)^7 \\ &= \log \left\{ 2 \times \left(\frac{16}{15} \right)^{16} \times \left(\frac{25}{24} \right)^{12} \times \left(\frac{81}{80} \right)^7 \right\} \\ &= \log \left\{ 2 \times \left(\frac{2^4}{3 \times 5} \right)^{16} \times \left(\frac{5^2}{2^3 \times 3} \right)^{12} \times \left(\frac{3^4}{2^4 \times 5} \right)^7 \right\} \\ &= \log \left\{ 2 \times \frac{2^{64}}{3^{16} \times 5^{16}} \times \frac{5^{24}}{2^{36} \times 3^{12}} \times \frac{3^{28}}{2^{28} \times 5^7} \right\} \\ &= \log \frac{2^{65} \times 5^{24} \times 3^{28}}{2^{64} \times 5^{23} \times 3^{28}} = \log (2 \times 5) = \log 10 = 1 \end{aligned}$$

Note. The base of the logarithm is 10.

Ex. 5. Find the value of $\frac{2 \log 6 + 6 \log 2}{4 \log 2 + \log 27 - \log 9}$

$$\begin{aligned} \text{L. H. S.} &= \frac{2 \log (2 \times 3) + 6 \log 2}{4 \log 2 + \log 3^3 - \log 3^2} \\ &= \frac{2(\log 2 + \log 3) + 6 \log 2}{4 \log 2 + 3 \log 3 - 2 \log 3} \\ &= \frac{8 \log 2 + 2 \log 3}{4 \log 2 + \log 3} = \frac{2(4 \log 2 + \log 3)}{4 \log 2 + \log 3} = 2 \end{aligned}$$

Otherwise,

$$\begin{aligned} \text{L. H. S.} &= \frac{\log 6^2 + \log 2^6}{\log 2^4 + \log 27 - \log 9} = \frac{\log (6^2 \times 2^6)}{\log \left(\frac{2^4 \times 27}{9} \right)} \\ &= \frac{\log 48^3}{\log 48} = \frac{3 \log 48}{\log 48} = 3 \end{aligned}$$

6. Common Logarithms

Any number may be used as the base of logarithms. In the Napierian System the base is e , an incommensurable quantity of immense mathematical importance. In the Common System, introduced by Henry Briggs, the base is 10. The base 10 is usually omitted. *i.e.*, by the word 'logarithm' we generally mean 'common logarithm'.

7. Characteristic and Mantissa

The logarithm of any number has two parts, the integral and the decimal. The integral or the whole number is called the *characteristic* and the decimal portion is called the *mantissa*. The former is determined by inspection only, while the latter is obtained from logarithm table.

8. All numbers having the same significant digits and arranged in the same order have the same mantissa.

Let x and y be any two numbers having the same significant digits in the same order.

Then $x = 10^n y$ where n is positive or negative *integer*.

$$\therefore \log x = \log 10^n + \log y = n + \log y.$$

$\therefore$ the logarithms of x and y differ by an *integer* only and so they differ in characteristics and have the same mantissa.

This is illustrated by the following examples :

$\log$ of 2580 = 3.41162	$\log$ of 2.58 = 0.41162
$\log$ of 258 = 2.41162	$\log$ of 0.258 = 9.41162—10
$\log$ of 25.8 = 1.41162	$\log$ of 0.0258 = 8.41162—10

9. Positive Characteristics

Since, $10^0 = 1$	$\therefore \log 1 = 0$
$10^1 = 10$	$\therefore \log 10 = 1$
$10^2 = 100$	$\therefore \log 100 = 2$
$10^3 = 1000$	$\therefore \log 1000 = 3$ etc.

It is clear that if a number is a positive integral power of 10, its logarithm is a positive integer.

All numbers lying between 1 and 10, *i.e.*, numbers, with 1 digit in the integral part have their logarithms lying between 0 and 1. Therefore, their integral parts are 0 only.

All numbers lying between 10 and 100 have two digits in their integral parts. Their logarithms lie between 1 and 2. Therefore, numbers with two digits in their integral parts have 1 as characteristic.

In general, the logarithm of a number containing n digits only in its integral part is $(n-1) + \text{a fraction}$.

Thus the characteristic of the logarithm of a number greater than unity is one less than the number of digits in its integral part, and is positive.

For example, the characteristics of $\log 46$, $\log 6982$, $\log 2.95$ are 1, 3, 0 respectively.

10. Negative characteristics

$$10^0=1, \quad \therefore \log 1=0$$

$$10^{-1}=\frac{1}{10}=.1, \quad \therefore \log .1=-1$$

$$10^{-2}=\frac{1}{100}=.01, \quad \therefore \log .01=-2 \text{ etc.}$$

Obviously, if a number is a negative integral power of 10, its logarithm is a negative integer.

All numbers lying between 1 and .1 have logarithms lying between 0 and -1, *i.e.*, greater than -1 and less than 0. Since the decimal part is always written positive, the characteristic is -1.

All numbers lying between .1 and .01 have their logarithms lying between -1 and -2, Therefore, such numbers have -2 as characteristic of their logarithms.

In general, the logarithm of a number having n zeroes just after the decimal point is $-(n+1)+\text{a fraction}$.

Thus the characteristic of the logarithm of a number less than unity is one more than the number of zeroes just after the decimal point, and is negative.

For example, the characteristics of $\log .46$, $\log .059$, $\log .007808$ are respectively -1, -2, -3.

Negative characteristics are usually written as $\bar{1}$, $\bar{2}$, $\bar{3}$ etc. and read as bar 1, bar 2, bar 3 etc.

In the common system of logarithm, the characteristic of logarithm may be positive or negative but the mantissa is *always positive*.

A negative mantissa must be converted into a positive mantissa before reference to logarithm table.

$$\begin{aligned} \text{Thus } -2.8972 &= -3 + 3 - 2.8972 = -3 + (3 - 2.8972) \\ &= -3 + .1028 = \bar{3}.1028. \end{aligned}$$

Note here that the mantissa has been made *positive* but the characteristic is negative. Negative characteristic is shown by the bar (—) overhead.

Note further that -3.1028 is quite different from $\bar{3}.1028$.

✓H. How to read logarithm tables

The table consists of 3 sets of columns.

The *first set* contains only one column. This column consists of 2 digits from 10 to 99.

The *second set* contains 10 columns under the heads 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The *third set* contains 10 columns under the heads 1, 2, 3, 4, 5, 6, 7, 8, 9.

✓Ex. 6. Find $\log 23$.

The number has two digits. Its characteristic is $2-1$ or 1,

To find mantissa move along 23 horizontally and note down the number under 0 ($23=23.0$). The number found is 36173

$$\therefore \log 23 = 1.36173$$

✓Ex. 7. Find $\log 237$

The number has three digits. Its characteristic is $3-1$ or 2.

To find mantissa move along 23 horizontally and note the number under 7. It is 37475

$$\therefore \log 237 = 2.37475$$

Ex. 8. Find $\log 2376$.

The characteristic $\bar{4}-1=3$. Move along 23 horizontally and note the number under 7 as before. For the 4th. digit consult the *mean difference table* and note the number under 6. This is 111. It is really .00111 in the 5-figure table. The cipher or ciphers (zeroes) after the

Ex. 13. Find the value of 578.25×89.529

Let $x = 578.25 \times 89.529$

Then $\log x = \log 578.25 + \log 89.529$

$$= 2.76212 + 1.95196$$

$$= 4.71408 = \log 51771$$

(The characteristic 4 determines the position of decimal point after 5 digits from the left)

$$\therefore x = 51771.$$

The scheme of work may be shown as below

$$\log 578.25 = 2.76212$$

$$\log 89.529 = 1.95196$$

$$4.71408$$

$$\text{antilog } 4.71408 = 51771$$

(2). Division. To divide one number by another, subtract the logarithm of the latter from the logarithm of the former. The difference of the logarithms is the logarithm of the quotient.

Ex. 14. Find the value of $578.25 \div 89.529$

$$\log 578.25 = 2.76212$$

$$\log 89.529 = 1.95195$$

$$\text{antilog } \overline{0.81017}$$

$$= 6.4591$$

The required quotient = 6.4591.

(3). Involution. To raise a given number to any power, multiply the logarithm of the number by the index of the power. The product is the logarithm of the power.

Ex. 15. Find the value of $(2.381)^5$

Let $x = (2.381)^5$

Then $\log x = 5 \log 2.381 = 5 \times 0.37677 = 1.88385$

$$= \log 76.535$$

$$\therefore x = 76.535$$

(4). **Evolution.** To extract any root of a given number, divide the logarithm of the number by the index of the root. The quotient is the logarithm of the root.

Ex. 16. Find the value of $(789.45)^{\frac{1}{8}}$

Let $x = (789.45)^{\frac{1}{8}}$

$$\begin{aligned}\text{Then } \log x &= \frac{1}{8} \cdot \log 789.45 = \frac{1}{8} \times 2.89733 \\ &= 0.36217 = \log 2.3023\end{aligned}$$

$$\therefore x = 2.3023$$

Ex. 17. Simplify $\frac{(74.37)^{\frac{1}{8}} - (952.6)^{\frac{1}{10}}}{(74.37)^{\frac{1}{8}} \div (952.6)^{\frac{1}{10}}}$

Let $x = (74.37)^{\frac{1}{8}}$

$$\begin{aligned}\text{Then } \log x &= \frac{1}{8} \log 74.37 = \frac{1}{8} \times 1.87140 = 0.233925 \\ &= \log 2.0507\end{aligned}$$

$$\therefore x = 2.0507$$

Let $y = (952.6)^{\frac{1}{10}}$

$$\begin{aligned}\text{Then } \log y &= \frac{1}{10} \log 952.6 = \frac{1}{10} \times 2.97891 = 0.297891 \\ &= \log 1.9855\end{aligned}$$

$$\therefore y = 1.9855.$$

$\therefore$ the required exp.

$$= \frac{2.0507 - 1.9855}{2.0507 \div 1.9855} = \frac{0.0652}{2.0507 \div 1.9855}$$

Let P be this expression, then

$$\begin{aligned}\log P &= \log 0.0652 - (\log 2.0507 - \log 1.9855) \\ &= \bar{2}.81425 - (0.31190 - 0.29788) \\ &= \bar{2}.81425 - 0.01402 \\ &= \bar{2}.80023 = \log .063129\end{aligned}$$

$$\therefore P = .063129$$

Ex. 18. Find the value of $\frac{1}{6} \cdot \sqrt{3 \log \frac{1728}{1 + \frac{1}{2} \log 36 + \frac{1}{3} \log 8}}$
without reference to table. (C. U. 1962)

we have $3 \log 1728$

$$\begin{aligned} &= 3 \log (2^6 \times 3^3) = 3(\log 2^6 + \log 3^3) \\ &= 3(6 \log 2 + 3 \log 3) = 9(2 \log 2 + \log 3) \end{aligned}$$

Also $1 + \frac{1}{2} \log 36 + \frac{1}{3} \log 8$

$$= 1 + \frac{1}{2} \log \frac{36}{100} + \frac{1}{3} \log 2^3$$

$$= 1 + \frac{1}{2} (\log 6^2 - \log 10^2) + \frac{1}{3} \cdot 3 \log 2$$

$$= 1 + \frac{1}{2} (2 \log 6 - 2 \log 10) + \log 2$$

$$= 1 + \log 6 - \log 10 + \log 2$$

$$= \log (2 \times 3) + \log 2$$

$$= \log 2 + \log 3 + \log 2$$

$$= 2 \log 2 + \log 3$$

$\therefore$ the required expression

$$= \frac{1}{6} \cdot \frac{\sqrt{9(2 \log 2 + \log 3)}}{2 \log 2 + \log 3} = \frac{1}{6} \sqrt{9} = \frac{1}{6} \times 3 = \frac{1}{2}$$

Ex. 19. Given $\log 2 = .3010300$ and $\log 3 = .4771213$,
find the value of $\log 13.5$.

$$\begin{aligned} \log 13.5 &= \log \frac{135}{10} = \log \frac{27}{2} = \log 27 - \log 2 \\ &= \log 3^3 - \log 2 = 3 \log 3 - \log 2 \\ &= 3(.4771213) - .3010300 \\ &= 1.4313639 - .3010300 = 1.1303339 \end{aligned}$$

Ex. 20. Find the number of digits in $(8.761)^{15}$.

$$\text{Let } x = (8.761)^{15},$$

$$\therefore \log x = 15 \log 8.761 = 15 \times 0.94255 = 14.13825$$

As the characteristic is 14, the number must have
15 digits in the integral part.

Ex. 21. Find the values of (a) $\frac{1}{\sqrt[7]{36 \cdot 21}}$
and (b) $30\{(1+0.035)^{15}-1\}$ (C.U. 1961)

(a) Let $x = \frac{1}{\sqrt[7]{36 \cdot 21}} = (36 \cdot 21)^{-\frac{1}{7}}$

$$\therefore \log x = -\frac{1}{7} \log 36 \cdot 21 = -\frac{1}{7} \times 1.55883$$

$$= -0.22269 = \bar{1}.77731 = \log .59883$$

$$\therefore x = .59883.$$

(b) Let $x = (1+0.035)^{15} = (1.035)^{15}$

$$\therefore \log x = 15 \log 1.035 = 15 \times 0.01496 = 0.2244$$

$$= \log 1.6764$$

$$\therefore x = 1.6764$$

$$\therefore \text{the required expression} = 30\{1.6764-1\}$$

$$= 30 \times .6764 = 20.292.$$

Exercise 9

1. Find, by inspection, the characteristics of the logarithms of

7940, 48.970, 7.38, .00438.

2. Find the logarithms of

896, 0.7508, $\frac{85.8}{27.9}$, $(1.035)^3$, $(1.035)^{-7}$.

3. Find the numbers whose logarithms are

4.872, -1.316 , 0.3782

4. Given $\log 2 = .3010300$, $\log 3 = .4771213$, find out the value of $\log 45$ (C.U. 1951)

5. Find the logarithm of

(i) 784 to the base $2\sqrt{7}$

(ii) 19683 to the base $3\sqrt{3}$

(iii) 1.728 to the base 1.2

6. Prove that $\log \frac{x}{y} + \log \frac{y}{z} + \log \frac{z}{x} = 0$

7. Prove that $\log_b a \times \log_c b \times \log_a c = \log_a a$.

(C.U. 1942)

8. Show that

(i) $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} + \log \frac{1}{2} = 0$

(ii) $\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243} = \log 2$. (C.U. 1951)

(iii) $\log \frac{384}{5} + \log \frac{81}{32} + 3 \log \frac{5}{3} - \log 9 = 2$

9. Evaluate

(i) $\frac{2 \log 2 + \log 9}{\log 3 + \log 2}$ (ii) $\frac{\log 125 + \log 54 - \log 2}{1 - \log 2 + \log 3}$

10. Find the value of

(i) $\frac{1}{6} \sqrt{\frac{3 \log 1728}{\frac{1}{2} \log 36 + \frac{1}{3} \log 8}}$

(ii) $\frac{3}{2} \sqrt{\frac{2(\log 6561 + \log 65536)}{\log 720 + \log 4 - \log 20}}$

11. Evaluate

(i) $\frac{400}{.06} \left[1 - \frac{1}{(1.06)^4} \right]$ (ii) $\frac{45}{.04} \{ (1.05)^{10} - 1 \}$

(iii) $\sqrt[7]{\frac{1}{1.235}}$ (P.U. 1948)

12. Find the values of

(a) $\frac{1}{\sqrt{36.21}}$ and (b) $30 \{ (1 + .035)^{18} - 1 \}$ (C. U. 1961)

13. Prove that

(i) $a^{\log b - \log c} \times b^{\log c - \log a} \times c^{\log a - \log b} = 1$

(ii) $\log_3 \log_3 512 = 2$

14. Simplify by using log. tables

(i) $\frac{.0405 \div 1.96 \times .00128}{.016 \times .081 \times 4.9}$

$$(ii) \sqrt[3]{\frac{9268 \times 4.573 \times .0964}{87.65 \times .5432}}$$

$$(iii) \frac{\sqrt[3]{.4165} \times \sqrt[5]{.8264}}{\sqrt[3]{.6438} \times \sqrt[3]{.0186}} \quad (P. U. 1945)$$

15. Find the number of digits in

$$(i) (3.896)^{18}, \quad (ii) (24098.62)^{\frac{1}{5}}$$

16. Evaluate

$$(i) \sqrt[3]{389.65} + \sqrt[3]{8231.5}$$

$$(ii) \frac{\sqrt{85.82} - \sqrt[4]{91.72}}{\sqrt[5]{125.7}}$$

17. Find n if

$$(i) (1 + .05)^n = 2 \quad (ii) 245.37(.82)^n = 200.$$

18. Find i if

$$(i) (1+i)^{20} = 3 \quad (ii) 75 = 60(1+i)^4$$

19. Solve for x , given

$$\frac{x}{(1.04)^7} + \frac{x}{(1.04)^8} = 20,000$$

20. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, show that

$$(i) xyz = 1 \quad (ii) x^x \cdot y^y \cdot z^z = 1.$$

Chapter VIII

SIMPLE INTEREST AND COMPOUND INTEREST

1. Interest is the consideration received by the lender from the borrower for the use of money lent. The money lent is called the *principal* and the sum of the principal and interest accrued on it is termed *amount*.

The interest per cent (hundred) per annum (year) is called the rate. Interest may be simple or compound.

2. Simple interest. In the case of simple interest only the original money earns interest.

If P be the principal, n the number of years, r the rate per cent per annum, I the interest accrued and A the amount, then

$$I = \frac{P \cdot n \cdot r}{100} \text{ and } A = P + I = P \left(1 + \frac{n \cdot r}{100} \right)$$

If i be the rate of interest per unit per year, then

$$i = \frac{r}{100} \text{ and } I = P \cdot n \cdot i, \quad A = P(1 + ni) \dots (1)$$

3. Compound interest

In the case of compound interest, each instalment of as interest and when due is added to the principal. This is unlike simple interest where interest is allowed to accumulate without itself earning any interest. The period after which interest is due is called *interest-period* or *period of rest*. This is also termed *conversion period*, for after every such period the interest is converted into or compounded with the principal. The interest-period may a year, halfyear, quarter-year, month and so on.

4. Formula for Compound Interest. If P be the principal, i the rate of interest per unit per period, n the

number of interest periods, then the amount A is given by

$$A = P(1+i)^n \dots \dots \dots (2)$$

5. The formula for compound interest admits of logarithmic computation. Taking logarithm we have

$$\log A = \log P + n \log (1+i)$$

The formula involves 4 quantities A , P , n , i .

Given any three of them, the fourth can be determined.

$$\text{Thus } \log P = \log A - n \log (1+i)$$

$$\log (1+i) = \frac{\log A - \log P}{n}$$

$$n = \frac{\log A - \log P}{\log (1+i)}$$

6. Effective Rate

If the rate of interest is 4% *p.a.*, then interest on 100 for 1 year is 4, if the interest is payable yearly. Hence the amount after one year is 104.

But if the interest is payable halfyearly, the interest for first halfyear is 2. The compounded capital after first halfyear is 102. For the second half, interest will be due on 102 and it will be 2.04. Hence the amount after second half-year = $102 + 2.04 = 104.04$

$$\therefore \text{the effective rate of interest percent per year} \\ = 104.04 - 100 = 4.04.$$

The flat rate of interest (4% here) is called *nominal rate* and the true rate of interest (4.04%) is called *effective rate*.

7. Formula for Effective Rate

If i be the nominal rate of interest per unit per year, and interest is convertible q times a year, then the effective rate of interest is

$$100 \left\{ \left(1 + \frac{i}{q} \right)^q - 1 \right\}$$

In particular, if interest is convertible halfyearly, effective rate is $100 \left\{ \left(1 + \frac{i}{2} \right)^2 - 1 \right\}$

Let P be the principal. Then

For the 1st period. Principal = P ,

interest for the 1st. period = $P.i$

$\therefore$ amount after 1st. period = $P + P.i = P(1+i)$

For the 2nd period. Principal = $P(1+i)$,

interest for the 2nd. period = $P(1+i).i$.

$\therefore$ amount after 2nd. period = $P(1+i) + P(1+i).i$
 $= P(1+i)(1+i) = P(1+i)^2$.

Proceeding thus, amount after n th period = $P(1+i)^n$.

$\therefore A = P(1+i)^n$.

Otherwise. Principal P amounts to $P(1+i)$ after 1st. period.

Thus the amount after one period = Principal $\times (1+i)$

So, amount after 2nd. period = $P(1+i) \times (1+i) = P(1+i)^2$.

Proceeding thus, the amount after n periods = $P(1+i)^n$.

In simple interest, interest is calculated directly on Principal. In compound interest, at first amount is calculated and compound interest is obtained as a difference

Thus **compound interest** = $A - P = P\{(1+i)^n - 1\}$.

If the interest is compounded **annually**, .

$$A = P(1+i)^n$$

If the interest is compounded **halfyearly**, $A = P\left(1 + \frac{i}{2}\right)^{2n}$

If the interest is compounded **quarterly**, $A = P\left(1 + \frac{i}{4}\right)^{4n}$

Generally, amount A after n conversion periods is $A = P(1+i)^n$

The above formula may also be written as

$A = P\left(1 + \frac{i}{q}\right)^{q.n}$, where the interest is compounded q times

a year, $\frac{i}{q}$ = interest per unit per period and $q.n$ = number of periods.

If interest is convertible quarterly, effective rate is

$$100 \left\{ \left(1 + \frac{i}{4} \right)^4 - 1 \right\}$$

8. Present value

If the principal P amounts to A in n years, then P is also called the *present value* or *present worth* of A due in n years.

Since $A = P(1+i)^n$

$$\therefore P = \frac{A}{(1+i)^n}$$

Treated logarithmically, it becomes

$$\log P = \log A - n \log (1+i)$$

Note. Unless otherwise stated interest is payable yearly

9. True Discount. The true discount on a sum due in n years is the interest on its present value.

Ex. 1. Find the compound interest on Rs. 10,000/- 4 years at 5% per annum. What will be the simple interest in the above case?

We have $A = P(1+i)^n$

Here $P = 10,000$, $n = 4$, $i = \frac{5}{100} = 0.05$.

$$\therefore A = 10,000 (1+0.05)^4 = 10,000 (1.05)^4$$

$$= 10,000 \times 1.215$$

$$= 12150$$

C. I = Compound interest

$$= A - P = \text{Rs } 2150/-$$

S. I = Simple interest

$$= P.n.i = 10,000 \times 4 \times \frac{5}{100}$$

$$= \text{Rs } 2000/-$$

$$\begin{aligned} x &= (1.05)^4 \\ \log x &= 4 \log 1.05 \\ &= 4 \times 0.0212 \\ &= 0.0848 \\ &= \log 1.215 \\ \therefore x &= 1.215 \end{aligned}$$

The compound interest can also be found out 'without logarithm. But its scope is rather limited.

Rs.

✓ Original Principal	= 10 000
Add int. 1st year	= 500
Principal 2nd year	= 10,500
Add int. 2nd year	= 525
Principal 3rd year	= 11,025
Add int. 3rd year	= 551.25
Principal 4th year	= 11,576.25
Add int. 4th year	= 578.8125
Amount after 4th year	= 12,155.0625
Less Principal	= 10,000
∴ C. I.	= 2155.0625
	= Rs 2155.06

Note 1. *Logarithm calculation gives only an approximate value.*

Note 2. *To obtain the result correct to the nearest nP. (to 2 decimal places) calculation is to be made correct to 4 decimal places.*

Ex. 2. Find the compound interest on Rs 6,950/- for 3 years. if interest is payable halfyearly, the rate for the first two years being 6% p.a, and for the third year 9% p.a. (C. U. 1928)

We have $A = P \left(1 + \frac{i}{2}\right)^{2n}$,

Hence $P = 6,950$, $n = 2$, $i = \frac{6}{100} = .06$

∴ $A = 6,950 \left(1 + \frac{.06}{2}\right)^{2 \times 2} = 6,950(1.03)^4$.

• Taking log,

$\log A = \log 6950 + 4 \log 1.03$

$= 3.8420 + 4 \times 0.0128$

$= 3.8420 + 0.0512 = 3.8932 = \log 7820$

∴ $A = 7820$, which is the principal for the third year.

$$\text{Again, } A = 7820 \left(1 + \frac{.09}{2}\right)^{2 \times 1},$$

$$(P = 7820, i = \frac{9}{100} = .09, n = 1)$$

$$\text{or } A = 7820(1.045)^2$$

Taking log,

$$\begin{aligned} \log A &= \log 7820 + 2 \log 1.045 = 3.8932 + 2 \times 0.0191 \\ &= 3.8932 + 0.0382 = 3.9314 = \log 8539 \end{aligned}$$

$$\therefore A = 8539$$

$$\therefore \text{C. I} = 8539 - 6950 = \text{Rs } 1589$$

Note : Instead of using logarithm in two stages, we may use it once only. Thus

$$A = 6950 (1.03)^4 (1.045)^2$$

$$\begin{aligned} \therefore \log A &= \log 6950 + 4 \log 1.03 + 2 \log 1.045 \\ &= 3.8420 + 4 \times 0.0128 + 2 \times 0.0191 \\ &= 3.8420 + 0.0512 + 0.0382 = 3.9314 = \log 8539 \end{aligned}$$

$$\therefore A = 8539$$

$$\therefore \text{C. I} = \text{Rs } 1589.$$

Otherwise	Rs
Principal, 1st half year	= 6950
Int. halfyear	= 208.50 1% = 69.50, 3% = 208.50
Principal, 2nd half	= 7158.50
Int. halfyear	= 214.7550 1% = 71.585, 3% = 214.7550
Principal, 3rd half	= 7373.2550
Int. halfyear	= 221.1978 1% = 73.7326, 3% = 221.1978
Principal, 4th half	= 7594.4528
Int. halfyear	= 227.8335 1% = 75.9445, 3% = 227.8335
Principal, 5th half	= 7822.2863
Int. halfyear	= 352.0032 1% = 78.2229, 4½% = 352.0032
Principal, 6th half	= 8174.2895
Int. halfyear	= 367.8432 1% = 81.7429, 4½% = 367.8432
Amount after 4th year	= 8542.1327
Less principal	= 6950
C. I	= 1592.1327
$\therefore \text{C. I}$	= Rs 1592.13.

Ex. 3. Which is the better investment, simple interest at 5% per annum or compound interest at 4% per annum payable quarterly?

Let principal = 100.

Then simple interest on 100 for 1 year = 5.

Compound interest.

$$A = P \left(1 + \frac{i}{4}\right)^{4 \times n} = 100 \left(1 + \frac{0.05}{4}\right)^{4 \times 1} = 100(1.0125)^4.$$

$$\text{Here } P = 100, i = \frac{5}{100} = 0.05, n = 1.$$

$$\begin{aligned} \therefore \log A &= \log 100 + 4 \log 1.0125 \\ &= 2 + 4 \times 0.0054 = 2 + 0.0216 = \log 105.1 \end{aligned}$$

$$\therefore A = 105.1$$

$$\therefore C.I. = 105.1 - 100 = 5.1.$$

Hence effective rate of compound interest = 5.1% p.a.

$\therefore$ rate of C.I. is higher than that of S.I.

Ex. 4. What is the present value of Rs 1000 due in 2 years @ 5% p.a., C.I. according as the interest is paid (a) yearly or (b) half-yearly? (P.U.1945)

$$(a) A = P(1+i)^n, \text{ Here } A = 1000, i = \frac{5}{100} = 0.05, n = 2$$

$$\therefore P = \frac{A}{(1+i)^n} = \frac{1000}{(1.05)^2}$$

$$\begin{aligned} \therefore \log P &= \log 1000 - 2 \log 1.05 \\ &= 3 - 2 \times 0.0212 = 3 - 0.0424 = 2.9576 = \log 906.9 \end{aligned}$$

$$\therefore P = \text{Rs. } 906.90$$

[or, $\log A = \log P + n \log (1+i)$

$$\therefore \log 1000 = \log P + 2 \log 1.05$$

$$\therefore \log P = \log 1000 - 2 \log 1.05 \text{ etc.]}$$

$$(b) A = P \left(1 + \frac{i}{2}\right)^{2n}. \text{ Here } A = 1000, \frac{i}{2} = \frac{0.05}{2} = 0.025,$$

$$n = 2$$

$$\therefore P = \frac{1000}{(1.025)^{2 \times 2}}$$

$$\therefore \log P = \log 1000 - 4 \log 1.025$$

$$= 3 - 4 \times 0.0107 = 3 - 0.0428 = 2.9572 = \log 906.1$$

$$\therefore P = \text{Rs. } 906.10.$$

Ex. 5. If the difference between the simple and compound interest on a certain sum invested for 8 years at 4 p. c. p. a is Rs. 25.75, find the sum.

$$\text{The simple interest on 1} = \frac{1 \times 8 \times 4}{100} = \frac{8}{25} = .32$$

Amount on 1 at compound interest

$$= 1(1+i)^n = (1.04)^8 = 1.368$$

$$\therefore \text{compound interest on 1}$$

$$= 1.368 - 1 = .368$$

$$\therefore \text{difference between C.I. and S.I.,}$$

$$= .368 - .32$$

$$= .048$$

$$\begin{aligned} x &= (1.04)^8 \\ \log 1.04 &= .0170 \\ &\quad \times 8 \\ \log x &= .1360 \\ x &= 1.368 \end{aligned}$$

Difference

.048

Rs. 25.75

Sum

1

?

$$? = \text{Rs. } \frac{25.75}{.048} = \text{Rs. } 536.46.$$

Ex. 6. What sum at compound interest will amount to £ 1210 at the end of the second year and £ 1331 at the end of the third year? What is the rate of interest per cent per annum?

We have $A = P(1+i)^n$, where A = amount, P = principal, i = rate per unit per year and n = number of years.

$\therefore$ from the given conditions,

$$1210 = P(1+i)^2 \dots \dots \dots (1)$$

$$1331 = P(1+i)^3 \dots \dots \dots (2)$$

Dividing (2) by (1), we have

$$\frac{1331}{1210} = 1+i, \text{ or } 1+i = \frac{11}{10} \dots \dots (3)$$

$$\therefore \text{ from (1) } 1210 = P \left(\frac{11}{10} \right)^2$$

$$\text{or } P = 1210 \times \left(\frac{10}{11} \right)^2 = 1000$$

$$\text{Since } i = \frac{11}{10} - 1 = \frac{1}{10}$$

$$\text{The required rate per cent} = 100 \times i = 10$$

Ex. 7. C. P. Mookherjee left Rs 20,000 to be divided between his son Amit and daughter Arati. Arati's share was to amount to a certain sum of money at the end of 5 years; and Amit's share was to amount to an equal sum at the end of 7 years. Calculate the sum of money each should receive if the interest rate was 4% per annum. (C. U. 1960)

Let each receive Rs x

Then Amit could receive Rs x after 7 years.

$$\therefore \text{ present value of Rs } x \text{ due 7 years after} = \text{Rs } \frac{x}{(1.04)^7}$$

$$\left[P = \frac{A}{(1+i)^n} \quad \text{here } A=x, i=\frac{4}{100}=0.04, n=7 \right]$$

Arati would receive Rs x after 5 years

$$\therefore \text{ present value of Rs } x \text{ due 5 years after} = \text{Rs } \frac{x}{(1.04)^5}$$

Sum of these present values = Rs 20,000

$$\therefore \frac{x}{(1.04)^7} + \frac{x}{(1.04)^5} = 20,000$$

$$\text{or } x \{ (1.04)^{-7} + (1.04)^{-5} \} = 20,000$$

$$\begin{aligned} a &= (1.04)^{-7} \\ \log a &= -7 \log 1.04 \\ &= -7 \times 0.0170 \\ &= -0.1190 \\ &= -1 + (1 - 0.1190) \\ &= \bar{1}.8810 \\ \therefore a &= 0.7603 \end{aligned}$$

$$\begin{aligned} b &= (1.04)^{-5} \\ \log b &= -5 \log 1.04 \\ &= -5 \times 0.0170 \\ &= -0.0850 \\ &= -1 + (1 - 0.0850) \\ &= \bar{1}.9150 \\ \therefore b &= 0.8222 \end{aligned}$$

$$\text{or } x(7603 + 8222) = 20,000 \quad \text{or } x \times 15825 = 20,000$$

$$\therefore x = \frac{20,000}{15825} = \text{Rs. } 12701. \text{ (app)}$$

Ex. 8. What is the effective rate of interest corresponding to the nominal rate of 5 per cent per annum convertible (i) halfyearly (ii) quarterly?

(i) Let principal = 100
Add halfyearly int. = 2.5

Amount after 1st halfyear = 102.5
Add halfyearly int. = 2.5625 $1\% = 1.025$, $2\frac{1}{2}\% = 2.5625$

Amount after 1 year = 105.0625
Less principal = 100

C.I. = 5.0625
 $\therefore$ the effective rate = 5.0625%

Otherwise. $A = P \left(1 + \frac{i}{2} \right)^{2n}$

Here $P = 100$, $i = \frac{5}{100} = .05$, $n = 1$

$\therefore A = 100(1 + .025)^{2 \times 1}$
 $= 100(1.025)^2$
 $= 100 \times 1.051 = 105.1$

$\therefore$ C.I. = $105.1 - 100 = 5.1$
 $\therefore$ the effective rate = 5.1%

(ii) Let principal = 100
Add quarterly int. = 1.25

Amount after 1st quarter = 101.25
Add quarterly int. = 1.2656 $1\% = 1.0125$, $1\frac{1}{4}\% = 1.2656$

Amount after 2nd quarter = 102.5156
Add quarterly int. = 1.2815 $1\% = 1.0252$, $1\frac{1}{4}\% = 1.2815$

Amount after 3rd quarter = 103.7971
Add quarterly int. = 1.2975 $1\% = 1.0380$, $1\frac{1}{4}\% = 1.2975$

Amount after 1 year = 105.0946
Less principal = 100

C. I. = 5.0946
 $\therefore$ the effective rate = 5.0946%

$x = (1.025)^2$ $\log 1.025 = 0.0107$ $\times 2$ $\log (1.025)^2 = 0.0214$ $x = 1.051$
--

Otherwise. $A = P \left(1 + \frac{i}{4} \right)^{4n}$

Here $P=100$, $i = \frac{5}{100} = .05$, $n=1$.

$$\begin{aligned} \therefore A &= 100(1 + .0125)^{4 \times 1} \\ &= 100(1.0125)^4 \\ &= 100 \times 1.0511 \\ &= 105.11 \end{aligned}$$

$$\begin{aligned} \therefore C.I. &= 105.11 - 100 \\ &= 5.11 \end{aligned}$$

$x = (1.0125)^4$
$\log 1.0125 = 0.00541$
$\times 4$
$\log x = 0.02164$
$\therefore x = 1.0511$

$\therefore$ the effective rate $= 5.11\%$

✓ **Ex. 9.** In what time will a sum of Rs. 1234 amount to Rs. 5678 at 8% *p. a.* compound interest, payable quarterly?

(given $\log 1.234 = 0.0913$, $\log 56.78 = 1.7542$, $\log 10.2 = 1.0086$)

We have $A = P \left(1 + \frac{i}{4} \right)^{4n}$

$$\therefore 5678 = 1234 \left(1 + \frac{.08}{4} \right)^{4n} = 1234 (1.02)^{4n}$$

Taking log,

$$\log 5678 = \log 1234 + 4n \log 1.02$$

$$\text{or } 4n \log 1.02 = \log 5678 - \log 1234$$

$$\therefore n = \frac{\log 5678 - \log 1234}{4 \log 1.02} = \frac{3.7542 - 3.0913}{4 \times 0.0086}$$

$$= \frac{0.6629}{0.0344} = 19.27 \text{ years.}$$

Note. In finding time no antilog. table is required.

Ex. 10. Find the number of years and the fraction of a year in which a sum of money will treble itself at compound interest at 4 per cent per annum. (C.U. 1935)

We have $A = P(1+i)^n$.

Let $P=100$, then $A=300$, $i = \frac{4}{100} = .04$,

$$\therefore 300 = 100(1.04)^n \text{ or } 3 = (1.04)^n$$

Taking log,

$$n \log 1.04 = \log 3$$

$$\therefore n = \frac{\log 3}{\log 1.04} = \frac{.4771}{.0170} = \frac{4771}{170} = 28.06 \text{ years.}$$

Ex. 11. S. Sen borrowed Rs 9000 from a moneylender but he could not repay any amount in a period of 4 years. Accordingly the money lender demands now Rs11,250 from him. What rate per cent per annum compound interest did the later require for lending his money?

(C.U. 1957)

We have $A = P(1+i)^n$.

Here $A = 11,250$ $P = 9000$, $n = 4$, $i = ?$.

$$\therefore 11,250 = 9000(1+i)^4.$$

$$\text{or } 1,125 = 900(1+i)^4$$

Taking log,

$$\log 1,125 = \log 900 + 4 \log (1+i)$$

$$\text{or } 3.0512 = 2.9542 + 4 \log (1+i)$$

$$\text{or } 4 \log (1+i) = .0970$$

$$\therefore \log (1+i) = \frac{.0970}{4} = .0243 = \log 1.058$$

$$\therefore 1+i = 1.058 = 1 + .058$$

$$\therefore i = .058$$

Hence the required rate per cent

$$= 100 \times i = 100 \times .058 = 5.8$$

Ex. 12. (a) What is the rate per cent per annum if a sum doubles itself in 17 years at compound interest?

(C. U. 1947)

(b) What is the nominal rate per cent per annum, when interest is payable half yearly corresponding to the effective rate of 6% per annum?

(a) We have $A = P(1+i)^n$. here $A = 2P$, $n = 17$, $i = ?$.

$$\therefore 2P = P(1+i)^{17}$$

$$\text{or } 2 = (1+i)^{17}$$

Taking log,

$$17 \log (1+i) = \log 2$$

$$\therefore \log (1+i) = \frac{\log 2}{17} = \frac{.3010}{17} = .0177 = \log 1.042$$

$$\therefore 1+i = 1.042 = 1 + .042$$

$$\text{or } i = .042$$

$$\therefore \text{the rate per cent} = 100 \times i = 100 \times .042 = 4.2$$

(b) Let i be the nominal rate per unit per annum.
Then since interest is convertible halfyearly,

$$\left(1 + \frac{i}{2}\right)^2 = 1.06$$

$$\therefore 1 + \frac{i}{2} = \sqrt{1.06} = 1.0296 = 1 + .0296$$

$$\text{or } \frac{i}{2} = .0296$$

$$\text{or } i = .0592$$

$$\therefore \text{the nominal rate per cent per annum} \\ = 100 \times i = 100 \times .0592 = 5.92$$

10. Uniform increase or growth

The formula for compound interest viz $A = P(1+i)^n$ in equally applicable to all cases of uniform increase or growth.

11. Uniform decrease or depreciation

In case of uniform decrease ' i ' is to be substituted by ' $-i$ ' and the formula is reduced to

$$A = P(1-i)^n$$

Ex. 13. If the annual increase in the population of a state is 25 per thousand and the present number of inhabitants is 26, 24, 000, what will the population be in three years' time? What was it a year ago?

(C. U. 1940)

$$25 \text{ per thousand} = \frac{25}{1000} \text{ per unit} = .025 \text{ per unit}$$

$$\text{We have } A = P(1+i)^n.$$

In the first case, $P = 26,24,000$, $i = .025$, $n = 3$, $A = ?$

$$\therefore A = 26,24,000 (1.025)^3$$

Taking log,

$$\begin{aligned} \log A &= \log 26,24,000 + 3 \log 1.025 \\ &= 6.4190 + 3 \times .0107 = 6.4190 + .0321 = 6.4511 \end{aligned}$$

$$\therefore A = 28,26,000$$

In the second case, $A = 28,26,000$, $n = 3$, $i = .025$, $P = ?$

$$\therefore 28,26,000 = P(1+i)^n = P(1.025)^3$$

$$\text{or } P = \frac{28,26,000}{1.025^3} = 25,60,000$$

Ex. 14. If the population of a town increases every year by 1.5 p.c. of the population at the beginning of that year, in how many years will the total increase be 25 per cent? (30 %)

$$\text{We have } A = P(1+i)^n.$$

$$\text{If } P = 100, \text{ then } A = 130, i = \frac{1.5}{100} = .015, n = ?$$

$$\therefore 130 = 100 (1.015)^n.$$

$$\text{or } (1.015)^n = \frac{130}{100} = 1.3$$

Taking log,

$$n \log 1.015 = \log 13 - \log 10.$$

$$\therefore n = \frac{\log 13 - \log 10}{\log 1.015} = \frac{1.1139 - 1}{.0064}$$

$$= \frac{.1139}{.0064} = \frac{1139}{64} = 17.8 \text{ years}$$

Ex. 15. A machine, the life of which is estimated to be 10 years, costs Rs 5600. Calculate its scrap value at the end of its life, depreciation on the reducing instalment system being charged at 10% per annum.

We have $A = P(1-i)^n$.

Here $P = 5,600$, $i = \frac{10}{100} = \frac{1}{10}$, $n = 10$

$$\therefore A = 5,600 \left(1 - \frac{1}{10}\right)^{10} = 5,600 \left(\frac{9}{10}\right)^{10}$$

Taking log,

$$\begin{aligned} \log A &= \log 5,600 + 10 \log \frac{9}{10} \\ &= \log 5,600 + 10 (\log 9 - \log 10) \\ &= 3.7482 + 10 (.9542 - 1) \\ &= 3.7482 + 9.542 - 10 \\ &= 13.2902 - 10 = 3.2902 \end{aligned}$$

$$\therefore A = \text{Rs } 1951$$

Ex. 16. A machine becomes depreciated in such a way that the value of the machine at the end of any year is 90% of the value at the beginning of the year. The cost of the machine is £20, and it was sold eventually as waste metal for £ 7 10s. Obtain the number of years during which the machine was in use. (C. U. 1936)

We have $A = P(1-i)^n$.

Here $P = 20$, $A = 7.5$, $i = \frac{10}{100} = \frac{1}{10}$, $n = ?$

$$\therefore 7.5 = 20 \left(1 - \frac{1}{10}\right)^n = 20 \left(\frac{9}{10}\right)^n$$

$$\text{or} \quad \left(\frac{9}{10}\right)^n = \frac{7.5}{20}$$

Taking log,

$$\begin{aligned} n(\log 9 - \log 10) &= \log 7.5 - \log 20 \\ \text{or} \quad n(.9542 - 1) &= .8751 - 1.3010 \\ \text{or} \quad n \times (-.0458) &= -.4259 \\ \therefore n &= \frac{-.4259}{-.0458} = \frac{4259}{458} = 9.3 \text{ years.} \end{aligned}$$

Ex. 17. The machinery in a factory is valued at £ 24537 and it is decided to reduce the estimated value at the end of each year by 18% of the value at the beginning of that year. When will the value be (a) £ 20,000, (b) one-tenth of original value? (C. U. 1942)

We have $A = P(1-i)^n$.

(a) Here $P = 24537$, $A = 20,000$, $i = \frac{18}{100}$, $n = ?$.

$$\therefore 20,000 = 24537 \left(1 - \frac{18}{100}\right)^n = 24537 \left(\frac{82}{100}\right)^n$$

$$\therefore \left(\frac{82}{100}\right)^n = \frac{20,000}{24,537}$$

Taking log,

$$n(\log 82 - \log 100) = \log 20,000 - \log 24,537$$

$$\begin{aligned} \text{or } n &= \frac{\log 20,000 - \log 24,537}{\log 82 - \log 100} = \frac{4.3010 - 4.3899}{1.9138 - 2} \\ &= \frac{-0.0889}{-0.0862} = \frac{889}{862} = 1.03 \text{ years} \end{aligned}$$

(b) Here if $P = 100$, $A = \frac{1}{10} \times 100 = 10$. (it is not necessary to take $P = \text{£ } 24537$), $i = \frac{18}{100}$, $n = ?$

$$\therefore 10 = 100 \left(\frac{82}{100}\right)^n$$

$$\text{or } \left(\frac{82}{100}\right)^n = \frac{10}{100} = \frac{1}{10}$$

$\therefore$ taking log,

$$n(\log 82 - \log 100) = \log 1 - \log 10$$

$$\begin{aligned} \therefore n &= \frac{\log 1 - \log 10}{\log 82 - \log 100} = \frac{0 - 1}{1.9138 - 2} = \frac{1}{0.0862} \\ &= \frac{10,000}{862} = 11.6 \text{ years.} \end{aligned}$$

Exercise 10

• 1. (a) Find the compound interest on Rs 1000 for 4 years at 5% per annum.

*(b) What will be the simple interest in the above case ?
(C. U. 1950)

• 2. Find the difference between simple and compound interest on Rs 5,000 invested for 4 years at 5% per annum, interest payable yearly.
(C. U. 1949)

3. What is the difference in the amounts of interest on a loan of Rs 11,765 for 3 months, if interest at 7 percent simple, and compound is reckoned.
(C. U. 1946)

[Here interest is convertible monthly]

• 4. (i) The difference between simple and compound interest on a sum put out for 2 years at 5% *p. a.* was £ 2-5s. Find the sum.
(C. U. 1947)

• (ii) The difference between the simple and compound interest on a certain sum put out for 2 years at 5% *p. a.* is Rs 7. Find the sum.

✕ 5. The difference between the compound and simple interest on £ 800 for 2 years is £ 1 12s 5d to the nearest penny. Find the difference between compound and simple interest on £ 100 for 10 years at the same rate percent. per annum.
(L. C. C)

• 6. A debt of Rs 510 is left unpaid for three years. What amount is due to the creditor at the end of that time, reckoning compound interest at $4\frac{1}{2}\%$ *p. a.*
(P. U. 1957)

• 7. Find the compound interest on Rs 6,950 for 3 years, if interest is payable half yearly, the rate for the first two years being 6% *p. a.* and for the third year 9% *p. a.*

• 8. Find out the amount which will produce £ 810 6s 9d in 4 years at 5 per cent compound interest.
(C. U. 1948)

★ 9. What principal will amount to £ 720, 6s. 6d in 4 years at 3% *p. a.* compound interest? If invested at simple interest, what principal would be required?

*10. (i) What sum will amount to £ 520 in 6 years at a rate of simple interest at which Rs 1,706-6-0 will amount to Rs 3,412-12-0 in 20 years?

• (ii) What sum at compound interest will amount to £ 650 at the end of the first year and to £ 676 at the end of the second year? (C. U. 1945)

, 11. A father left £ 11067 to his two sons so that the elder should receive his share in three years' time and the younger in seven years' time, the two shares being equal when received. Reckoning compound interest at 4% *p. a.*, find the amount of each share. (L. C. C.)

• 12. A man left Rs 18000 with the direction that it should be divided in such a way that his 3 sons aged 9, 12, and 15 years should each receive the same amount when they reached the age of 25. If the rate of interest is $3\frac{1}{4}\%$ *p. a.*, what should each son receive when he is 25 years old? (C. U. 1938)

13. A owes B Rs 1600 but it is not due for payment till the end of 3 years from this date. How much should A pay B if he is willing to accept now in order to clear off the debt (a) taking money to be worth 5% per annum simple interest (b) taking it to be worth 5% per annum compound interest, payable yearly? (C. U. 1951)

• 14. In what time will a sum of money double itself at 5% *p. a.*, compound interest? (C. U. 1949)

• 15. In what time will a sum of money treble itself at 3% *p. a.* compound interest payable halfyearly? given $\log 30 = 1.47712$, $\log 103 = 2.01284$.

• 16. What is the rate per cent per annum if a sum doubles itself in 17 years at compound interest? (C. U. 1947)

, 17. What rate of interest per annum does a person get who is paid at the rate of 5 percent compound interest payable halfyearly? (C. U. 1941)

, 18. S. Sen borrowed Rs 6000 from a money lender but he could not pay any amount in a period of 4 years. Accordingly the money lender demanded now Rs 7500 from

him. What rate per cent per annum compound interest did the latter require for lending his money? (C. U. 1957)

• 19. Find the effective rate of interest corresponding to the nominal rate of 4% *p. a.* per annum convertible (i) halfyearly (ii) quarterly.

• 20. What is the nominal rate per cent per annum when interest is payable halfyearly that would give an effective rate of 4 per cent per annum?

• 21. (i) If the population of a town increases every year by 1.8% of the population at the beginning of that year, in how many years will the total increase of population be 30 per cent? (C. U. 1935)

[Hints take $P=100$, then $A=130$]

• (ii) If the population of a town increases every year by 2% of the population at the beginning of that year, in how many years will the total increase of population be 40%?

• 22. The population of a certain town in 1951 was 276,000 and in 1961 it was 345,120. What was the annual increase per thousand, assuming it to be uniform?

• 23. A machine, the life of which is estimated to be 10 years, costs Rs 10,000. Calculate its scrap value at the end of its life, depreciation on the reducing instalment system being charged at 10% per annum. (C. U. 1956)

• 24. The value of a machine at the end of each year after charging depreciation was 90% of the value at the beginning of the year. If the cost of the machine was Rs 12,000 and if its scrap value was Rs 200, find out the number of years during which it was in use. (C. U. 1948)

• 25. A machinery in a factory is valued at Rs 49,074 and it is decided to reduce the estimated value of the end of each year by 15 per cent of the value at the beginning of that year. When will the value be (a) Rs 20,000 (b) $1/10$ th of the original value?

• 26. The wear and tear of a machine is taken each year to be one-tenth of the value at the beginning of the year for the first ten years and one-fiftieth each year for the next five years. Find the scrap value after 15 years.

(P. U. 1950)

Chapter IX

Annuities

1. **Annuity.** An annuity is a series of payments, ordinarily of a fixed amount payable regularly at equal intervals. The intervals may be a year, a halfyear, a month and so on.

Annuities may be divided into two classes—*Annuity Certain* and *Annuity Contingent*.

In **Annuity Certain** payments are to be made unconditionally, for a certain or fixed number of years..

In **Annuity Contingent** the payments are to be made till the happening of some contingent event such as the death of a person, the marriage of a girl, the education of a child or a person reaching a specified age.

Life Annuity is an example of Annuity Contingent. Annuity Certain may be divided into

(1) *Annuity-due* (2) *Annuity immediate* (3) *Deferred annuity*.

Annuity Due is one where the first payment falls due at the *beginning* of the first interval and so all payments are made at the beginning of successive intervals.

Immediate Annuity is one where the first payment falls due at the *end* of the first interval.

Deferred Annuity is that which does not begin until the lapse of certain period. When the annuity is deferred for n years, it is said to commence after n years, and the first payment is made at the end of $(n+1)$ years. A deferred annuity is also called a **Reversion**. An annuity left unpaid for a certain number of years is said to be **foreborne** for that number of years.

If the annuity is to continue for ever it is called a **Perpetuity**, but if it does not commence at once it is called a **Deferred Perpetuity**.

2. Present Value. The present value of an annuity is the sum of the present values of its instalments.

In finding the present value of annuity it is always customary to reckon compound interest.

3. Present value of an Immediate Annuity.

Let A be the annuity, V the present value, i the rate of interest per unit per year and n the number of years to continue.

Then the present value of A due in 1 year $= \frac{A}{1+i}$,

present value of A due in 2 years $= \frac{A}{(1+i)^2}$

... ..

present value of A due in n years $= \frac{A}{(1+i)^n}$

$\therefore$ the sum of the present values of the different payments is given by

$$V = \frac{A}{1+i} + \frac{A}{(1+i)^2} + \dots + \frac{A}{(1+i)^n}$$

$$= \frac{A}{(1+i)^n} \{ (1+i)^{n-1} + (1+i)^{n-2} + \dots + 1 \},$$

which is in G.P;

$$V = \frac{A}{(1+i)^n} \left\{ \frac{(1+i)^n - 1}{(1+i) - 1} \right\}.$$

$$V = \frac{A}{i} \left\{ \frac{(1+i)^n - 1}{(1+i)^n} \right\} = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\} = \frac{A}{i} \left\{ 1 - (1+i)^{-n} \right\}$$

The present value of an immediate annuity of 1 per annum for n years is usually denoted by the symbol a_n which stands for $\frac{1-v^n}{i}$ where $v = \frac{1}{1+i}$. i.e., $a_n = \frac{1 - (1+i)^{-n}}{i}$

Hence using this symbol the present value of an immediate annuity A may be written as $V = A \cdot a_n$.

4. Present value of an Annuity Due.

An annuity due for n years is equivalent to an immediate payment of A plus an immediate annuity for $(n-1)$ years. Hence present value of an annuity due is

$$\begin{aligned} V &= A + \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^{n-1}} \right\} \\ &= \frac{A}{i} (1+i) \left\{ 1 - \frac{1}{(1+i)^n} \right\} \checkmark \\ &= \frac{A}{i} (1+i) \left\{ 1 - (1+i)^{-n} \right\}. \end{aligned}$$

5. Present value of a Perpetuity.

In this case the number of instalments is *infinite* and by taking n indefinitely large we can make $(1+i)^{-n}$ as small as we please.

i.e. $(1+i)^{-n}$ tends to zero.

$$\therefore V = \frac{A}{i} \{1 - 0\} = \frac{A}{i}. \quad \text{Symbolically, } a_{\infty} = \frac{A}{i} \checkmark$$

6. Present value of a Deferred Annuity to commence at the end of p years and to continue for n years.

Let A be the annuity, i the rate of interest per unit per year and V the present value.

The first payment is made at the end of $(p+1)$ years.

$\therefore$ the present values of the first, second..... n th

payments are $\frac{A}{(1+i)^{p+1}}, \frac{A}{(1+i)^{p+2}}, \dots, \frac{A}{(1+i)^{p+n}}.$

$$\therefore V = \frac{A}{(1+i)^{p+1}} + \frac{A}{(1+i)^{p+2}} + \dots + \frac{A}{(1+i)^{p+n}}$$

$$= \frac{A}{(1+i)^{p+1}} \{ (1+i)^{-1} + (1+i)^{-2} + \dots + 1 \}, \text{ which is in G.P.}$$

$$\begin{aligned}
&= \frac{A}{(1+i)^{p+n}} \left\{ \frac{(1+i)^n - 1}{(1+i) - 1} \right\} \\
&= \frac{A}{i} \left\{ \frac{(1+i)^n - 1}{(1+i)^{p+n}} \right\} \\
&= \frac{A}{i} \left[\frac{1}{(1+i)^p} - \frac{1}{(1+i)^{p+n}} \right] \\
&= \frac{A}{i} \left[\left\{ 1 - \frac{1}{(1+i)^{p+n}} \right\} - \left\{ 1 - \frac{1}{(1+i)^p} \right\} \right]
\end{aligned}$$

$\therefore p/a_n = \text{P.V. for } (p+n) \text{ years} - \text{P.V. for } p \text{ years.}$

7. Present value of Deferred Perpetuity.

By making n indefinitely large, we can make $\frac{1}{(1+i)^{p+n}}$ tend to zero.

$$\therefore V = \frac{A}{i} \left\{ \frac{1}{(1+i)^p} - 0 \right\} = \frac{A}{i} \cdot \frac{1}{(1+i)^p}$$

8. If the interest i per unit per annum is converted into principal p times a year, then the annual instalment A is payable p times a year, so that the interest per unit *per period* is $\frac{i}{p}$ and the number of instalments in n years becomes $n.p$.

Thus the present value of an immediate annuity A payable halfyearly for n years is given by

$$V = \frac{A}{\frac{i}{2}} \left\{ 1 - \frac{1}{\left(1 + \frac{i}{2}\right)^{2n}} \right\}$$

Other formulas undergo similar modifications.

9. Amount of an Immediate Annuity to continue for n years.

Let A be set aside at the end of every year for n years. Then at the end of n years-

the first payment will amount to $A(1+i)^{n-1}$

the second payment will amount to $A(1+i)^{n-2}$

... ..

the n th payment will amount to $A(1+i)^{n-n}$ or A

$\therefore$ the amount at the end of n years is given by

$M = A(1+i)^{n-1} + A(1+i)^{n-2} + \dots + A$, which is in G.P.

$$= A \cdot \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{A}{i} \{(1+i)^n - 1\}$$

The amount of an immediate annuity of 1 per annum payable for n years is usually denoted by the symbol S_n

$$S_n = \frac{A}{i} \{(1+i)^n - 1\}$$

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 650 @ Commercial models
 680 @ Mathematics
 650 @ System analysis

11. The present value of a sum and its amount are connected by the relation $A = P(1+i)^n$. Hence given P we

can find out A and conversely. Hence to find amount of an annuity from its present value, we are simply to multiply the present value by $(1+i)^n$.

Thus,	a_n	$S_n = a_n \times (1+i)^n$
	Present value	Amount
1. Immediate Annuity	$\frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$	$\frac{A}{i} \{ (1+i)^n - 1 \}$
2. Annuity Due	$\frac{A(1+i)}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$	$\frac{A(1+i)}{i} \{ (1+i)^n - 1 \}$
3. Deferred Annuity	$\frac{A \{ (1+i)^n - 1 \}}{i (1+i)^{p+n}}$	$\frac{A \{ (1+i)^n - 1 \}}{i (1+i)^p}$

A Free hold Estate is an estate which yields a perpetual annuity called rent. So the value of a freehold estate is equal to the present value V of a perpetual annuity A (perpetuity) equal to the rent.

$$\therefore V = \frac{A}{i}$$

A Lease hold Estate is held only for a fixed time after which it reverts to the original proprietor.

If an estate be worth P , the present value of reversion after n years is equal to the present value of P .

$$\therefore \text{present value of reversion} = \frac{P}{(1+i)^n}$$

If mA is the present value of an annuity A , the annuity is said to be worth m years' purchase.

So in the case of a perpetual annuity

$$mA = \frac{A}{i}$$

$$\therefore m = \frac{1}{i} = \frac{100}{\text{rate per cent.}}$$

Thus the number of years' purchase of a perpetual annuity is obtained by dividing 100 by the rate per cent.

Conversely, given the number of year's purchase that a tenant pays, we get the rate of interest per cent per annum by dividing 100 by the number of years' purchase.

The amount of money required to be paid by a lessee for renewal of the lease is called **fine**.

Let the annual value of an estate be A and that a tenant has paid for m out of $m+p$ years of lease. If he now wishes to renew the lease for a term of $m+n$ years, the fine is equal to the present value of deferred annuity A , to begin after p years and to run for n years.

$$\therefore \text{ the fine } = \frac{A}{i(1+i)^p} - \frac{A}{i(1+i)^{p+n}}.$$

Sinking Fund. When a sum is set aside annually or by any other periodic contributions to accumulate at compound interest so as to provide fund for the liquidation of a loan, debenture stock and the like or for replacement of an wasting asset such as plant and machinery, such a fund is called Sinking Fund.

If A be the sum set aside at the end of each year to accumulate for n years at i per unit per annum compound interest, then the fund M accumulated after n years is the amount of an n -year annuity of A .

$$M = \frac{A}{i} \{ (1+i)^n - 1 \}$$

$$\text{or } A = \frac{M \cdot i}{(1+i)^n - 1}$$

Endowment Fund. - It is equal to the present value of a perpetuity. If A be the annual value of an endowment held at i per unit per year, then the

$$\text{endowment fund} = \frac{A}{i}.$$

Ex. 1. Find the amount and present value of an annuity certain of Rs. 150 for 12 years, reckoning interest at $3\frac{1}{2}\%$ *p. a.* (C. U. 1927)

$$[\text{Given } (1.035)^{12} = 1.511066]$$

Formula for
amount

$$M = \frac{A}{i} \{ (1+i)^n - 1 \}$$

$$A = 150, \quad i = \frac{3.5}{100} = .035, \quad n = 12.$$

$$\begin{aligned} \therefore M &= \frac{150}{.035} \{ (1 + .035)^{12} - 1 \} \\ &= \frac{150}{.035} \{ 1.511066 - 1 \} = \frac{150}{.035} \times .511066 \\ &= \frac{30,000}{7} \times \frac{511066}{10,00,000} = 2190.28 \end{aligned}$$

$$\therefore \text{Amount} = \text{Rs. } 2190.28$$

Formula for
present value

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

$$\begin{aligned} V &= \frac{150}{.035} \left\{ 1 - \frac{1}{(1 + .035)^{12}} \right\} \\ &= \frac{150}{.035} \left\{ 1 - \frac{1}{1.511066} \right\} \\ &= \frac{30,000}{7} \times \frac{1.511066 - 1}{1.511066} \\ &= \frac{30,000}{7} \times \frac{.511066}{1.511066} \\ &= \frac{30,000}{7} \times \frac{511066}{1511066} = 1,449.50 \end{aligned}$$

$$\therefore \text{present value} = \text{Rs. } 1,449.50$$

Ex. 2. A person borrows £ 1500 promising to repay the sum borrowed and the proper interest by ten equal yearly instalments, the first to fall due in one year's time.

Reckoning compound interest at 5% *p.a.*, find the value of the annual instalment.

$[(1.05)^{10}]$ may be taken as 1.629 (C.U. 1940)

Let £ A = annual instalment.

Then £ 1500 is the present value of an immediate annuity of A payable for 10 years.

Using the formula for present value

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

where $V=1500$, $i=\frac{5}{100}=.05$, $n=10$, we have

$$\begin{aligned} 1500 &= \frac{A}{.05} \left\{ 1 - \frac{1}{(1.05)^{10}} \right\} \\ &= \frac{100.A}{5} \left\{ 1 - \frac{1}{1.629} \right\} \\ &= 20.A \times \frac{1.629-1}{1.629} = 20.A \times \frac{.629}{1.629} \\ &= 20.A \times \frac{629}{1629} \end{aligned}$$

$$\therefore A = \frac{1500 \times 1629}{20 \times 629} = \frac{75 \times 1629}{629} = \text{£ } 194 \text{ } 4\text{s } 9\text{d.}$$

Ex. 3. A man borrows Rs. 3000/- at 6% and promises to pay off the loan in 20 annual payments beginning at the end of the first year. What is the annual payment necessary? (C.U. 1954)

Let. A = annual payment.

Then Rs. 3000 is the *P.V.* of an immediate annuity of A payable annually for 20 years at 6% *p. a* compound interest.

$\therefore$ from the formula for present value

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

where $V = \text{Rs. } 3000$, $i = \frac{6}{100} = 0.06$, $n = 20$, we have

$$\begin{aligned} 3000 &= \frac{A}{0.06} \left\{ 1 - \frac{1}{(1.06)^{20}} \right\} \\ &= A \frac{100}{6} \left\{ 1 - \frac{1}{3.206} \right\} \\ &= A \frac{100}{6} \cdot \frac{2.206}{3.206} \\ &= A \frac{50}{3} \times \frac{2206}{3206} \end{aligned}$$

$$\begin{aligned} x &= (1.06)^{20} \\ \log 1.06 &= 20 \cdot \log 1.06 \\ &= 20 \times 0.0253 \\ &= 0.506 \\ x &= \text{antilog } 0.506 \\ &= 3.206 \end{aligned}$$

$$\therefore A = \frac{3000 \times 3 \times 3206}{50 \times 2206} = \text{Rs. } 261.60.$$

Ex. 4. A man wishes to buy a house valued at £ 2,550. He is prepared to pay £ 1600 now and the balance in 9 equal annual instalments. If interest is calculated at $4\frac{1}{2}\%$ *p. a.*, what should he pay annually?

The cost of the house = £ 2,550

less cash payment = £ 1,600

Balance to be paid = £ 950

$\therefore$ £ 950 is the *P. V.* of an immediate annuity to continue for 9 years.

Let A be the amount of each instalment. Then from the formula for present value

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

where $V = 950$, $i = \frac{4.5}{100} = 0.045$, $n = 9$, we have

$$\begin{aligned} 950 &= \frac{A}{0.045} \left\{ 1 - \frac{1}{(1.045)^9} \right\} \\ &= \frac{A}{0.045} \left\{ 1 - \frac{1}{1.486} \right\} \\ &= \frac{A \cdot 1000}{45} \times \frac{.486}{1.486} \\ &= A \cdot \frac{200}{9} \times \frac{486}{1486} \end{aligned}$$

$$\begin{aligned} x &= (1.045)^9 \\ \log x &= 9 \log 1.045 \\ &= 9 \times 0.0191 \\ &= 0.1719 \\ \therefore x &= \text{antilog } 0.1719 \\ &= 1.486 \end{aligned}$$

$$\therefore A = \frac{950 \times 9 \times 1486}{200 \times 486} = \text{£ } 130.14s.$$

$$\text{Or. } 950 = \frac{A}{.045} \{1 - (1.045)^{-9}\}$$

$$= A \cdot \frac{200}{9} \{1 - .6732\}$$

$$= A \cdot \frac{200}{9} \times .3268 = A \cdot \frac{2}{9} \times \frac{3268}{100}$$

$$\therefore A = \frac{950 \times 9 \times 100}{2 \times 3268} = \text{£ } 130.16s$$

Note : The above difference in the values of the amount is because of approximation at different stages of calculation.

Ex. 5. A company sets aside a sum of Rs. 5000 annually for 10 years to pay off a debenture issue of Rs. 60,000. If the fund accumulates at 5% per annum compound, find the surplus after full redemption of the debenture issue.

The fund accumulated after 10 years is the amount of the annuity of Rs. 5000.

From the formula for amount

$$M = \frac{A}{i} \{(1+i)^n - 1\}$$

where $A = 5000$, $i = \frac{5}{100} = .05$, $n = 10$, we have

$$M = \frac{5000}{.05} \{(1.05)^{10} - 1\}$$

$$= \frac{5000 \times 100}{5} \{1.629 - 1\}$$

$$= \frac{5000 \times 100}{5} \times .629$$

$$= 62900$$

$$\text{surplus} = \text{Rs. } (62,900 - 60,000)$$

$$= \text{Rs. } 2900.$$

$$x = (1.05)^{10}$$

$$\log x = 10 \log 1.05$$

$$= 10 \times 0.0212$$

$$= 0.212$$

$$x = \text{antilog } 0.212$$

$$= 1.629$$

Ex. 6. An equal sum is set aside every year for 25 years to pay off a debenture issue of Rs. 100,000. If the fund accumulates at 2% per annum compound, find the value of the annual payment. Given that the amount of an annuity of 1 per annum for 25 years at 2% *p.a.* is equal to 32.03030. Answer to the nearest rupee.

(C.U. 1952)

Let A = annual payment.

Then from the formula for amount

$$M = \frac{A}{i} \left\{ (1+i)^n - 1 \right\}$$

where $M = 100,000$, $i = \frac{2}{100} = .02$, $n = 25$, we have

$$100,000 = \frac{A}{.02} \left\{ (1.02)^{25} - 1 \right\}$$

$$\text{But } 32.03030 = \frac{1}{.02} \left\{ (1.02)^{25} - 1 \right\}$$

Hence by division

$$\frac{100,000}{32.03030} = A$$

$$\therefore A = \frac{100,000 \times 10,000}{320303} = \text{Rs. } 3122.$$

Ex. 7. A man borrows Rs. 10,000 on the understanding that it is to be paid back with interest of $2\frac{1}{2}\%$ *p.a.* compound in 12 equal instalments paid halfyearly, the first payment to be made six months after the money is borrowed. Calculate the amount of each instalment.

Let A = halfyearly payment.

Then Rs. 10,000 is the *P. V.* of the annuity A payable halfyearly in 12 instalments (6 years).

$\therefore$ from the formula for present value

$$V = \frac{A}{\frac{i}{2}} \left\{ 1 - \frac{1}{\left(1 + \frac{i}{2}\right)^{2n}} \right\}$$

where $V=10,000$, $i=\frac{2.5}{100}=0.025$ $\frac{i}{2}=0.0125$, $n=6$, we have

$$\begin{aligned}
 10,000 &= \frac{A}{0.0125} \left\{ 1 - \frac{1}{(1.0125)^{12}} \right\} \\
 &= \frac{A \times 10,000}{125} \left\{ 1 - \frac{1}{1.161} \right\} & \left. \begin{aligned} x &= (1.0125)^{12} \\ \log x &= 12 \log 1.0125 \\ &= 12 \times 0.0054 \\ &= 0.0648 \\ x &= \text{antilog } 0.0648 \\ &= 1.161 \end{aligned} \right| \\
 &= A.80 \times \frac{.161}{1.161} \\
 \therefore A &= \frac{10,000 \times 1161}{80 \times 161} = \text{Rs. } 901.40
 \end{aligned}$$

Ex. 8. A sinking fund is created for the redemption of debentures of Rs 1,00,000/- at the end of 25 years. How much money should be provided out of profits each year for the sinking fund if the investment can earn interest @ 4% per annum? (C. U. 1958)

Let A be the annual instalment. Then Rs 1,00,000 is the amount of the annuity A to continue for 25 years.

$$M = \frac{A}{i} \{ (1+i)^n - 1 \}$$

where $M=1,00,000$, $i=\frac{4}{100}=0.04$, $n=25$, we have

$$\begin{aligned}
 1,00,000 &= \frac{A}{0.04} \{ (1.04)^{25} - 1 \} \\
 &= \frac{A.100}{4} \{ 2.661 - 1 \} & \left. \begin{aligned} x &= (1.04)^{25} \\ \log x &= 25 \log 1.04 \\ &= 25 \times 0.0170 \\ &= 0.4250 \\ x &= \text{antilog } 0.4250 \\ &= 2.661. \end{aligned} \right| \\
 &= A.25 \times 1.661 \\
 \therefore A &= \frac{1,00,000 \times 1000}{25 \times 1661} \\
 &= \frac{4000 \times 1000}{1661} = \text{Rs } 2408.19.
 \end{aligned}$$

Ex. 9. A machine costs the company Rs 48,500 and its effective life is estimated to be 12 years. If the scrap realises Rs 1000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% per annum?

Let A be the annual instalment.

Evidently the amount of the annuity A to continue for 12 years = Rs 48,500 — Rs 1,000 = Rs 47,500.

Hence from the formula

$$M = \frac{A}{i} \{ (1+i)^n - 1 \}$$

where $M = 47,500$, $i = \frac{5}{100} = 0.05$, $n = 12$, we have

$$47,500 = \frac{A}{0.05} \{ (1.05)^{12} - 1 \}$$

$$= \frac{A \cdot 100}{5} (1.797 - 1)$$

$$= A \times 20 \times 0.797$$

$$\therefore A = \frac{47,500 \times 1000}{20 \times 797} = \frac{2375000}{797}$$

$$= \text{Rs } 2980.$$

$$\begin{aligned} x &= (1.05)^{12} \\ \log x &= 12 \log 1.05 \\ &= 12 \times 0.0212 \\ &= 0.2544 \\ \therefore x &= \text{antilog } 0.2544 \\ &= 1.797. \end{aligned}$$

Ex. 10. What sum will buy an annuity of £ 60 payable for 4 years, the rate of interest being $3\frac{1}{2}$ per cent per annum?

(C.U. 1945)

The sum is equal to the present value of an immediate annuity £ 60 for 4 years at $3\frac{1}{2}\%$ p.a.

From the formula

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

where $A = 60$, $i = \frac{3.5}{100} = 0.035$, $n = 4$, we have

$$V = \frac{60}{0.035} \left\{ 1 - \frac{1}{(1.035)^4} \right\}$$

$$= \frac{60 \times 1000}{35} \{ 1 - (1.035)^{-4} \}$$

$$= \frac{12 \times 1000}{7} (1 - 0.8718)$$

$$= \frac{12 \times 1000}{7} \times 0.1282$$

$$= \text{£ } 219 \text{ } 16s$$

$$\begin{aligned} x &= (1.035)^{-4} \\ \log x &= -4 \log 1.035 \\ &= -4 \times 0.0149 \\ &= -0.0596 \\ &= -1 + (1 - 0.0596) \\ &= \bar{1}.9404 \\ \therefore x &= 0.8718 \end{aligned}$$

Ex. 11. A man retires at the age of 60 years and his employer gives him a pension of Rs 600/- a year paid in halfyearly instalments for the rest of his life. Reckoning his expectation of life to be 13 years and that interest is at 4% per annum payable halfyearly, what single sum is equivalent to this pension?

Clearly the single sum is the present value of an immediate annuity of Rs 300 to continue for 13 years.

Hence from the formula

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

where A = halfyearly instalment = 300

i = halfyearly rate of interest = $\frac{2}{100} = .02$

n = number of halfyears = 26, we have

$$\begin{aligned} V &= \frac{300}{.02} \left\{ 1 - \frac{1}{(1.02)^{26}} \right\} \\ &= \frac{300 \times 100}{2} \left\{ 1 - \frac{1}{1.673} \right\} & \left| \begin{array}{l} x = (1.02)^{26} \\ \log x = 26 \log 1.02 \\ = 26 \times .0086 \\ = 0.2236 \\ x = 1.673 \end{array} \right. \\ &= 300 \times 50 \times \frac{.673}{1.673} \\ &= 15000 \times \frac{673}{1673} \\ &= \text{Rs } 6034. \text{ (app)} \end{aligned}$$

Ex. 12. Which is better, an annuity of £ 150 to last for 10 years or the reversion of a free-hold estate of £ 79 4s per annum to commence 7 years hence, the rate of interest being 5 p.a. (C.U. 1934)

The first one is an immediate annuity and the second one is a deferred perpetuity.

Clearly the better one has greater present value.

The P.V. of the first

$$= \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\} \quad [A=150; i=.05, n=10]$$

$$= \frac{150}{\cdot 05} \left\{ 1 - \frac{1}{(1 \cdot 05)^{10}} \right\}$$

$$= \frac{150 \times 100}{5} \left\{ 1 - \frac{1}{1 \cdot 629} \right\}$$

$$= 3000 \times \frac{\cdot 629}{1 \cdot 629}$$

$$= 3000 \times \frac{629}{1629} = \text{£ } 1158 \quad 8s$$

∴

$$x = (1 \cdot 05)^{10}$$

$$\log x = 10 \log 1 \cdot 05$$

$$= 10 \times \cdot 0212$$

$$= 0 \cdot 212$$

$$\therefore x = 1 \cdot 629.$$

The *P.V.* of the second = $\frac{A}{i(1+i)^p}$ $\left[A = 79\frac{1}{8}, i = \cdot 05, p = 7 \right]$

$$= \frac{79\frac{1}{8}}{\cdot 05(1 \cdot 05)^7}$$

$$= \frac{396}{5} \times \frac{100}{5} \cdot \frac{1}{(1 \cdot 05)^7}$$

$$= \frac{396 \times 4}{1 \cdot 407} = \text{£ } 1125 \quad 15s$$

$$x = (1 \cdot 05)^7$$

$$\log x = 7 \log 1 \cdot 05$$

$$= 7 \times \cdot 0212$$

$$= 0 \cdot 1484$$

$$\therefore x = 1 \cdot 407$$

Hence the immediate annuity is better.

Ex. 13. What sum will buy an annuity of £ 60 payable for 4 years, rate of interest being $3\frac{1}{2}$ per cent per annum ? (C.U. 1945)

The sum is clearly the *P. V.* of an immediate annuity of £ 60 for 4 years at $3\frac{1}{2}\%$ *p.a.*

Hence from the formula

$$V = \frac{A}{i} \left\{ 1 - \frac{1}{(1+i)^n} \right\}$$

where $A = 60$, $i = \frac{3 \cdot 5}{100} = \cdot 035$, $n = 4$, we have

$$V = \frac{60}{\cdot 035} \left\{ 1 - \frac{1}{(1 \cdot 035)^4} \right\}$$

$$= \frac{60 \times 1000}{35} \left\{ 1 - \frac{1}{1 \cdot 148} \right\}$$

$$= \frac{12000}{7} \times \frac{\cdot 148}{1 \cdot 148}$$

$$x = (1 \cdot 035)^4$$

$$\log x = 4 \log 1 \cdot 035$$

$$= 4 \times \cdot 0149$$

$$= 0 \cdot 0596$$

$$\therefore x = 1 \cdot 148$$

$$= \frac{12000}{7} \times \frac{148}{1148} = \text{£ } 221$$

Ex. 14. A lease for 10 years is taken at Rs. 60,000. What annual contribution should be made to raise a sinking fund against it, if the investment can earn interest at 4% per annum?

The annual contribution A is given by

$$M = \frac{A}{i} \{ (1+i)^n - 1 \}$$

where M = value of the asset to be replaced = 60,000

A = annual contribution

i = rate of interest per unit per year = .04

n = 10

$$\begin{aligned} A &= \frac{Mi}{(1+i)^n - 1} = \frac{60,000 \times .04}{(1.04)^{10} - 1} \\ &= \frac{2400}{1.479 - 1} \\ &= \frac{2400}{.479} \\ &= \text{Rs. } 5010.44 \end{aligned}$$

$$\begin{aligned} x &= (1.04)^{10} \\ \log x &= 10 \log 1.04 \\ &= 10 \times .0170 \\ &= 0.170 \\ \therefore x &= 1.479 \end{aligned}$$

Ex. 15. Find the value of a freehold estate that yields an annual income of Rs. 455, if money is worth $3\frac{1}{2}$ p.c. p.a. compound.

From the formula

$$V = \frac{A}{i}$$

where $A = 455$, $i = \frac{3.5}{100}$, we have

$$V = \frac{455}{\frac{3.5}{100}} = \frac{455 \times 100}{3.5} = \frac{455 \times 1000}{35} = \text{Rs. } 13000.$$

Ex. 16. For endowing an annual scholarship of Rs. 500 I made five equal annual contributions. The first

award is to be made two years after the last of my contributions. What was the value of each contribution, assuming interest at $3\frac{1}{2}\%$ p.a.?

Let the first contribution A be made now, so that the *fifth* contribution is made at the beginning of the *fifth* year.

The first award of scholarship is made at the beginning of the *7th* year. So the accumulated fund must be available at the beginning of the *6th* year (i.e., one year before the first award).

Clearly the amount of the fund

= amount of annuity due for 5 years

$$= \frac{A(1+i)}{i} \{(1+i)^5 - 1\} \quad \left[i = \frac{3.5}{100} = .035, \quad n = 5 \right]$$

$$= \frac{A \times 1.035}{.035} \{(1.035)^5 - 1\}$$

This is the *P.V.* of the perpetuity of Rs. 500 and is equal to

$$\begin{aligned} \frac{500}{i} &= \frac{500}{.035} \\ \therefore \frac{A \times 1.035}{.035} \{(1.035)^5 - 1\} &= \frac{500}{.035} \\ \therefore A &= \frac{500}{1.035 \{(1.035)^5 - 1\}} \\ &= \frac{500}{1.035 \times .187} \\ &= \text{Rs. } 2588 \end{aligned}$$

$$\begin{aligned} x &= (1.035)^5 \\ \log x &= 5 \log 1.035 \\ &= 5 \times .0149 \\ &= .0745 \\ \therefore x &= 1.187 \\ \log 500 &= 2.6990 \\ \log 1.035 &= .0142 \\ \log .187 &= \overline{1.2718} \\ \log A &= 3.4130 \\ \therefore A &= 2588 \end{aligned}$$

Note. We may find the present value of the annuity due and the present value of the perpetuity, deferred 5 years and equate them.

Exercise 11

1. Find the present value of an annuity of Rs. 1000 *p. a.* for 14 years allowing compound interest at 5% *p. a.*
(*P.U. 1948*)

2. Calculate the amount and present value of an annuity of Rs. 3000 for 15 years if the rate of interest be $4\frac{1}{4}\%$ *p. a.*
(*C.U. 1959*)

3. Find the present value of an annuity—due of 1 per annum payable annually for 2 years, the rate of interest being 5% *p. a.*

4. A man borrows Rs. 6,000 at 6% and promises to pay off the loan in 20 annual payments beginning at the end of the first year. What is the annual payment necessary ?

5. A member of a society obtains a loan of £ 400 and agrees to repay it with compound interest at 5 per cent in five annual equal instalments, the first payment to be made at the end of a year. What would each instalment be ?
(*C.U. 1943*)

6. A person borrows £ 45,00 promising to repay the sum borrowed and the proper interest by ten equal yearly instalments, the first to fall due in one year's time. Reckoning compound interest at 5% *p. a.*, find the value of the annual instalment, $(1.05)^{10}$ may be taken as 1.629

7. A man wishes to buy a house valued at £1750. He is prepared to pay £800 now and the balance in nine equal annual instalments. If interest is calculated at $4\frac{1}{2}\%$ *p. a.* per annum, what should he pay annually ? (*C.U. 1939*)

8. A man borrows Rs. 1000 on the understanding that it is to be paid back in four equal instalments at intervals of six months, the first payment to be made six months after the money was borrowed. Calculate the amount of each instalment, reckoning compound interest at $2\frac{1}{2}\%$ *p. a.* per half year.
(*C.U. 1931*)

9. An overdraft of Rs. 30,000 is to be paid back in equal annual instalments over a period of 30 years.

Find the value of the payment reckoning compound interest at 4% *p. a.* (P. U. 1943)

Ex. 10. A man retires at the age of 60 years and his employer gives him a pension of Rs. 1200 a year paid in halfyearly instalments for the rest of his life. Reckoning his expectation of life to be 13 years and that interest at 4% per annum payable halfyearly, what single sum is equivalent to his pension? (C.U. 1954)

• 11. S. Roy wishes to buy a Printing Machinery valued at Rs. 17000. He is prepared to pay now Rs. 9000 and the balance in 8 annual instalments. If interest is calculated at $3\frac{1}{2}\%$ per annum, how much should he pay annually? (C.U. 1957)

• 12. The annual subscription for the membership of a club is Rs. 25 and a person may become a life member by paying Rs. 1000 in a lump sum. Find the rate of interest charged.

13. What sum should be paid for an annuity of Rs. 2,400 for 20 years at $4\frac{1}{2}\%$ compound interest *p. a.* (given $\log 1.045 = 0.0191$ and $\log 4.150 = 0.6180$)

• 14. A man wishes to create an endowment fund to provide an annual prize of Rs. 500 out of its income. If the fund is invested in $2\frac{1}{2}\%$ *p. a.*, find the amount of this fund.

15. The price of a radio is Rs. 750 cash or it may be paid for by 20 equal monthly instalments, the first instalment to be paid at the time of purchase. Find the value of each instalment reckoning 5% *p. a.* compound interest payable monthly.

• 16. How many years' purchase should be given for freehold estate at $2\frac{1}{2}\%$ compound interest?

• 17. What sum will buy an annuity of £ 60 payable for 4 years, the rate of interest being $3\frac{1}{2}\%$ *p. a.*?

18. A man borrowed Rs. 20,000 at 6% compound interest, promising to repay Rs. 5,000 at the end of each

of the first four years to reduce principal and interest and to pay the balance at the end of the fifth year. Find the amount of his final payment. (C.U. 1950)

19. What annuity should be paid for 10 years if 10,000 is paid now, the first payment of annuity being made in one year's time and the subsequent payments in yearly instalments at the end of each subsequent year? Compound interest is to be taken at 5% per annum. (C. U. 1956)

20. If I buy a sewing machine with cash down I pay Rs. 190. If I buy it on instalment basis I pay Rs. 10 down and 20 monthly instalments of Rs. 10. What rate of interest is calculated in the second case? (C.U. 1946)

21. A limited company intends to create a depreciation fund to replace, at the end of the 25th year, assets costing Rs. 1,00,000. Calculate the amount to be retained out of profits every year if interest rate is 3%. (C.U. 1961)

22. At the beginning of a year Mitra decided to deposit Rs. 45.50 nP at the end of each year in a Bank which paid 4% interest. If the instalments were thus allowed to accumulate, what would be the total amount standing at his credit at the end of the 12th year? (C. U. 1959)

Ex. 23. A machine costs the company Rs. 97,000 and its effective life is estimated to be 12 years. If the scrap realises Rs. 2,000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% per annum? (C.U. 1960)

Ex. 24. For endowing an annual scholarship of Rs. 1000, I make five equal annual instalments. The first award is to be made two years after the last of my contributions. What was the value of each contribution assuming interest at $3\frac{1}{2}\%$ per annum? (C. U. 1946)

Chapter X

Bill Discounting—True & Banker's Discount

1. **Bill of Exchange**—A bill of exchange is an unconditional order in writing signed by the maker, directing a certain person to pay on demand, or at a fixed or determinable future time, a certain sum of money to or to the order of a certain person or to bearer.

Three persons are involved in a bill.

The maker of the bill is called the **Drawer**. The person who is directed to pay is called the **Drawee** and the person to whom or to whose order the money is to be paid is called the **Payee**. The payee may be the drawer himself.

A bill is prepared by the drawer and sent to the drawee for acceptance. After drawee has signed his consent on the face of the bill by writing the word 'accepted' and putting his signature and date, it becomes legally valid and called an **acceptance**. The acceptance is there after delivered to the drawee or payee as per directions.

A bill of exchange is called a **Negotiable Instrument**. By endorsement and delivery the full title of the benefit of the bill may be transferred.

A person who has the possession of the bill is called its **Holder**. The date on which the bill is due is called its **Maturity**.

Form of Bill of Exchange

Stamp	Calcutta, 4th. May, 1962
Rs 100	
Three months after date pay to D. Sen., or order the sum of Rupees One Thousand for value received.	
To Messrs Frank & Wood Calcutta.	K. Saraogi

In the above case K Saraogi is the drawer, D. Sen is the payee and Frank & wood are the drawees, who become the acceptors when they write the words

“Accepted payable at the United Bank of India Ltd. Frank & Wood” across the face of the bill.

The bill has now attained legal status and is called an *acceptance*. An unaccepted bill has no value.

If the drawee and the payee are the same, then *“pay to me or my order”* should be written instead of *“pay to D. Sen, or order”*. After accepting the bill the drawee sends it to the payee.

2. Inland bill—If a bill is drawn in India and also payable in India, it is called an Inland Bill, any other bill is called **Foreign Bill**.

Accommodation Bill—A bill drawn, accepted or endorsed without receiving value in exchange for it but simply to assist a party by negotiating it, is called an *Accommodation bill*.

3. Date of Drawing or Sight—The date on which a bill is drawn is called the *date of drawing*, and the date on which it is accepted is called the *date of acceptance or sight*. The date of drawing and the date of sight may or may not be the same.

Due Date—The date on which a bill is payable or due is called the *due date*. A bill drawn at 3 months is *nominally due* after 3 *calender months* and not after 90 days. A bill drawn at 90 days is *nominally due* exactly after 90 days and not after 3 months.

Days of Grace—Every bill of exchange, unless it is payable at sight or on presentation or on demand, is *legally due* or *at maturity* on the *third day* after the nominally due date. This practice is not followed everywhere. In U. S. A no days of grace are allowed.

When no definite date is mentioned the days of grace are ignored.

Days after Date—Where a bill is drawn *after date*, the period begins to run from the date of drawing the bill.

Days after Sight or Acceptance—When a bill is drawn *after sight*, the term of the bill begins to run from the date of sighting the bill.

Illustration—A bill drawn on the 4th January, 1961 at 3 months after date and accepted on the 6th January will be nominally due on the 4th April and legally due on the 7th April. But if the same bill is drawn at 3 months after sight it will be nominally due on the 6th April and legally due on the 9th April.

Counting of Days—In counting number of days from one date to another, it is usual to *exclude* the day from which the time begins to run and to *include* the last day of payment.

4. Discounting of Bills :—A bill of exchange is a *Bill Receivable* (B/R) to the drawer as he receives the payment and a *Bill Payable* (B/P) to the acceptor as he pays the bill.

The holder of a Bill Receivable (*i.e.*, a bill-holder) may deal with a bill in any one of the ways

(1) He may *keep the bill to himself* till the due date and receive payment on presentation.

(2) He may *endorse the bill* in payment of his own debt.

(3) He may *cash the bill before its due date*. To do so, he is to present the bill to a banker or a bill-broker. The latter pays the bill holder not exactly the bill value (*i.e.*, the face value of the bill) but something less. The amount deducted from the bill value is called the Banker's Discount and the sum of money that the holder receives is called the Present value of the bill.

5. Commercial or Banker's Discount :—After the bill is presented, the banker calculates the *unexpired period* of

the bill which is the period from the date on which the bill is presented till the date on which it becomes legally due.

6. The **Banker's Discount (B.D)** is equal to the interest on the face value of the bill or bill value (**B.V.**) for the unexpired period.

The **Present Worth** or **Present Value (P.V.)** or **Discounted Value** of the bill is the sum paid by the banker to the holder and is, therefore, equal to the difference of bill value and banker's discount.

Hence, **B.D.=interest on B.V.**

(Banker's) **P.V.=B.V.—B.D.**

Theoretical or True Discount. If a sum of money is due after a given time, then *True Discount* is equal to the int. on the Present value of the sum. The difference between the sum due and the true discount is called the **True Present Value** of the sum.

Hence, **T.D.=interest on P.V.**

(True) **P.V.=B.V.—T.D.**

To show diagrammatically,

(True P.V.) $\overbrace{\text{T.P.V.} + \text{T.D.}} = \text{B.V.}$

(Banker's P.V.) $\overbrace{\text{B.P.V.} + \text{B.D.}} = \text{B.V.}$

First bracket shows that T.D. is the interest on T.P.V.

Second bracket shows that B.D. is the interest on B.V.

Illustration—Let a person borrow Rs. 1000 to be paid back with interest at 10% *p.a* after six months. Then the interest due after six months is Rs. 50 and the amount to be paid is Rs. 1050.

Here Rs. 1000 is the true P.V. and Rs. 1050 is the sum due. Now if a bill for Rs. 1050 due 6 months after

is discounted at 10% p.a., six months before it is due, then the amount of bill discount is the interest on Rs. 1050 for 6 months at 10% p.a.

$$\therefore \text{B.D.} = \text{Rs. } 1050 \times \frac{6}{12} \times \frac{10}{100} = \text{Rs. } 52.50.$$

$$\therefore \text{Banker's P.V.} = \text{Rs. } 1050 - \text{Rs. } 52.50 = \text{Rs. } 997.50$$

Clearly, B.D. is greater than T.D and B.P.V. is less than T.P.V.

The two present values can never be equal.

7. Banker's Gain. The difference between B.D. and T.D. is called the *Banker's Gain* (B.G.).

$$\therefore \text{B.G.} = \text{B.D.} - \text{T.D.}$$

In the above example, $\text{B.G.} = \text{B.D.} - \text{T.D.} = \text{Rs. } 2.50$

Further, $\text{B.D.} = \text{interest on B.V.}$

$$= \text{interest on (true P.V.} + \text{T.D.)}$$

$$= \text{interest on true P.V.} + \text{interest on T.D.}$$

$$= \text{T.D.} + \text{interest on T.D.}$$

$$\therefore \text{B.D.} - \text{T.D.} = \text{interest on T.D.}$$

Hence, $\text{B.G.} = \text{B.D.} - \text{T.D.} = \text{interest on T.D.}$

✓ **Ex. 1.** Calculate T.D., B.D., and B.G. on a bill for Rs. 6,100 due 6 months hence at 4% p.a. Show that the difference between B.D. and T.D. is equal to the interest on T.D. for 6 months at 4% p.a.

Let true P.V. = 100

Then T.D. = int. on 100 for 6 months at 4% p.a.

$$= 100 \times \frac{6}{12} \times \frac{4}{100} = 2.$$

$$\therefore \text{B.V.} = 100 + 2 = 102.$$

Then B.D. = int. on 102 for 6 months at 4% p.a.

$$= 102 \times \frac{6}{12} \times \frac{4}{100} = 2.04$$

T.D.	B.V.	$? = \text{Rs. } \frac{6,100 \times 2}{102} = \text{Rs. } 119.61$
2	102	
?	Rs. 6,100	

B.D.	B.V.	$? = \text{Rs. } \frac{6,100 \times 2.04}{102} = \text{Rs. } 122$
2.04	102	
?	Rs. 6,100	

$$\text{B.G.} = \text{Rs } 122 - \text{Rs } 119.61 = \text{Rs } 2.39$$

Interest on T.D. for 6 months at 4% *p.a.*

$$= \text{Rs } 119.61 \times \frac{6}{12} \times \frac{4}{100} = \text{Rs } 2.39$$

✓**Ex. 2.** If the difference between the banker's discount and the true discount on a sum due 4 months hence at 3% *p.a.* is 4s, find the sum due, banker's discount and true discount.

Let true P.V. = 100

$$\text{Then T.D.} = 100 \times \frac{4}{12} \times \frac{3}{100} = 1$$

$$\therefore \text{sum due} = 100 + 1 = 101$$

$$\therefore \text{B.D.} = 101 \times \frac{4}{12} \times \frac{3}{100} = \frac{101}{100}$$

$$\text{Hence B.D.} - \text{T.D.} = \frac{101}{100} - 1 = \frac{1}{100}$$

Difference	Sum	$? = 101 \times 4 \times 100s = \text{£ } 2020.$
$\frac{1}{100}$	101	
4s	?	

Difference	B.D.	$? = \frac{101}{100} \times 4 \times 100s = 404s = \text{£ } 20, 4s.$
$\frac{1}{100}$	$\frac{101}{100}$	
4s	?	

Difference	T.D.	$? = 4 \times 100s = \text{£ } 20.$
$\frac{1}{100}$	1	
4s	?	

Ex. 3. A man agrees to pay Rs 2000 for the lease of a house, dating from January 1, 1962, and to pay for

the same on July 15, 1962 subject to a discount of 5 p.c. *p.a.* How much should he pay?

The amount to be paid in advance is the true present value of Rs 2000.

Number of days from July 15

to January 1 = 170 days

Let true Present Value = 100

$$\text{Then T.D.} = 100 \times \frac{170}{365} \times \frac{5}{100} = \frac{170}{73}$$

$\therefore$ the sum due

$$= 100 + \frac{170}{73} = \frac{7470}{73}$$

July	16	days
Aug.	31	"
Sep.	30	"
Oct.	31	"
Nov.	30	"
Dec.	31	"
Jan.	1	"
	170	"

T.P.V. Sum due

$$\begin{array}{r|l} 100 & \frac{7470}{73} \\ ? & \text{Rs } 2000 \end{array} \quad ? = \text{Rs } \frac{2000 \times 100}{\frac{7470}{73}} = \text{Rs } 1954.48$$

✓Ex. 4. A owes B Rs 1600 but it is not due for payment till the end of 3 years from this date. How much should A pay B if he is willing to accept now in order to clear off the debt—(a) taking money to be worth 5% per annum simple interest (b) taking it to be worth 5% per annum compound interest, payable yearly? (C. U. 1951)

Here we are required to find the true present value of Rs 1600 as it does not involve payment by bill.

(a) Let true P.V. = 100.

$$\text{Then T.D.} = 100 \times 3 \times \frac{5}{100} = 15.$$

$\therefore$ the sum due = 100 + 15 = 115

$$\begin{array}{r|l} \text{T.P.V.} & \text{Sum due} \\ 100 & 115 \\ ? & \text{Rs } 1600 \end{array} \quad ? = \text{Rs } 1600 \times \frac{100}{115} = \text{Rs } 1391.30$$

(b) Let $T. P. V. = 100$,

Then amount after 3 years $= 100 (1.05)^3 = 115.8$.

$T. P. V.$ Sum due

100 115.8

? Rs 1600

$$? = \text{Rs } 1600 \times \frac{100}{115.8} = \text{Rs } 1381.70$$

$$\begin{aligned} x &= (1.05)^3 \\ \log x &= 3 \log 1.05 \\ &= 3 \times .0212 \\ &= .0636 \\ \therefore x &= 1.158 \end{aligned}$$

✓ **Ex. 5.** A bill of exchange for £ 713, 15s was drawn on 3rd April and made payable 3 months after date. It was discounted on 15th April at $4\frac{1}{4}\%$; What was the discounted value of the bill? (P. U. 1945, C. U. 1929)

Date of maturity—6th July, date of discounting—15th April

$\therefore$ unexpired period of the bill

= number of days from the 15th April to 6th July.

= $15 + 31 + 30 + 6 = 82$ days.

$$\therefore B. D. = \text{int. on } B. V. = \text{£ } 713\frac{3}{4} \times \frac{82}{365} \times \frac{4\frac{1}{4}}{100}$$

$$= \text{£ } 6 \text{ } 16\text{s } 3.5 \text{ d.}$$

$\therefore$ the discounted value of the bill

$$= \text{£ } 718. 15\text{s} - \text{£ } 6. 16\text{s. } 3.5\text{d} = \text{£ } 706. 18\text{s. } 1.5 \text{ d.}$$

Ex. 6. A bill of value Rs 1064 drawn on 23rd September, 1951 at 3 months after date is discounted on the 14th October, 1951. What is the discounted value of the bill? Answer to the nearest anna. What return does the rate represent to the banker? Answer to two decimal places correct. (C. U. 1952)

Bill legally due on - Dec 26

Bill discounted on Oct 14

$\therefore$ the bill has to run for 73 days. As no rate of interest is given, assume that as $5\% p. a.$

Oct.	17	days
Nov.	30	"
Dec.	26	"
	73	"

$$\text{Then } B. D. = \text{Rs } 1064 \times \frac{73}{365} \times \frac{5}{100} = \text{Rs } 10.64.$$

$$\therefore \text{discounted value of the bill} = \text{Rs } 1064 - \text{Rs } 10.64 \\ = \text{Rs } 1053.36$$

Thus the banker pays Rs. 1053.36 and receives Rs. 1064 after 73 days.

Hence he receives an interest of Rs. 10.64 in 73 days.

$$\therefore \text{Int. on Rs. } 1053.36 \text{ for 73 days} = \text{Rs. } 10.64$$

$$\therefore \text{,, ,, Rs. } 100 \text{ ,, } 365 \text{ ,, } = \text{Rs. } \frac{10.64}{1053.36} \times 100 \times \frac{365}{73} \\ = \text{Rs. } 5.05.$$

Hence the banker gets a return of 5.05%

Ex. 7. A bill for Rs. 12,000 was payable 60 days after sight. If was accepted on 4th March, 1953 and discounted on 20th March, 1953. What was the discounted value of the bill, if the rate of discount was $4\frac{1}{2}\%$.

(C.U. 1954)

Bill legally due on 63 days after 4th March. i.e., on 6th May.

Bill discounted on 20th March.

Mar.	27	days
Apr.	30	
	57	"
May.	6	"
	63	"

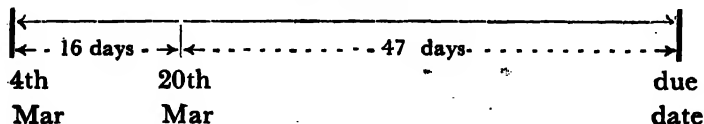
$\therefore$ number of days to run
= number of days from the 20th.
March to the 6th. May
= 47 days.

Mar.	11	days
Apr.	30	"
May.	6	"
	47	"

The number of days may be easily calculated as follows

The bill is legally due 63 days after 4th March. It is discounted on 20th March.

63 days.



From 4th to 20th March there are 20—4 or 16 days.

∴ the unexpired period = 63—16 or 47 days.

∴ B.D. = int. on Rs. 12,000 for 47 days at $4\frac{4}{5}\%$ p. a

$$= \text{Rs. } 12,000 \times \frac{47}{365} \times \frac{4\frac{4}{5}}{100} = \text{Rs. } \frac{27072}{365} = \text{Rs. } 74.17$$

Hence the discounted value of the bill

$$= \text{Rs. } 12,000 - \text{Rs. } 74.17 = \text{Rs. } 11,925.83$$

Ex. 8. A trader unable to pay the cash price of £ 495 of goods purchased by him makes out a bill payable 4 months after date, which will realise the amount if immediately discounted at 3%. Find the face value of the bill.

£ 495 is the P.V. of the new bill.

Let 100 be the face value of the new bill.

$$\text{Then B.D.} = 100 \times \frac{4}{12} \times \frac{3}{100} = 1$$

$$\therefore \text{P.V.} = 100 - 1 = 99.$$

P.V.	B.V.	
99.	100	?
£ 495	?	?

$$? = £ 495 \times \frac{100}{99} = £ 500$$

✓ **Ex. 9.** A banker discounts a bill for a certain amount which has 32 days to run before it matures at 5%. The discounted value of the bill is Rs. 995—9 as —10 P. Find the face value of the bill. (C. U. 1953)

Let the bill value = 100.

Then B.D. = int. on 100 for 32 days at 5%.

$$= 100 \times \frac{32}{365} \times \frac{5}{100} = \frac{32}{73}$$

$$\therefore \text{P.V. or discounted value of bill} = 100 - \frac{32}{73} = \frac{7268}{73}$$

P.V.	B.V.	
$\frac{7268}{73}$	100	?
Rs. 995 $\frac{59}{96}$	?	?

$$? = \text{Rs. } \frac{95579}{96} \times 100 \times \frac{73}{7268} = \text{Rs. } 1000 \text{ (app)}$$

✓ **Ex. 10.** *A* holds a bill of exchange from *B* for Rs. 1000 payable 6 months hence and *B* holds a bill for Rs. 1,750 payable 4 months hence. They agree to cancel their bills and *A* pays Rs. 125 in cash to *B* and gives a new bill payable 3 months hence.

Determine the amount of the new bill if the rate of discount is 5% per annum.

$$\text{B.D. on the 1st. bill} = \text{Rs. } 1000 \times \frac{6}{12} \times \frac{5}{100} = \text{Rs. } 25.$$

$$\therefore \text{P.V. of the 1st. bill} = \text{Rs. } (1000 - 25) = \text{Rs. } 975$$

$$\text{B.D. on the 2nd. bill} = \text{Rs. } 1,750 \times \frac{4}{12} \times \frac{5}{100} = \text{Rs. } \frac{175}{6}$$

$$\begin{aligned} \therefore \text{P.V. of the 2nd. bill} &= \text{Rs. } \left(1,750 - \frac{175}{6}\right) \\ &= \text{Rs. } \frac{10325}{6} = \text{Rs. } 1720.83 \end{aligned}$$

After cancellation of the bills the amount outstanding
= Rs. 1720.83 - Rs. 975 = Rs. 745.83

As *A* pays Rs. 125 cash down, the outstanding balance = Rs. 745.83 - Rs. 125 = Rs. 620.83, which is the Present Value of the new bill to be drawn at 3 months.

Let now the amount of the new bill = 100

$$\text{Then B.D.} = 100 \times \frac{3}{12} \times \frac{5}{100} = \frac{5}{4}.$$

$$\therefore \text{P.V. of the new bill of 100} = 100 - \frac{5}{4} = \frac{395}{4}$$

P.V.	B.V.	?
<u>395</u>	100	?
4		= Rs. 620.83
Rs. 620.83	?	= Rs. 628.69

Exercise 12

1. Find the Banker's Discount and Present Value of a bill for

(i) £ 2020 due 4 months hence and discounted at $3\frac{3}{4}\%$ *p. a.*

(ii) £ 1427. 10s drawn on April 5 at 3 months and discounted on April 17 at $4\frac{1}{4}\%$ *p. c. p. a.*

(iii) £ 292 which has 25 days to run and discounted at $5\frac{1}{16}\%$ *p. c. p. a.*

2. Find the true discount and true P.V. on a bill for

(i) £ 1407 for 73 days and discounted at $2\frac{1}{2}\%$ per annum.

(ii) Rs 2518.75 drawn on 9th. March for 4 months and discounted on April 30 at $3\frac{3}{4}\%$ per cent per annum.

3. Find B.D. and T.D. on a bill for

(i) Rs 9000 due in 6 months and discounted at 5% *p. a.*

(ii) £ 812 payable 90 days after sight and accepted on May 10 and discounted on May 30 at $7\frac{1}{2}\%$ *p. a.*

4. Calculate commercial discount at $7\frac{3}{4}\%$ on a bill of £ 75. 13s. 8d. (C. U. 1952)

5. Calculate Banker's Gain on £ 4040 due 4 months hence and discounted at 3% per annum. Show that this gain is equal to the interest on true discount for 4 months at 3% *p. a.*

6. A bill for Rs 4,500 is due in 6 months. Find the difference in the amount between the Banker's Discount and True Discount, the rate of interest being 5% . (C. U. 1950)

7. A bill of exchange drawn at 3 months on 11th. October, 1934 was discounted at $5\frac{1}{4}\%$ per cent on 19th. November 1934. Find the discount. (C.U. 1936)

8. (a) What do you understand by banker's discount and true discount? If the difference between true and banker's discount on a sum due in 4 months at 3 *p. c.* is Rs 10, find the amount of the bill. (C.U. 1957)

(b) The difference between Commercial and True Discount on a bill due after $7\frac{1}{2}$ months at 5% p.a. is Rs 9. Find the amount of the bill.

9. If the difference between True and Commercial Discount on a bill due 6 months hence at 5% p.a. is Rs 2.75, find the True Discount, Commercial Discount and the amount of the bill.

10. A banker discounts a bill which has 25 days to run before it is legally due at $5\frac{1}{4}\%$ per annum. This discount amounts to £1. 0s. 3d. For what sum was the bill drawn? (C.U. 1935)

11. A bill for Rs 12000 was payable 60 days after sight. It was accepted on 4th. March 1953 and discounted on 20th. March 1953. What was the discounted value if the rate of discount be $4\frac{4}{5}\%$ p. a. ?

12. (a) A merchant has a four months' bill for Rs 2,000 which the broker discounts at $2\frac{1}{4}\%$ per annum. Find what rate per cent per annum he should earn on the discounted value so that he may not suffer any loss on discounting the bill. (C.U. 1958)

(b) What is the actual rate of interest which a banker gets for the money when he discounts a bill legally due in 6 months at 4% p. a. ? (P.U. 1950)

13. At the rate of 4% p.a., find the Banker's discount, the true discount and the Banker's gain on a bill of exchange for Rs 650 due four months hence. (C.U. 1960)

14. If the true discount on a bill for 3 months is Rs 40 and the banker's discount for the same time and at the same rate is Rs 40.50, find the rate of discount and the amount of the bill.

15. The true discount on a bill for 5628 was Rs 28 at $2\frac{1}{2}\%$ per annum. When was the bill due?

16. The Banker's Discount on a bill discounted at $\frac{1}{2}\%$ p. a. was one-twentieth of the bill value. Find the unexpired term of the bill.

17. A bill for £ 228. 2s. 6d. drawn at 6 months was discounted by a banker at $5\frac{1}{5}\%$ on 15th. January, 1936. If the banker paid £ 225. 18s. 9d., on what date was the bill drawn? (I.M.C)

18. On 4th. February 1961, *A* drew and *B* accepted a bill at 4 months for Rs 1600. On 15th. February *A* discounted the bill at 5% . How much did he receive?

19. On 7th. January, 1960 *A* drew a bill on *B* at 3 months and discounted the acceptance with the Bank receiving Rs 2,475 in cash. For what amount was the bill drawn, if the rate of discount was 4% p. a.?

20. *A* owes *B* Rs 1800 due one month hence. But *A* finds that he would have funds after 2 months, accepts a bill drawn by *B* for such an amount as would fetch Rs 1800 if discounted after one month at 6% . Find the amount.

21. *A* owes *B* £ 600 and wishes to pay him in nine months. *B* demands bill at 9 months for an amount that, if immediately discounted at 4 per cent, would yield £ 600 due. For how much must the bill be drawn?

22. *A* holds a bill of exchange from *B* for Rs 2000 payable 6 months hence and *B* holds a bill from *A* for Rs 3500 payable 4 months hence. They agree to cancel their bills and *A* pays Rs 250 in cash to *B* and gives a new bill payable 3 months hence. Determine the amount of the new bill accepted by *A* if the rate of discount be 5% per annum. (C.U. 1956)

Chapter XI

Average Due Date and Equation of Payments

1. Average Due Date. When different payments are due on different dates to the same person by the same party it is found convenient by businessmen to settle the accounts by one single payment on one single date. Clearly this date should be such that neither party loses or gains in the transaction.

2. Method of ascertaining the average due date

1. Take a suitable date, say, the first due date of payment as the starting point or **zero date**.

2. Calculate the number of days from the zero date to the due date of each payment.

3. Multiply the amount of each payment by the corresponding number of days from the zero date.

4. Divide the total of these products by the total amount of the payments.

5. The quotient gives the number of days of the average due date from the zero date.

The number of days of the average due date from the zero date is called the **Equated time** or **Equated period**.

Hence, $\text{zero time} + \text{equated time} = \text{average due date}$

Note 1. The average due date lies between the two due dates of first and last payments.

Note 2. The zero date is an arbitrarily chosen date.

3. Equation of Payments.

Let $P_1, P_2, \dots$ be the amounts of payments due after $n_1, n_2, \dots$ days calculated from the zero date.

Let the average due date be N from the zero date. Then clearly N will be greater than some n and less than the rest.

Let N be greater than $n_1, n_2, \dots$ and less than $n_3, n_4, \dots$

There will be loss of interest in respect of payments $P_1, P_2, \dots$ for $(N - n_1), (N - n_2), \dots$ days and gain of interest in respect of payments $P_3, P_4, \dots$ for $(n_3 - N), (n_4 - N), \dots$ days.

But *by hypothesis* there is neither loss nor gain.

Hence equalising loss and gain, we have

$$(N - n_1)P_1 + (N - n_2)P_2 + \dots = (n_3 - N)P_3 + (n_4 - N)P_4 + \dots$$

$$\text{or } N(P_1 + P_2 + P_3 + \dots) = n_1P_1 + n_2P_2 + n_3P_3 + \dots$$

$$\text{Hence } N = \frac{n_1P_1 + n_2P_2 + n_3P_3 + \dots}{P_1 + P_2 + P_3 + \dots}$$

Note 1. When the principle of average due date is applied to payments by bills, by interest we mean banker's discount.

Note 2. The average due date is independent of the rate of interest.

✓ **Ex. 1.** A buys goods from T, the due dates for payment being as follows

- | | |
|----------------------|---------|
| 1. Due Jan. 10, 1962 | Rs 850 |
| 2. Due Mar. 13, 1962 | Rs 1200 |
| 3. Due Apr. 1, 1962 | Rs 580 |

Find the average due date.

Take Jan 10 as the zero date.

The number of days of each payment from the zero point is as under

- | | |
|---------------------|-------------------------|
| 1. Jan 10 to Jan 10 | = 0 |
| 2. Jan 10 to Mar 13 | = 21 + 28 + 13 = 62 |
| 3. Jan 10 to Apr 1 | = 21 + 28 + 31 + 1 = 81 |

Amount	period (days)	product.
850	0	0
1200	62	74,400
580	81	46,980
2,630		1,21,380

$$\therefore \text{equated time} = \frac{121380}{2630} = 46 + \dots \text{days}$$

Ignoring fraction of days in favour of the creditor the average due date is 46 days from zero date i.e., the 25th Feb, 1962.

Ex. 2. A merchant has purchased goods and has accepted the following bills drawn at 3 months each

Rs 800 drawn on March 10

Rs 350 drawn on April 15

Rs 625 drawn on May 4

Rs 275 drawn on June 8

He wishes payment to be made by one bill, payable one month after the average due date. Find this date.

The first bill is due on the 13th June.

Choose June 13 as the average due date

Bill	due date	period (days)	product
800	June 13	0	0
350	July 18	35	12250
625	Aug 7	55	34375
275	Sep 11	90	24750
2050			71375

$$\therefore \text{equated time} = \frac{71375}{2050} = 34 + \dots \text{days}$$

Ignoring fraction of days the average due date is 17th June.

$\therefore$ the required date is the 17th July.

Ex. 3. Henry has accepted the following bills

1. £ 450 drawn on 1st. Sept. 1961 at 3 months after date
2. £ 700 drawn on 3rd. Oct. 1961 at 2 months after date
3. £ 325 drawn on 11th. Nov. 1961 at 1 month after date
4. £ 800 drawn on 17th. Nov. 1961 at 4 months after date

He wants to retire the bills under rebate at 5% *p.a* on 15th. Oct. 1961. What funds does he require for the purpose ?

Take 15th Oct. as the zero date.

	Bill	due on	period (days)	product
1.	450	4th. Dec. 1961	50	22500
2.	700	6th. Dec. 1961	52	36400
3.	325	14th. Dec. 1961	60	19500
4.	800	20th. Mar. 1962	156	124800
	2,275			2,03,200

$$\therefore \text{equated time} = \frac{203200}{2275} = 89.3 \text{ days or } 89 \text{ days.}$$

$\therefore$ amount of rebate

= interest on £ 2275 for 89 days at 5% *p. a.*

$$= £ 2275 \times \frac{89}{365} \times \frac{5}{100}$$

$$= £ 27. 14s. 9d.$$

$$\therefore \text{ necessary funds } = £ 2275 - £ 27. 14s. 9d$$

$$= £ 2247. 5s. 3d.$$

Ex. 4.

Bills Payable		Bills Receivable	
Rs. 1,000	due on Jan 12, 1961.	Rs. 600	due on Jan 28
Rs. 800	... Feb 18	Rs. 350	... Mar 3
Rs. 650	... Mar 4	Rs. 100	... Mar 15

On which date may the accounts be settled without loss or gain to either party ?

Take Jan 12 as zero date.

Debits			Credits		
Bill	Period	Product	Bill	Period	Product
1,000	0	0	600	16	9,600
800	37	29,600	350	50	17,500
650	51	33,150	100	62	6,200
2,450		62,750	1,050		33,300

$$\therefore \text{equated time} = \frac{62,750 - 33,300}{2,450 - 1,050} = \frac{29,450}{1,400} = 21 + \dots \text{days.}$$

$\therefore$ the date of settlement 21 days after Jan 12 i.e., Feb 2.

Ex. 5. One dealer offers to sell a radio on the following terms: Rs. 200 cash down and Rs. 20 per mensem for 15 months following. Another offers the terms: Rs. 250 down and Rs. 25 per mensem for 10 months following.

Which offer is cheaper to the customer and by how much, if money is worth 6% per annum?

Take the day of cash payment as zero date:—

First dealer

Equivalent amount

$$= 200 \times 0 + 15 \text{ payments each Rs. 20 due after}$$

$$1, 2, \dots, 15 \text{ months}$$

$$= 200 \times 0 + 15(1 + 2 + \dots + 15) \text{ due after 1 month}$$

$$= 0 + 15 \cdot \frac{15 \times 16}{2} \quad \quad \quad \text{''} \quad \text{''} \quad \text{''}$$

$$= 1800 \quad \quad \quad \text{''} \quad \text{''} \quad \text{''}$$

Second dealer

Equivalent amount

$$= 250 \times 0 + 10 \text{ payments each Rs. 25 due after}$$

$$1, 2, \dots, 10 \text{ months}$$

$$= 250 \times 0 + 25(1 + 2 + \dots + 10) \text{ due after 1 month}$$

$$= 0 + 25 \cdot \frac{10 \times 11}{2} \quad \quad \quad \text{''} \quad \text{''} \quad \text{''}$$

$$= 1375. \quad \quad \quad \text{''} \quad \text{''} \quad \text{''}$$

Although, the first series of payments is equal to the second series, the customer by accepting the second offer gains interest on Rs. (1800—1375) or Rs. 425 for 1 month.

∴ the second offer is cheaper to him and his gain in respect of interest

= interest on Rs. 425 for 1 month at 6% *p. a.*

$$= \text{Rs. } 425 \times \frac{1}{12} \times \frac{6}{100} = \text{Rs. } 1.12$$

Ex. 6. A man borrowed Rs 5000 on 15th. of January, 1959 and agreed to pay Rs 1500 after 1 month, Rs 1600 after 3 months, Rs 2000 after 4 months and Rs 500 after 5 months in full redemption of his debts. Find the rate of interest charged.

Take 15th Jan. as zero date.

The average due date

$$\begin{aligned} &= \frac{1500 \times 1 + 1600 \times 3 + 2000 \times 4 + 500 \times 5}{1500 + 1600 + 2000 + 500} \\ &= \frac{16800}{5600} = 3 \text{ months.} \end{aligned}$$

Hence the equivalent amount is Rs 5600 payable 3 months after the 15th January.

∴ interest on Rs 5000 for 3 months

$$= \text{Rs } 5600 - \text{Rs } 5000 = \text{Rs } 600.$$

Hence the rate of interest per cent per annum

$$= \frac{600}{5000} \times 100 \times \frac{12}{3} = 48.$$

4. Interest on Instalment sale.

Ex. 7. If I buy a sewing machine with cash down I pay Rs 190. If I buy it on instalment basis I pay Rs 10 cash down and 20 monthly instalments of Rs 10. What rate of interest is calculated in the second case?

(C. U. 1946)

Take date of purchase as zero date.

Then average due date of 20 instalments of Rs 10 each

$$= \frac{10 \times 1 + 10 \times 2 + 10 \times 3 + \dots + 10 \times 20}{10 + 10 + 10 + \dots + 10}$$

$$= \frac{10(1+2+3+\dots+20)}{10 \times 20} = \frac{10 \times 20 \times 21}{10 \times 20 \times 2} = 10\frac{1}{2} \text{ months.}$$

$\therefore$ Rs 190 cash down is equivalent to Rs 10 cash down + Rs $10 \times 20 (=Rs 200)$ payable after $10\frac{1}{2}$ months.

$\therefore$ interest on Rs 180 for $10\frac{1}{2}$ months

$$= Rs 200 - Rs 180 = Rs 20$$

$\therefore$ the rate of interest per cent per annum

$$= \frac{20}{180} \times 100 \times \frac{12}{10\frac{1}{2}} = \frac{800}{63} = 12.7$$

5. The average rate of interest charged can be found from the following formula

$$\text{Rate of interest per unit per period} = \frac{2}{n+1} \cdot \frac{D}{C-h}$$

where n = number of equal periodical instalments

D = difference between instalment and cash price

and $C-h$ = outstanding balance after the first instalment is paid (c = cash price, h = first instalment)

In the foregoing example,

$$n=20, \quad D=20, \quad C-h=190-10=180$$

$$\therefore \text{rate per unit per month} = \frac{2}{20+1} \times \frac{20}{180}$$

$\therefore$ rate per cent per annum

$$= \frac{2}{21} \times \frac{20}{180} \times 100 \times 12 = 12.7$$

The problem of *true* rate of interest is, however, very complicated and beyond the scope of this elementary book.

Exercise 13

1. Find the average due date for the following payments :

Rs. 500 payable on 1st Jan ; Rs. 1500 on 18th Jan ;
Rs. 700 on 11th March ; Rs. 1100 payable on 3rd May and
Rs. 1800 payable on 30th June. (C. U. 1931)

2. What is the average due date for the following instalments payable under a contract

June	1st	Rs	3200	
July	1st	Rs	4000	
3 Aug	31st	Rs	1800	(C. U. B. Com)

3. Find the average due date for the following payments :

Rs. 300 payable on 10th Nov, 1956 ; Rs. 700 payable
on 21st Nov. 1956 ; Rs. 1200 payable on 4th Dec, 1956 ;
Rs. 650 payable on 22nd Dec, 1956 ; Rs. 2100 payable on
6th Jan. 1957 and Rs 850 payable on 20th Jan, 1957
(C. U. 1957)

4. An up-country merchant requests his suppliers to
to make out a single bill in lieu of the following :—

Bill for Rs.	2585	payable	6th	Jan.	1944
"	"	3860	"	13th	Feb. 1944
"	"	5000	"	26th	Feb. 1944
"	"	1650	"	6th	Mar. 1944

What would be the due date of the new bill ?

(P. U. 1949)

5. What do you understand by "Average Due Date" ?
How is it calculated ? Determine the Average Due Date
of the following :—

Rs.	500	due	on	6th	July
Rs.	800	due	on	9th	August
Rs.	1,000	due	on	2nd	September
Rs.	700	due	on	21st	November
Rs.	3,400	due	on	31st	December

(C. U. 1955)

6. Calculate the date on which the total amount of the following bills of exchange may be paid by a single cheque without any loss to any one of the parties :—

<i>Date of the Bill of Exchange</i>	<i>Period of the bill</i>	<i>Amount Rs</i>
1.1.61	2 months	250
15.1.61	3 months	300
30.1.61	2 months	600
2.2.61	3 months	300
15.2.61	2 months	400

(C. U. 1961)

7. A man has purchased goods, the due dates for payments are as follows :—

<i>Date</i>	<i>Amount</i>	<i>Due Date</i>
Jan. 5	500/-	Mar. 8
Jan. 17	300/-	Apr. 20
Jan. 29	700/-	Feb. 1
Feb. 10	1,800/-	Mar. 13
Mar. 23	2,000/-	July. 26
Apr. 3	200/-	May. 6
Apr. 29	600/-	July. 2

the average due date.

(C. U. B. Com

8. A trader accepted the following bills due on different dates, but he desired to have the bills cancelled and to accept a new bill for the whole amount payable on a single date. Find this date.

The bills were

Jan. 4	Rs. 200	due Apr 7, 1958
Feb. 2	Rs. 300	due Mar 5, "
Feb. 10	Rs. 700	due Mar 13, "
Mar. 5	Rs. 250	due May 8, "
Apr. 14	Rs. 500	due June 17, "

9. A trader accepted the following bills. But he desired to retire the bills under rebate at 5% *p. a.* on 17th. Jan, 1961. What funds did he require for the purpose ?

The bills were

Bill for Rs. 400	payable 1st	Jan. 1961
" " Rs. 650	" 10th	Feb. 1961
" " Rs. 825	" 4th	Mar. 1961
" " Rs. 525	" 12th	Mar. 1961

• 10. On June 15, 1962 a merchant had the following bills due as under

Bills Payable	Bills Receivable
Rs. 600 after 10 days	Rs. 950 after 15 days
Rs. 750 " 25 "	Rs. 300 " 19 "
Rs. 1000 " 13 "	Rs. 700 " 22 "

On which date the balance might be paid in a lump sum so that neither party would gain or lose?

• 11. The price of a gramophone is Rs. 650 cash or 5 equal monthly instalments of Rs. 115 each, the first instalment is due 1 month after purchase. Find the rate of interest charged.

Chapter XII

Foreign Exchange

1. **Mint Par of Exchange.** Gold is the international medium of exchange. The monetary units of different countries having gold standard are compared by the intrinsic values of their gold contents. The relation between monetary units of two gold standard countries is known as **rate of exchange**. This is also called **mint par of exchange** as the rate of exchange depends on the weight and fineness of the standard gold the countries might use. A Pound Sterling contains 123·27447 grains of (British) standard gold $\frac{1}{12}$ ths fine, i.e., the British standard gold, an alloy of pure gold and a base metal, contains 11 parts of pure gold to 12 parts of alloy. Similarly, \$ 10 (10 dollars) contained 258 grains of (U. S. A) standard gold $\frac{9}{10}$ ths fine.

2. In case of paper currencies the rate of exchange depends on their **Purchasing power parity**. By purchasing power parity is meant a parity point of exchange where the respective paper currencies have the same purchasing power in their own countries. For this purpose a base year with normal index numbers at 100 for both the countries is selected. Then the relative values of the two currencies would be *inversely proportional* to their current index numbers.

For example, considering 1929 as base year (i.e., with index number 100) when the rate of exchange between India and Germany was Rs. 100=153·22 R.M., we may determine the purchasing power parity in any subsequent year when the respective index numbers were 180 and 130.

The rate of exchange was

$$\text{Rs. } 100 = \frac{180}{100} \times 153 \cdot 22 \text{ R. M.} = 110 \cdot 67 \text{ R. M.}$$

$$\text{or Re. } 1 = 1 \cdot 1067 \text{ R. M.}$$

3. Specie Points.

The actual rate of exchange is, however, variable being subject to the conditions of parity and the demand and supply of bills of exchange or drafts and transmission of coin (specie) or metal bars (bullions). Normally the rate of exchange does not fluctuate beyond two well-defined points. For in contrary case, actual flow of specie or bullion would take place.

Theoretically, the rate of exchange in buying and selling foreign bills varies between *mint par of exchange plus cost of freight and insurance* and *mint par of exchange minus cost of freight and insurance*. These two limits are called *gold exporting* or *upper specie point* and *gold importing* or *lower specie point* respectively.

4. Arbitrated Rate of Exchange. Sometimes it is cheaper to have exchange between two countries through the medium of one or more other countries. Indirect rate of exchange arrived through one or more intermediate countries is known as *arbitrated rate of exchange*, whereas the direct rate of exchange between two countries is called the *cross rate*.

Ex. 1. Find, which method is cheaper for a Calcutta merchant to pay 1000 francs

by (a) direct exchange at Rs 100=201.5 fr.

or (b) payment through London, Re 1=1s 6d and £ 1=27 fr ?

Under the indirect method of payment

$$\text{Re } 1 = 1\frac{1}{2}\text{s} = \text{£ } \frac{3}{40} = \frac{3}{40} \times 27 \text{ fr} = \frac{81}{40} \text{ fr} = 2.025 \text{ fr.}$$

$$\therefore \text{Rs } 100 = 202.5 \text{ fr.}$$

Thus by paying Rs 100 in India the merchant can settle a debt of 201.5 fr in France by direct exchange and 202.5 fr. by indirect exchange.

Hence the second method is cheaper and his saving = 1 fr. for every Rs 100

5. The arbitrated rate of exchange can be calculated by a suitable method, called the Chain Rule, which has been fully discussed in Chapter VII, Part I. The following example illustrates the method.

Ex. 2. An Indian merchant ordered silk in France at 50 francs per metre. If the import duty is 20% *ad valorem*, find the cost of silk in rupees per yard, given

Re 1 = 1s 6d, £ 1 = 120 fr and 1 metre = 39·37 inches

Let Rs. x = cost of 1 yard. Then

$$\begin{aligned}\text{Rs. } x &= 1 \text{ yard} \\ 1 \text{ yard} &= 36 \text{ inch} \\ 39\cdot37 \text{ inch} &= 1 \text{ metre} \\ 1 \text{ metre} &= 50 \text{ fr} \\ 120 \text{ fr} &= \text{£ } 1 \\ \text{£ } 1 &= 20\text{s} \\ 1\frac{1}{2} \text{ s} &= \text{Re } 1\end{aligned}$$

$$\therefore x \times 1 \times 39\cdot37 \times 1 \times 120 \times 1 \times 1\frac{1}{2} = 1 \times 36 \times 1 \times 50 \times 1 \times 20 \times 1$$

$$\text{or, } x = \text{Rs } \frac{36 \times 50 \times 20}{39\cdot37 \times 120 \times 1\frac{1}{2}} = \text{Rs } \frac{200}{39\cdot37} = \text{Rs } 5\cdot08.$$

Ex. 3. Express 3s 6d per yard in francs per metre, given 1 metre = 39·37 inch, £ 1 = 20·25 R. M., Re 1 = 6·25 fr, 150 R. M. = Rs. 100.

Let x fr = cost of 1 metre. Then

$$\begin{aligned}x \text{ fr} &= 1 \text{ metre} \\ 1 \text{ metre} &= 39\cdot37 \text{ inch} \\ 36 \text{ inch} &= 1 \text{ yard} \\ 1 \text{ yard} &= 3\frac{1}{2} \text{ s} \\ 20 \text{ s} &= \text{£ } 1 \\ \text{£ } 1 &= 20\cdot25 \text{ R.M.}\end{aligned}$$

$$150 \text{ R.M.} = \text{Rs. } 100$$

$$\text{Re } 1 = 6\cdot25 \text{ fr.}$$

$$\begin{aligned}\therefore x \times 1 \times 36 \times 1 \times 20 \times 1 &= 1 \times 39 \cdot 37 \times 1 \times 3\frac{1}{2} \times 1 \times 20 \cdot 25 \\ \text{or } x &= \frac{39 \cdot 37 \times 3\frac{1}{2} \times 20 \cdot 25}{36 \times 20} \text{ fr,} \\ &= \frac{558 \cdot 06975}{144} \text{ fr} = 3 \cdot 88 \text{ fr.}\end{aligned}$$

The chain rule may also be expressed as in the following example.

Ex. 4. Determine the par of exchange between London and New York from the following data. £ 1 contains 123·27447 grains of standard gold which is $\frac{11}{16}$ ths fine and £ 10 contain 258 grains of standard gold (U.S.A) which is $\frac{9}{10}$ ths fine. (C.U. 1944)

The chain may be arranged thus

£	Br. St. gold	Pure gold	U. S. St. gold	\$
1	→ 123·27447			
	12	→ 11		
		9	→ 10	
			258	→ 10

$$£ 1 = \$ \frac{123 \cdot 27447 \times 11 \times 10 \times 10}{1 \times 12 \times 9 \times 258} = \$ 4 \cdot 867$$

6. Short and long rates. The term 'Short Exchange' denotes cheques and bills payable at sight and the rate quoted for them is called *Short Rate*. The term 'Long Exchange' denotes bills which have a term to run before maturity. The rate for the latter bills is known as *Long Rate*. The difference between the two rates is due to various factors, such as, cost of stamps, discount and risk.

7. Foreign bills are generally *tenor bills* and hence for *spot exchange* the rate must include the effect of discount.

Ex. 5. A 6 months' bill on Paris is sold at Calcutta at Rs. 100=165 fr. If the rate of discount for a 6 months' bill in Paris be 5%, find the rate for spot exchange.

Rate of 6 months' bill for Rs. 100=165 fr.

Less discount at 5% = 4·13 fr.

Spot exchange=160·87 fr.

Ex. 6. If the exchange in Glasgow on Paris be 124·21 and the rate of banker's discount in Glasgow be 4 per cent per annum, what debt can be discharged in Paris by a person who holds a 3 months' bill in Glasgow for £ 1, 000 ?

(C.U. 1941)

The bill is a tenor bill.

Banker's discount = int. on £ 1,000 for 3 months at 4% *p. a.*

$$= £ 1000 \times \frac{3}{12} \times \frac{4}{100} = £ 10$$

∴ discounted value of bill = £ 1,000 - £ 10 = £ 990

Now, rate of exchange is £ 1 = 124·21 fr.

∴ £ 990 = 124·21 × 990 fr = 122967·9 fr.

Ex. 7. An Indian trader buys 22000 Swiss francs worth of machinery. He buys Sterling bills of value just sufficient to meet the cost of machinery. If exchange stands at Re 1 = 1s 6d and £ 1 = 35·20 Swiss francs, find the value of bills in rupees. Neglect charges. Answer to the nearest pie.

(C.U. 1952)

$$£ 1 = 35·20 \text{ fr.}, \text{ Re } 1 = 1\frac{1}{2} \text{ s} = £ \frac{3}{40}$$

$$\therefore 1 \text{ fr.} = £ \frac{1}{35·20} = \text{Rs } \frac{1}{35·20} \times \frac{40}{3} = \text{Rs } \frac{25}{66}$$

$$\therefore 22,000 \text{ fr.} = \text{Rs } \frac{25}{66} \times 22,000 = \text{Rs. } 8333 \text{ 5a } 4\text{p.}$$

8. Sometimes speculators borrow money and speculate in foreign exchange. In such a case interest on the loan is to be calculated. *The exchange profit less interest is the net profit.*

Ex. 8. A broker wishing to buy a bill for 3,500 dollars drawn on New York borrowed £ 1,250 from a money lender for 15 days at 5½%. He bought the bill at 3½ exchange and sold it 15 days later at 3½ exchange and repaid the loan. How much did he gain or lose?

The cost of a bill for $3\frac{7}{16}$ dollars = £ 1.

∴ the cost of a bill for 3,500 dollars = £ $\frac{1}{3\frac{7}{16}} \times 3,500$

$$= £ \frac{16}{55} \times 3,500$$

The selling price of the bill = £ $\frac{1}{3\frac{1}{3}} \times 3,500 = £ \frac{3}{10} \times 3,500$

∴ his gain = £ $\frac{3}{10} \times 3,500 - £ \frac{16}{55} \times 3,500$

$$= £ \left(\frac{3}{10} - \frac{16}{55} \right) \times 3,500$$

$$= £ \frac{1}{110} \times 3,500 = £ 31 \text{ 16s } 4\text{d}$$

Interest paid to money lender

= int. on £ 1,250 for 15 days at $5\frac{3}{4}\%$ p. a.

$$= £ 1,250 \times \frac{15}{365} \times \frac{23}{400} = £ 2 \text{ 19s } 1\text{d}.$$

∴ his net gain = £ 31 16s 4·4d - £ 2 19s 1d

$$= £ 28 \text{ 17s } 3\text{d}$$

9. Remittance to a Foreign Country.

There are various methods of remittance or transfer of money from one country to another of which the following are the most important.

(1). **Bill of Exchange.** It is the most convenient instrument for foreign payment. A bill is 'an unconditional order in writing signed by the maker, directing a certain person to pay on demand, or at a fixed or determinable future time, a certain sum of money to or to the order of a certain person or to bearer'

Usance. It is the usual time allowed at which Foreign Bills are made payable in a particular country. The usance between London and Bombay is 30 days after sight.

Functions of a bill of Exchange. Remittance and accommodation are the two main functions of a bill of exchange.

As already stated three persons are involved in a bill—the *drawer* who writes out and signs the bill, the *drawee* on whom it is drawn and the *payee* to whom or to whose order, the money is payable.

Bill Brokers. They buy and sell bills of exchange and act as a connecting link between the traders who hold foreign bills and require cash and those who wish to remit money to foreign countries and require bills as such. Thus bill brokers reduce transfer of specie or bullion between different countries to a minimum and render invaluable service in international trade.

(2). **Telegraphic Transfers (T. T).** It is the prompt method of remittance. Under this arrangement, the foreign branch of an Exchange Bank is advised by cable or telegraph for payment of an equivalent sum of money already received to some specified person. The charge for T.T. is higher than ordinary remittance.

(3). **Bank Draft or Demand Draft.** These are issued by local banks on foreign banks after receiving money locally and forwarded to the payee.

10. Quotations. The rates at which foreign bills can be bought and sold are subject to daily fluctuations. Long Rate includes (a) cost of stamps, (b) discount for the time between the dates of quotation and maturity, and (c) the risk that the bill may not be paid when due. This is the cause for difference between Short and Long Rates. These rates are daily published by Exchange Banks and called *quotations*.

Generally, currencies of foreign countries are quoted for a unit of the Standard currencies of the country of quotation.

In England the unit of exchange is £ for most European countries.

Thus £ 1 = 124·21 franc. (London—Paris)
 £ 1 = 25·225 franc (London—Geneva)
 £ 1 = 92·46 Lira (London—Rome) etc.

Quotations are made for the most important trade centres of the world, where foreign bills can be bought and sold.

For *Eastern Countries*, the foreign monetary unit is generally expressed in British money.

Thus Re 1 = 1s 6 d (India), \$ = 2s. 4d. (Singapore) etc.

In *New York* the unit of exchange is dollar (\$) except that on London which is quoted at so many dollars to £.

The rates are different for different classes of remittance.

The T. T. Rate sometimes includes cost of cable.

The rate for cheques or D. D. is calculated from T. T. Rate by deducting the interest on it for the term between the date of cheque and date of payment.

Two rates are usually quoted for each class—one for buying and another for selling. If the quotation be in foreign currency the lower rate is the selling rate for banks or the buying rate for customers, the higher rate is the buying rate for banks or the selling rate for customers.

Thus on a certain day Calcutta Market, quoted T. T. rates Calcutta—London as $1/5\frac{3}{8}d$ — $5\frac{1}{2}d$. This means that the bank's selling rate on that day at Calcutta on London was 1s $5\frac{3}{8}d$ per rupee and bank's buying rate was 1s $5\frac{1}{2}d$ per rupee.

If the quotations are in the country's own currency, the rule is reversed.

Thus for the quotation Bombay-New York 475—476, a Bombay merchant can buy bills on New York at 476 rupees to 100 dollars.

The rate for T. T. is highest, that for D. D. is lower and that for usance bills still lower.

11. The following is a list of some important countries arranged in alphabetical order with respective monetary units.

Country		Monetary Unit
Argentina		Peso (= 100 centavos)
Austria		Schilling (= 100 groschen)
Australia	(£)	Pound (= 20 shilings)
Belgium		Belga (= 5 francs)
Brazil		Milreis (= 1000 reis)
Canada	(\$)	Dollar (= 100 cents)
Ceylon		Rupee (= 100 cents)
Chile		Peso (= 100 centavos)
China		Dollar or yuan (= 100 cents)
Denmark		Krone (= 100 ore)
Egypt		Pound (= 100 piastres)
France	(fr)	Franc (= centimes)
Germany	(R. M.)	Reichsmark (= 100 pfennige)
Great Britain	(£)	Pound (= 20 shillings)
Greece		Drachma (= 100 lepta)
Holland		Florin (= 100 cents)
Hungary		Pengo (= 100 filler)
India	(Re)	Rupee (= 100 nP)
Italy		Lira (= 100 centesimi)
Japan		Yen (= 100 sen)
Korea		Won (= 100 chon)
Norway		Krone (= 100 ore)
Poland		Zloty (= 100 grosz)
Portugal		Escudo (= 100 centavos)
Russia		Roubles
South Africa	(£)	Pound (= 20 shillings)
Switzerland	(fr)	Franc (= 100 centimes)
U. S. A.	(\$)	Dollar (= 100 cents)
Yugoslavia		Dinar (= 100 paras)

12. Bank's profit. If the bank desires to make a margin of profit that must be *deducted* from bank's selling rate.

In the previous example the bank was selling at $1s\ 5\frac{3}{8}d$.

If it desired to keep a margin of $\frac{1}{4}d$ as profit,

Bank's actual selling rate $= 1s\ 5\frac{3}{8}d - \frac{1}{4}d$ or $1s\ 5\frac{1}{8}d$ per rupee.

The Bank's golden rule is "buy high, sell low".

Ex. 9. Calculate 2 months Bank's rate for buying bills in London,

given T.T. London—Bombay buying rate $1s\ 6d$
 T.T. Bombay—London buying rate $1s\ 6\frac{3}{8}d$
 Discount rate in London 2%
 Discount rate in Bombay $2\frac{1}{2}\%$
 Risk plus stamp $\frac{1}{8}d$ in the rate.

T.T. London—Bombay bank's rate on London $\underline{= 18d.}$

Deduct (1) discount for 2 m at 2% $\cdot 06$

(2) risk plus stamp $\cdot 125$ $\underline{\cdot 185d}$
 $17\cdot 815d.$

$= 1s\ 5\frac{13}{16}d.$ (app)

Ex. 10. A calcutta merchant has to pay \$ 300 dollars at New York at the end of 3 months. If the T.T. rates are 20 cents— $20\frac{1}{4}$ cents and discount rate in New York is 4% , find the cost of a 3 months' bill if the banker keeps a margin of $\frac{1}{2}$ cent.

T. T. selling rate $= 20$ cents

Less Bank's margin

$\frac{1}{2}$ cent $= 0\cdot 5$ cents
 $\underline{\hspace{1.5cm}}$
 $19\cdot 5$ cents

Add 3 months' int. $= \cdot 2$

Bank's rate $= 19\cdot 7$ cents

$\therefore$ Cost of \$ 300 = Rs.	$\frac{300 \times 100}{19\cdot 7}$	Re	dollar
		1	$\frac{19\cdot 7}{100}$
	$= Rs. 1522\cdot 84$	?	300

Ex. 11. A merchant in New York has to pay for goods bought in London valued at 5000 dollars, a dollar being equal to 4s 6d. For how much in English money should he draw a bill when bills on London are at a premium of $9\frac{1}{2}$ per cent ? (C.U. 1945)

$$1 \text{ dollar} = 4\frac{1}{2}\text{s}$$

$$\therefore 5000 \text{ dollars} = 5000 \times 4\frac{1}{2}\text{s} = \text{£ } 1125$$

To pay £ 1125 the merchant requires a bill for £ 1125

$$\text{Cost of £ 100 bill on London} = \text{£ } 109\frac{1}{2}.$$

$$\begin{aligned} \therefore \text{£ } 1125 &= \text{£ } \frac{1125}{100} \times 109\frac{1}{2} \\ &= \text{£ } 1231.875 \\ &= \text{£ } 1231 \text{ } 17\text{s } 6\text{d} \end{aligned}$$

Ex. 12. A Calcutta merchant has to remit \$ 4,000 to New York. T.T. are selling at Rs. 4.75 per dollar and sight drafts at \$.215 per rupee. Which is the cheaper method of remittance and how much does he gain by adopting the cheaper method ?

$$\text{T.T. 1 dollar} = \text{Rs. } 4.75$$

$$\text{S.D. 1 dollar} = \text{Rs. } \frac{1}{.215} = \text{Rs. } \frac{1000}{215} \text{ (4.65)}$$

$\therefore$ Sight draft is cheaper costing less rupees

$$\text{Gain on 1 dollar} = \text{Rs. } \left(4.75 - \frac{1000}{215} \right) = \text{Rs. } \frac{17}{4 \times 43}$$

$$\therefore \text{gain on 4000 dollars} = \text{Rs. } \frac{17}{4 \times 43} \times 4000 = \text{Rs. } 395.35.$$

Ex. 13. On 20th Jan. 1959, Indo British Traders advised their Bankers to remit £ 25 to their Agent in London by T.T. and to debit their account current with them. On that date Calcutta Market quoted T.T. rates Calcutta—London to be 1s $5\frac{1}{2}$ d— $5\frac{1}{2}$ d. Calculate the amount that would be debited by the Bank who kept a margin of $\frac{1}{4}$ d only.

(C.U. 1960)

The Banker's buying rate is

$$\text{Re } 1 = 1\text{s} - 5\frac{1}{8}\text{d}$$

$$\text{Less gain } \frac{1}{2}\text{d}$$

$$\text{Actual rate Re } 1 = 1\text{s. } 4\frac{5}{8}\text{d} = 16\frac{5}{8}\text{d.} = \text{£ } \frac{133}{8 \times 12 \times 20}$$

∴ the amount to be debited

= amount to be paid in rupees

$$= \text{Rs. } 25 \div \frac{133}{8 \times 12 \times 20}$$

$$= \text{Rs. } \frac{25 \times 8 \times 12 \times 20}{133}$$

$$= \text{Rs. } 360.90.$$

Rs.	£
1	$\frac{133}{8 \times 12 \times 20}$
?	25

Ex. 14. Which is more advantageous of the following methods which a London merchant might adopt to settle a debt in Paris?

(a) Buying French 4 months' bills at 25.76

(b) Buying Vienna bills at 24.30 and selling these at Paris at 1.05 (brokerage 1 per mille, discount 3 per cent per annum)? (C.U. 1943)

mille = 1 thousand.

(a) Let amount of 4 months' French Bill = 100 fr.

$$\text{Then bill discount} = 100 \times \frac{4}{12} \times \frac{3}{100} = 1 \text{ fr.}$$

∴ discounted value of bill = 100 - 1 fr = 99 fr.

To buy 100 fr bill, cost = 100 fr + brokerage 1 fr per thousand
 = 100 fr + .1 per hundred
 = 100.1 fr.

Hence cost of paying a debt of 99 fr = £ $\frac{100.1}{25.76}$ (£ 1 = 25.76 fr)
 = £ 3.886

(b) To buy 100 sch bill, the cost

= 100 sch + brokerage 1 per thousand.

= 100 + .1 sch

= £ $\frac{100.1}{24.30}$ (£ 1 = 24.30 sch)

Sale proceeds of the 100 sch bill

$$= 100 \text{ sch} - \text{brokerage at 1 per thousand}$$

$$= 100 \text{ sch} - 1 \text{ sch} = 99 \cdot 9 \text{ sch}$$

$$= 99 \cdot 9 \times 1 \cdot 05 \text{ fr (1 sch} = 1 \cdot 05 \text{ fr)}$$

Hence the cost of discharging a debt of $99 \cdot 9 \times 1 \cdot 05$ fr

$$= \text{£} \frac{100 \cdot 1}{24 \cdot 30}$$

∴ cost of discharging a debt of 99 fr.

$$= \text{£} \frac{100 \cdot 1 \times 99}{24 \cdot 30 \times 99 \cdot 9 \times 1 \cdot 05}$$

$$= \text{£} \frac{1001 \times 11 \times 200}{243 \times 111 \times 200} = \text{£} 3 \cdot 888$$

Hence the first method is cheaper.

Exercise 14

1. Calculate the par exchange between *lire* and dollars, expressing 1 dollar in lire and 1 lire in dollars, given 4·85 dollars = 92·50 lire.

2. Given that an Indian rupee contained standard Silver $\frac{5}{16}$ ths fine, find the bullion value of an Indian rupee, given that the value of standard silver is Rs. 160 per 100 tolas. Neglect cost of alloy.

3. Find the cost of remitting £ 240 from Bombay to London if the rate of exchange Re 1 = 1s 6 d.

4. Determine the par of exchange between London and Berlin, assuming £ 1 = 123·27 grains of gold $\frac{1}{16}$ ths fine and 10 marks = 61·4 grains of gold $\frac{9}{16}$ ths fine. (P. U. 1949)

5. If 16·1 Marks = 100 francs, 19 dollars = 80 Marks, 25 francs = £ 1 and 1s 4d = Re 1, find how many rupees are equivalent to 6118 dollars?

6. Coal is sold in England f. o. b. for 17s a ton and in France the price realised is 30 francs per tonneau (1000 Kg). What is the total amount in francs per tonneau of freight and profit?

7. Safety razor blades are quoted at German ports at RM 175 per Kg. Find the imported cost of blades per gross, being given £ 1=19.50 RM ; Re 1=1s 6d ; 1 Kg=2.2 lb : weight of a blade=2.20 grains ; average cost of transport, insurance etc.=8as 7.4 p per gross.

8. A man gives 25 twenty-franc pieces in exchange for 20 sovereigns. Calculate his gain in English money and in French money. What is his percentage gain ? (assume £ 1=25.22 fr) (C.U. 1930)

9. An Indian trader buys 25000 Swiss francs worth of machinery. He buys Sterling bills of value just sufficient to meet the cost of machinery. If exchange stands at Re 1=1s 6d and £ 1=35.20 Swiss francs, find the value of bills in rupees. Neglect charges. Answer to the nearest pie. (C.U. 1952)

10. An Indian importer ordered 2000 yds of Linen at 2s 6d a yard. He paid for freight £ 2. 3s, insurance £1 7s 6d and dock charges Rs. 26 4 as. If exchange stands at 1s 4d, at what price per yard should the importer sell to earn $17\frac{1}{3}\%$ profit on his cost ? (C.U. 1954)

11. A person has to remit \$ 10,000 to New York. Telegraphic Transfers are selling at Rs. 4.76 per dollar and sight drafts at \$.213 to the rupee. Which is the cheaper method of remittance and how much does he gain by adopting the cheaper method ? (C.U. 1955)

12. A merchant in New York has to pay for goods bought in London valued at 15,000 dollars, a dollar being equal to 4s 6d. For how much £ should he draw a bill when bills on London are at a premium of $8\frac{1}{4}\%$ per cent ?

13. If the exchange in Glasgow on Paris be 124.21 and the rate of banker's discount in Glasgow be 4% p. a., what debt can be discharged in Paris by a person in Glasgow who holds a 3 months' bill in Glasgow for £ 1000 ?

(C.U. 1941)

14. A merchant in New York buys goods in Geneva to the value of 4,004 fr. Find the value of the goods in dollars, when the rate of exchange between London and New York is \$ 4.865 to the £, and between London and Geneva 25.48 francs to the £ (C.U. 1938,42)

15. The rate of exchange between London and New York is 5s 2d per dollar, between London and Calcutta 1s 5d per rupee and between Calcutta and New York Rs. 4/12 nP. per dollar. If a merchant of Calcutta remits a debt of 720 dollars to New York through London, instead of direct, does he gain or lose thereby? And by how much? (C.U. 1958)

16. A broker wishing to buy a bill for 7000 dollars drawn on New York borrowed £ 2,500 from a money lender for 15 days at $5\frac{3}{4}\%$. He bought the bill at $3\frac{7}{18}$ exchange and sold it 15 days later at $3\frac{1}{8}$ exchange and repaid the loan. How much did he gain or lose? (C.U. 1959)

17. On 1st March, 1962 Anglo-Indian Traders remitted £ 150 to British Exporters Corporation by Telegraphic Transfer through their Bankers who debited their account with them. On that day the market quotation for T. T. rates to London was $1/5\frac{1}{4}$ d— $5\frac{5}{8}$ d. Calculate the amount the Bank should debit to keep a margin of $\frac{5}{8}$ d only.

(C.U. B.Com.I, 1962)

Chapter XIII

Elementary Notions of Probability and Chance Calculation

1. **Definition.** If an event can happen in a ways and fail (to happen) in b ways, and if all these ways are equally likely, then the probability or chance of its happening is $\frac{a}{a+b}$ and that of its failing or non-happening is $\frac{b}{a+b}$.

For clearly the ratio of the chance of happening to that of failing is $a : b$.

$\therefore$ the chance of happening is Ka , and that of failing is Kb , where K is some constant.

But the event must either happen or fail and there is no third alternative.

Therefore, the sum of the chance of happening and chance of failing must represent certainty. If we take unity to represent certainty, we have

chance of happening + chance of failing = 1

$$\therefore Ka + Kb = 1 \quad \text{or} \quad K(a+b) = 1$$

$$\text{Hence } K = \frac{1}{a+b}.$$

$$\text{Thus the chance of happening} = \frac{a}{a+b}$$

$$\text{and the chance of failing} = \frac{b}{a+b}.$$

Note. The probability is thus represented by a *frequency ratio* of which the numerator represents the number of cases favourable to the event and the denominator represents the total number of possible cases.

2. Another way of saying that 'the probability' of happening' of an event is $\frac{a}{a+b}$ is that the odds are a to b in favour of the event or b to a against it.

3. The probability of happening of an event is generally denoted by p and probability of failing by q .

$$\text{Then } p+q = \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Hence if p is the chance of happening of an event, the chance of its failing is $1-p$.

4. Mutually exclusive events.

Two events are said to be mutually exclusive when they do not concur, *i.e.*, if one happens the other can not happen.

When a coin is tossed, either the head or the tail will turn up. They can not turn up together, *i.e.*, they are two mutually exclusive events.

5. Alternative definition of probability.

If an event can occur in n mutually exclusive ways all equally likely to happen, and if of these a are favourable to the event, then the probability or chance of it happening is $\frac{a}{n}$ and probability or chance of its failing or not happening is $1 - \frac{a}{n}$.

6. Alternative method of proof $p+q=1$.

Let n be the totality of cases of which a represents the number favourable to an event and b the number unfavourable to it. Then the sum of favourable and unfavourable cases is equal to the total number of cases.

$$\therefore a+b=n$$

Dividing both sides by n , we have

$$\frac{a}{n} + \frac{b}{n} = 1$$

$$\text{i.e., } p+q=1$$

7. It is obvious that if the probability p is impossible, the ratio $p = \frac{a}{a+b}$ is zero, and if the probability is a certainty, the ratio is unity. In the former case $a=0$, in the latter $b=0$.

Thus the scale of probability ranges from 0 representing impossibility to 1 representing certainty.

Mathematical probability is an abstract conception. If we toss a coin either head or tail will turn up. These are the only two possibilities and if they are equally likely we should get probability of head or tail turning up $=\frac{1}{2}$. But this does not mean that out of every 2 tossings, one will be head and the other tail, or in 4 tossings, 2 will be head and 2 tail. But what we mean is that if a very large numbers of tossings is taken, we should have the probability of a head approaching $\frac{1}{2}$, and that of a tail approaching $\frac{1}{2}$.

Clearly $p+q=1$.

Ex. 1. If a die (a cube of six faces marked 1 to 6) is rolled without bias, it may show up any one of the six spots 1 to 6.

$\therefore$ the total number of ways in which the spots can show up $=6$

But any one spot can show up only in 1 way

$\therefore$ the probability of showing up 1 $=\frac{1}{6}$

the probability of showing up 2 $=\frac{1}{6}$

the probability of showing up 3 $=\frac{1}{6}$ etc.

Hence the probability of not showing up 1 $=1-\frac{1}{6}=\frac{5}{6}$

the probability of not showing up 2 $=1-\frac{1}{6}=\frac{5}{6}$

the probability of not showing up 3 $=1-\frac{1}{6}=\frac{5}{6}$ etc.

Ex. 2. From a pack of 52 cards 1 card is drawn at random. Find the chance of drawing or drawing not a spade, drawing or drawing not an ace.

Since a full pack consists of 13 hearts, 13 diamonds, 13 spades and 13 clubs and there are only one ace of hearts, one ace of diamonds, one ace of spades and one ace of clubs

(i) the chance of drawing a spade $=\frac{13}{52}$.

(ii) " " drawing not ... $=1-\frac{13}{52}=\frac{39}{52}$.

(iii) " " drawing an ace $=\frac{4}{52}$.

(iv) " " drawing not an ace $=1-\frac{4}{52}=\frac{48}{52}$

Similarly

(v) the chance of drawing an ace of spades = $\frac{1}{52}$

(vi) " " " drawing not " " " = $1 - \frac{1}{52} = \frac{51}{52}$.

(vii) " " " an ace not of spades = $\frac{51}{52}$. etc.

Ex. 3. From a pack of 52 cards, 4 cards are drawn at random. What is the probability that

(a) all of them are spades

(b) one is of each suit?

(a) The total number of ways in which 4 cards are drawn is ${}^{52}C_4$ and the number of ways of drawing 4 spades is ${}^{13}C_4$.

Hence the required probability

$$= \frac{{}^{13}C_4}{{}^{52}C_4} = \frac{13.12.11.10}{52.51.50.49} = \frac{11}{4165}$$

(b) The total number of ways of drawing 4 cards = ${}^{52}C_4$

But 1 card of each suit can be drawn in ${}^{13}C_1$ or 13 ways

$\therefore$ 4 cards, one of each suit, can be drawn in $13 \times 13 \times 13 \times 13$ or 13^4 ways

$$\text{Hence the required probability} = \frac{13^4}{{}^{52}C_4} = \frac{2197}{20825}$$

Ex. 4. From a bag containing 5 white and 4 black balls, 4 balls are drawn at random. (a) What are the odds against these all white? (b) What is the chance that 2 are white and 2 black?

The total number of ways in which 4 balls can be drawn out of 5+4 or 9 balls is 9C_4 .

The number of ways of drawing 4 white balls = 5C_4

Hence the probability of drawing 4 white balls

$$= \frac{{}^5C_4}{{}^9C_4} = \frac{5.4.3.2}{9.8.7.6} = \frac{5}{126}$$

$\therefore$ the odds against the particular event are 121 to 5.

(b) 2 white balls can be drawn in 5C_2 ways and 2 black balls can be drawn in 4C_2 ways.

$\therefore$ 2 white and 2 black balls can be drawn in ${}^5C_2 \times {}^4C_2$ ways.

$$\text{Hence the required chance} = \frac{{}^5C_2 \times {}^4C_2}{{}^9C_4} = \frac{10 \times 6}{9 \times 7 \times 2} = \frac{10}{21}$$

Ex. 5. What is the chance of throwing a number greater than 3 in a single throw with a die?

There are 6 ways in which the die can fall. It may show up 1 to 6. Of these 3 are favourable to the event.

$$\therefore \text{ the required chance} = \frac{3}{6} = \frac{1}{2}.$$

Ex. 6. What is the chance of throwing (i) a double-four (ii) a four-five in a single throw with two dice?

The total number of cases $= 6 \times 6 = 36$

(i) Out of these four-four can turn up in 1 way

$$\therefore \text{ the required chance} = \frac{1}{36}$$

(ii) There are two ways of showing a four-five, viz- (4,5) and (5,4)

die A	die B
4	5
5	4

$$\therefore \text{ the required chance} = \frac{2}{36} = \frac{1}{18}.$$

Ex. 7. What is the chance of showing at least one ace in a single throw with two dice?

Two dice can fall in 6×6 or 36 ways.

Each die can fall in 5 ways so as not to show an ace. Hence 25 falls of the two dice will not show any ace.

$$\therefore \text{ the chance of showing at least an ace} \\ = 1 - \frac{25}{36} = \frac{11}{36}.$$

Ex. 8. In a single throw with 3 dice, find the chance of showing (i) three aces (ii) two aces (iii) an ace, and (iv) no ace.

Three dice can be thrown in $6 \times 6 \times 6$ or 216 ways.

(i) the three aces can turn up in 1 way

$\therefore$ the required chance $= \frac{1}{216}$

(ii) Two aces and any other particular number, say 6 can turn up in 3 ways, for this is clearly equal to the number of permutations of 3 things of which 2 are alike, taken all together $= \frac{3!}{2!} = 3$

Since two aces of two dice can occur with each of the 5 numbers 2, 3, 4, 5, 6, of the third, the total number of ways in which two aces may occur

$$= 5 \times 3 = 15.$$

$\therefore$ the required chance $= \frac{15}{216} = \frac{5}{72}$.

(iii) An ace on one die can be associated with any of the 5 numbers on each of the other two dice, so this can happen in 5×5 ways.

$\therefore$ the total number of favourable cases $= 3 \times 5 \times 5 = 75$

Hence the required chance $= \frac{75}{216} = \frac{25}{72}$.

(iv) There are 5 ways in which each die can be thrown so as not to give an ace. Hence $5 \times 5 \times 5$ or 125 throws of the three dice will exclude aces.

$\therefore$ the required chance $= \frac{125}{216}$.

Note. Clearly the sum of all the four chances is a certainty.

$$\frac{1}{216} + \frac{5}{72} + \frac{25}{72} + \frac{125}{216} = \frac{1+15+75+125}{216} = \frac{216}{216} = 1.$$

8. Addition Law of Probability. If several events are *mutually exclusive*, the probability that one or more of them will happen is the sum of their separate probabilities.

For example, when a die is thrown, any of the six numbers may turn up. If these cases are equally likely, the probability that 1 will turn up $= \frac{1}{6}$

" " 2 " " " $= \frac{1}{6}$

" " 3 " " " $= \frac{1}{6}$

Hence the probability that either 1 or 2 or 3 will turn up

$$=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{1}{1}$$

Ex. 9. What is the chance of picking a spade or an ace not of spade from a pack of 52 cards?

The chance of picking a spade $=\frac{13}{52}$

The chance of picking an ace, not of spade $=\frac{3}{52}$

$\therefore$ the chance of picking a spade or an ace, not of spade

$$=\frac{13}{52}+\frac{3}{52}=\frac{16}{52}=\frac{4}{13}$$

Ex. 10. Find the chance of picking an even number or a number ending with 0 from the series of natural numbers 1 to 100.

The number of even numbers in the series 1 to 100
= number of odd numbers = 50.

$\therefore$ the chance of picking an even number $=\frac{50}{100}=\frac{1}{2}$

But every number ending with 0 is even.

Hence the required chance $=\frac{1}{2}$

Alternative Method.

The chance of picking a number ending with 0

$$=\frac{10}{100}=\frac{1}{10}$$

But all numbers ending with 0 are even.

$\therefore$ the events are not mutually exclusive and so Addition law of Probability is not applicable.

But we can avoid overlapping cases as follows :

The chance of picking an even number, not ending with 0

$$=\frac{1}{2}-\frac{1}{10}=\frac{2}{5}$$

∴ the chance of picking an even number or a number ending with 0

$$= \frac{2}{5} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}$$

Note. But the chance of picking up an odd number or a number ending with 0

$$= \frac{1}{2} + \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$$

Ex. 11. What is the chance of throwing 15 in a single throw with 3 dice?

15 can be made up as follows :

(6, 6, 3) which can occur in 3 ways, viz. (6, 6, 3), (6, 3, 6), (3, 6, 6)

(6, 5, 4) which can occur in 6 ways, viz. (6, 5, 4), (6, 4, 5), (5, 6, 4), (5, 4, 6), (4, 5, 6), (4, 6, 5)

(5, 5, 5) which can occur in 1 way.

Therefore, the number of favourable cases

$$= 3 + 6 + 1 = 10.$$

But total number of cases $= 6^3 = 216$.

Hence the required chance $= \frac{10}{216} = \frac{5}{108}$.

Ex. 12. What is the chance of throwing not less than 15 in a single throw with three dice?

The number of points may be 15, 16, 17 or 18.

(i) 15 can occur in 10 ways. (see Ex. 11)

(ii) 16 can occur in 3 ways, viz (6,6,4), (6,4,6), (4,6,6)
and also in 3 ways, viz (6,5,5), (5,6,5), (5,5,6)

(iii) 17 can occur in 3 ways, viz (6, 6, 5), (6, 5, 6), (5, 6, 6)

(iv) 18 can occur in 1 way (6, 6, 6)

∴ the total number of favourable cases

$$= 10 + 3 + 3 + 1 = 20$$

Hence the required chance $= \frac{20}{216} = \frac{5}{54}$.

9. Compound Event. When two or more events can occur together, their *concurrence* is called a *compound event*. Thus the throw of a dice does not affect the result of the next throw. These two events form a compound event.

Two Events are said to be *independent* or *dependent* according as the occurrence of one does not or does affect the occurrence of the other. Dependent events are also called *contingent* events.

10. Multiplication Law of Probability.

The probability of a compound event is equal to the product of the probabilities of the separate events, *i.e.*, if two or more events are independent of one another, then the probability of their concurrence is the product of their separate probabilities.

Ex. 13. Find the chance of throwing head and tail alternately in 3 successive tossings of a coin.

The first toss must give either head or tail

The chance that the second toss gives opposite to the first = $\frac{1}{2}$
 " " " " third " " " " second = $\frac{1}{2}$

As the two events are *independent*,

the chance of the compound event = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Ex. 14. A die is rolled twice. Find the chance that the first throw gives an ace and the second a 3-spot.

The chance that the first throw gives an ace = $\frac{1}{6}$
 " " " " second " " a 3-spot = $\frac{1}{6}$

$\therefore$ the chance of the compound event = $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Ex. 15. Find the chance of getting exactly 6 in the toss of two dice.

	Die A	Die B
I.	1	5
II.	2	4
III.	3	3
IV.	4	2
V.	5	1

The chance of securing an ace with die $A = \frac{1}{6}$

" " " a 5 " " $B = \frac{1}{6}$

$\therefore$ the chance of the compound event $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

Similarly, the chance of the second compound event $= \frac{1}{36}$

" " third " " $= \frac{1}{36}$

" " fourth " " $= \frac{1}{36}$

" " fifth " " $= \frac{1}{36}$

Hence the chance of getting 6 points

$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$$

Note. The above example shows that in finding the probability of a compound event it may be necessary to *multiply and add* probabilities.

Ex. 16. Two balls are drawn at random from a bag containing 6 white and 4 black balls. Find the chance that one is white and the other black.

We may regard the occurrence as a *simple event*.

1 white ball can be drawn from 6 in 6C_1 or 6 ways

1 black " " " " 4 " 4C_1 or 4 ways

$\therefore$ the chance of drawing 1 white and 1 black ball

$$= \frac{6 \times 4}{{}^{10}C_2} = \frac{8}{15}$$

We may regard the occurrence as a *compound event*.

1 white ball can be drawn from 6 in 6C_1 or 6 ways

the chance of drawing a white ball $= \frac{6}{10}$

When 1 ball has been drawn *and not replaced*, a black ball can be drawn in 4C_1 or 4 ways.

$\therefore$ the chance of drawing a black ball $= \frac{4}{9}$

Hence the chance of drawing a white ball and then a black ball

$$= \frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$$

Similarly, the chance of drawing a black ball and then a white ball

$$= \frac{4}{10} \times \frac{6}{9} = \frac{4}{15}$$

These events being *mutually* exclusive, we have, by the Addition Law of Probability, the required chance

$$= \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$$

Ex. 17. What is the chance of drawing without bias four aces successively from a pack of 52 cards?

A pack of 52 cards contains 4 aces.

The chance of drawing an ace at the first drawing = $\frac{4}{52}$

When 1 ace is drawn, 3 aces and 51 cards are left.

$\therefore$ the chance of drawing an ace at the second drawing = $\frac{3}{51}$

Now 2 aces and 50 cards are left.

$\therefore$ the chance of drawing a third ace at the third drawing = $\frac{2}{50}$

Now 1 ace and 49 cards are left.

$\therefore$ the chance of drawing the fourth ace at the last drawing = $\frac{1}{49}$

Hence by the Multiplication Law of Probability, the chance of the compound event

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = \frac{1}{270725}$$

Ex. 18. Find the probability that out of 10 coins tossed, exactly 3 will be tails.

The total number of cases = 2^{10} .

Number of favourable cases is evidently equal to the number of arrangements of 10 things of which 3 are alike of one kind, 7 are alike of a different kind as shown below

T T T H H H H H H H

$$\therefore \text{ number of favourable cases} = \frac{10!}{3!7!} = \frac{8.9.10}{1.2.3} = 120$$

$$\text{Hence the required probability} = \frac{120}{2^{10}} = \frac{15}{128}$$

11. If the chance of two independent events A and B happening be p_1 and p_2 and their failing be q_1 and q_2 respectively, the

the chance that A and B will happen $= p_1 \cdot p_2$

“ “ “ “ “ “ “ fail $= q_1 q_2$

“ “ “ A will happen and B will fail $= p_1 \cdot q_2$

“ “ “ “ “ fail and B will happen $= q_1 \cdot p_2$

It is easy to verify that the sum of the four chances is a certainty

$$\text{For } p_1 + q_1 = 1 \text{ and } p_2 + q_2 = 1$$

$$\therefore q_1 = 1 - p_1 \text{ and } q_2 = 1 - p_2$$

Hence the sum of the chances

$$= p_1 p_2 + (1 - p_1)(1 - p_2) + p_1(1 - p_2) + (1 - p_1)p_2 = 1$$

The above principle may be generalised.

If $p_1, p_2, \dots, p_n$ be the chances of happenings of n independent events,

The chance that they will all happen $= p_1 p_2 \dots p_n$

“ “ “ “ “ all fail $= (1 - p_1)(1 - p_2) \dots (1 - p_n)$

Hence the chance that at least one will happen

$$= 1 - (1 - p_1)(1 - p_2) \dots (1 - p_n)$$

12. It is interesting to note that the probabilities of different combinations of independent events can be measured directly from Binomial Distribution, if we know the probabilities of the different independent events.

For example, if two coins are tossed

the chance of throwing two heads in a single throw $= \frac{1}{4}$

 " " one head, one tail " " $= \frac{2}{4}$

 " " two tails " " $= \frac{1}{4}$

Now the chance of the two independent events, *viz.* the chance of throwing a head $= \frac{1}{2}$, that of throwing a tail $= \frac{1}{2}$.

The chances of the compound events are given by the following successive terms of the binomial expansion.

$$\left(\frac{1}{2} + \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4}$$

The above process can be generalised.

Exercise

1. A die is cast. Find the chance of throwing an ace ?
 2. From a pack of 52 cards, thoroughly shuffled, a card is drawn at random. Find the chance of drawing (i) a spade (ii) the ace of spade (iii) a spade or a king not of spade, (iv) an ace, not of spade.
 3. From a pack of 52 cards two are drawn at random. What is the probability that one is a spade and the other is a club ?
 4. (i) A bag contains 3 green and 8 white balls. If one ball is drawn from it, find the chance that the ball drawn is green.
(C. U. B. Com I 1962)
 - (ii) From a bag containing 7 white and 5 black balls two balls are drawn at random. What is the probability that one is white and another black ?
 5. Find the probability of throwing (i) 9, (ii) 10, (iii) 11, (iv) 12 in a single throw with two dice.
- Hence find the chance of throwing at least 9 points in a single throw with two dice.

6. What is the chance of throwing (i) a double-two (ii) a three-four in a single throw with two dice?

7. What is the chance of showing *at least* one ace in a single throw with two dice?

8. Three dice are rolled. Find the chance of throwing a three-three-six.

9. In three throws with 2 dice, find the chance of throwing doublets *at least* once.

10. In a single throw with 3 dice, find the chance that the dice fall all different.

11. Find the chance of picking a vowel and consonant x from the 26 English alphabets.

12. Find the chance of throwing 13 in a single throw with three dice.

13. (a) Three cards are drawn at random from an ordinary pack. What is the chance that they will consist of an ace, a king, and a queen.

$$[{}^4C_1 \times {}^4C_1 \times {}^4C_1 / {}^{52}C_3 = \frac{4 \times 4 \times 4 \times 6}{1 \times 2 \times 3 \times 50}]$$

(b) A bag contains 5 white, 6 black and 7 green balls. Find the chance that three balls drawn at random are one of each colour.

14. Find the chance that out of 9 coins tossed, exactly 4 will be heads.

15. 9 coins are tossed once. Find the probability of getting 3 tails.

16. 10 coins are tossed once. Show that the probability of getting 7 heads is equal to the probability of getting 3 heads.

17. In a throw with three dice, the player desires at least 1 to turn up. Find his chance.

18. Show that the probabilities of throwing six with 2, 3 or 4 dice respectively are 18 : 6 : 1.

19. An urn contains 4 white and 5 black balls. One ball is drawn at random and replaced before a second ball is drawn. What is the chance that each time a black ball is drawn?

20. Two drawings, each of 4 balls are made from a bag containing 6 white and 7 green balls *being replaced* before the second draw. What is the chance that the first drawing will give 4 white, and the second 4 green balls?

21. Two drawings, each of 4 balls, are made from a bag containing 6 white and 7 green balls, the balls *not being replaced* before the second trial. What is the chance that the first drawing will give 4 white and the second 4 green balls?

22. (a) A bag contains 11 counters marked with the numbers 1 to 11. A counter is drawn and replaced; a second drawing is then made. What is the chance that the first number is even and the second odd?

(b) There are 11 tickets marked with the numbers 1 to 11. One is drawn at random. Find the chance that it is a multiple of 3 or 5?

23. If 4 whole numbers taken at random are multiplied together show that the chance that the last digit in the product is 1, 3, 7 or 9 is $\frac{16}{81}$.

24. One purse contains 1 rupee and 3 half-rupees, a second contains 2 rupees and 4 half-rupees, and a third contains 3 rupees and one half-rupee. If a coin is taken out of any one purse at random, find the chance that it is a rupee.

[Out of 3 purses, the chance of selecting the first = $\frac{1}{3}$. The first purse contains 4 coins and the chance of drawing a rupee = $\frac{1}{4}$. Hence the chance of drawing a rupee from the first purse = $\frac{1}{3} \times \frac{1}{4}$. The chance of drawing a rupee from the second purse = $\frac{1}{3} \times \frac{2}{6}$ and from the third purse = $\frac{1}{3} \times \frac{1}{4}$. The required chance is the sum of the chances.]

Chapter XIV

Stocks and Shares

Share. It is an individual portion of the capital of a Joint Stock Company. A Joint Stock Company generally issues different classes of shares of different denominations to suit the capacity of various classes of investors. Persons who buy shares are called *Shareholders*.

Kinds of Shares. Under Section 85 of the new Companies Act, 1956 only two kinds of shares—Preference shares and Ordinary shares—can be issued in India.

Preference Shares. These are entitled to a fixed dividend *only out of profits in preference* to the other shares. The preference may be in respect of dividend or capital or both. Preference shares are cumulative as regards dividend, unless otherwise specified.

Ordinary Shares. The holders of such shares are entitled to surplus profits after the preference shareholders are fully paid. Ordinary shares are also called *Equity Capital*.

Cumulative Preference Shares. These are a class of shares entitled to a fixed yearly dividend out of profits, present or future. The right to dividend, if not paid in any year or years, accumulates till all arrears of their claims are fully met.

Redeemable Preference Shares. These shares are fully paid up and their repayment is assured.

Deferred and Founders' shares are now made obsolete in India by law.

Cum-dividend and ex-dividend Prices. Shares as well as stocks and debentures are transferable assets. The right to receive the *next dividend* is sometimes transferred to the next

holder with the share. A share may be sold *ex-dividend* or *cum-dividend* according as the seller or the buyer has the right to receive the next dividend.

The cum—dividend price includes the price of the share *plus* the next expected dividend. This extra amount which the seller pays may be looked upon as a temporary advance which he later realises as dividend.

Hence Cum—dividend (cum-div) price = ex-dividend (ex-div) price + dividend.

2. Stock. Fully paid up shares may be converted into stock. Stock certificates are sometimes issued by Joint Stock Companies in lieu of shares.

Debentures. These are documents by which a company borrows money. They are dealt with in the similar way as stock.

Dividend. It is the amount of annual or periodic profits distributed to the shareholders according to their holdings. Dividend is paid only out of profits.

A stock-holder or Preference shareholder is entitled to a fixed dividend but an ordinary share-holder can claim no such fixed dividend. The latter dividend is variable from year to year. *The dividend on a share is paid on its paid-up value. Dividend on stock is the income on 100 stock.*

Brokerage. Stocks and shares are dealt with on the Stock Exchange by brokers who act on behalf of the sellers and buyers. The remuneration or commission charged by the brokers for their services in selling and buying stocks and shares is called *brokerage*.

Brokerage raises the cost price and lowers the selling price. Brokerage is charged on the face value or amount of stocks or shares and not on their market value.

For example, if Rs. 100 stock is quoted at 102 and brokerage is $\frac{1}{8}\%$, then

$$\begin{aligned}\text{Cost price of Rs. 100 stock} &= \text{market value} + \text{brokerage} \\ &= \text{Rs. } 102 + \text{Rs. } \frac{1}{8} \\ &= \text{Rs. } 102\frac{1}{8}\end{aligned}$$

$$\begin{aligned}\text{Sale proceeds of Rs. 100 stock} &= \text{market value} - \text{brokerage} \\ &= \text{Rs. } 102 - \text{Rs. } \frac{1}{8} \\ &= \text{Rs. } 101\frac{7}{8}.\end{aligned}$$

The difference between these two prices represents broker's margin of profit on Rs. 100 stock.

The distinction between *face value* and *market value* of a share or stock must be clearly understood.

While face or nominal value represents the amount of share or stock, the market value is the price at which it is offered for sale. Market value is also called *cash value*.

The prices of shares and stocks are always changing. Shares are quoted on their paid-up values.

A share is said to be *at par* when its market value is equal to its face value.

It is *below par* or *at a discount* when the market value is lower than its face value.

It is *above par* or *at a premium* when the market value is higher than its face value.

Newspapers daily publish the prices of Government stocks and shares issued by important companies.

Quotation. Stocks are quoted at so much per Rs. 100 stock, but shares are quoted at so much on their paid-up values.

Dividends are usually declared as a percentage of the face value of Rs. 100 stock or as a certain amount per share.

Example. 3% Rs. 5000 stock at 102 means

- (1) face value or amount of stock is Rs. 5000
- (2) market value or price of Rs. 100 stock is Rs. 102

(3) dividend on Rs. 100 stock is Rs. 3 per annum

(4) income on Rs 102 for 1 year = Rs. 3

$2\frac{1}{2}\%$, Rs. 300 stock at a discount of 2 means, among other things, that the market price of Rs. 100 stock = Rs. 100 - 2 = Rs. 98. etc.

Yield. It is the actual percentage return on cash invested.

$$\therefore \text{Yield} = \frac{\text{Rate of dividend}}{\text{Price of stock}} \times 100$$

Distinction between Stock and Share. Stock of a any amount (face value) can be bought or sold. But shares can be bought and sold only in complete units.

Distinction between stock-holder and share-holder

(1) A stock-holder is a loan creditor of the company, whereas a share-holder is a part-proprietor or a partner in the company.

(2) A stock-holder is entitled to agreed interest, but a shareholder can not claim any such dividend and this is payable only out of profits, if sanctioned by the general meeting.

Income-tax. Incometax is deducted at the source on income from dividends and other securities unless they are declared tax-free.

Ex. 1. How much 3% stock at $98\frac{1}{8}$ can be bought by investing Rs. 392.50 ?

Stock	Cash
100	$98\frac{1}{8}$
?	Rs. 392.50

$$? = \text{Rs. } \frac{392.50 \times 100}{98\frac{1}{8}} = \text{Rs. 400 stock.}$$

Note. To guard against error, the word 'stock' should be written after stock.

Ex. 2. How much money should I invest to buy £ 320 Stock in the 3 per cents at $101\frac{3}{8}$?

Stock	Cash
100	$101\frac{3}{8}$
£ 320	?

$$? = \frac{£ 320 \times 101\frac{3}{8}}{100} = £ \frac{320 \times 811}{100 \times 8} = £ 324.4 = £ 324 \text{ 8s}$$

Ex. 3. What stock should I sell out at $97\frac{1}{2}$ to realise £ 1,300?

Stock	Cash
100	$97\frac{1}{2}$
?	£ 1,300

$$? = \frac{£ 1,300 \times 100}{97\frac{1}{2}} = £ \frac{4000}{3} = £ 1333\frac{1}{3} \text{ stock.}$$

Ex. 4. How much will be realised by selling Rs. 10,000 Railway stock at $133\frac{1}{3}$?

Stock	Cash
100	$133\frac{1}{3}$
Rs. 10,000	?

$$? = \frac{\text{Rs. } 10,000 \times 133\frac{1}{3}}{100} = \text{Rs. } \frac{40,000}{3} = \text{Rs. } 13,333\frac{1}{3}.$$

Ex. 5. A man sells out £ 6,450 India $3\frac{1}{2}\%$ stock at $64\frac{1}{2}$. Find the amount realised and the income from original investment? (brokerage $\frac{1}{4}\%$)

£ 100 stock produces on sale £ ($64\frac{1}{2} - \frac{1}{4}$). or £ $64\frac{1}{4}$.

Stock	Cash (sale)	$? = \frac{£ 6,450 \times 64\frac{1}{2}}{100}$ $= £ 4,144\frac{1}{8} = £ 4,144 \text{ 2s 6d.}$
100	$64\frac{1}{2}$	
£ 6,450	?	

Stock	Income	$? = \frac{£ 6,450 \times 3\frac{1}{2}}{100}$ $= £ \frac{903}{4} = £ 225 \text{ 15s}$
100	$3\frac{1}{2}$	
£ 6,450	?	

Note: Addition and subtraction of brokerage can be easily effected as follows:

$$64\frac{1}{2} + \frac{1}{4} = 64 + (\frac{1}{2} + \frac{1}{4}) = 64 + \frac{3}{4} = 64\frac{3}{4}$$

$$64\frac{1}{2} - \frac{1}{4} = 64 + (\frac{1}{2} - \frac{1}{4}) = 64 + \frac{1}{4} = 64\frac{1}{4}$$

Ex. 6. Find which is more profitable investment, the 3 per cents at $98\frac{1}{8}$ or the 4 per cents at 105? Assume brokerage at $\frac{1}{8}$ per cent. (C.U. 1944)

First investment ✓

Rs $(98\frac{1}{8} + \frac{1}{8})$ produces Rs 3

$$\therefore \text{Rs } 100 \quad \text{,,} \quad \text{Rs } \frac{3}{98\frac{1}{8}} \times 100 \text{ or Rs } \frac{800}{263} \text{ or Rs } 3.04$$

$$\therefore \text{yield} = 3.04$$

Second investment

Rs $(105 + \frac{1}{8})$ produces Rs 4

$$\therefore \text{Rs } 100 \quad \text{,,} \quad \text{Rs } \frac{4}{105\frac{1}{8}} \times 100 \text{ or Rs } \frac{3200}{841} \text{ or Rs } 3.80$$

$$\therefore \text{yield} = 3.80$$

Hence second investment is better.

Ex. 7. A man earns Rs 375 by investing a sum of money in $4\frac{1}{2}\%$ papers when they are at 2% premium. How much money should he invest in 3% papers at 96 in order that his income may be doubled? (C.U. 1950)

Income from $4\frac{1}{2}\%$ stock = Rs 375

 " " 3% " = Rs 375×2 = Rs 750

Cash Income

96 3

? Rs 750

$$? = \text{Rs } \frac{750 \times 96}{3} = \text{Rs } 24,000$$

Ex. 8. Which is the more paying investment?

4% Improvement Trust Debentures at 120 or Bank of Bengal shares of Rs 100 each, half paid-up paying 6% dividend and quoted at 80. (Brokerage $\frac{1}{4}\%$ for debenture and 12 n.p. for Rs 100 share).

By investing Rs $(120 + \frac{1}{4})$ in 4% debentures, income = Rs. 4

 " Rs 100 " " " ?"

Investment Income

120 $\frac{1}{4}$ 4

Rs. 100 ?

$$? = \text{Rs } \frac{400}{120\frac{1}{4}} = \text{Rs } 3.33.$$

$$\therefore \text{Yield} = 3.33\%$$

$$\text{Paid-up value of a share} = \text{Rs. } 50$$

$$\text{Dividend at } 6\% = \text{Rs. } 3$$

$$\text{Price} = \text{Rs. } 80.12$$

$$\therefore \text{Yield} = \frac{3}{80.12} \times 100 = 3.74$$

$\therefore$ the more paying investment is 3% Bank of Bengal shares.

Ex. 9. Calculate brokerage at $\frac{1}{8}\%$ on a sale of stock at $97\frac{5}{8}$, if the total money realised was £ 2,440 12s 6d after paying brokerage.

$$\text{£ } 2,440 \text{ 12s 6d} = \text{£ } 2440\frac{5}{8}$$

$$\text{Sale proceeds from £ 100 stock} = \text{£ } (97\frac{5}{8} - \frac{1}{8}) = \text{£ } 97\frac{5}{8}$$

Net sale brokerage	?	$= \frac{\text{£ } 2,440\frac{5}{8} \times \frac{1}{8}}{97\frac{5}{8}}$
$97\frac{5}{8} \quad \frac{1}{8}$	?	$= \frac{25}{8} = \text{£ } 3\frac{1}{8}$
£ 2,440 $\frac{5}{8}$	?	

Ex. 10. How much stock at $97\frac{1}{2}$ must I sell in order to realise £ 1,300? If by investing this £ 1,300 in $4\frac{1}{4}$ per cent stock at 102, I increase my income by £ 8 6s 8d per annum, what rate of interest was I receiving on the former stock? (C.U. 1936, 40)

Stock	Cash		
100	$97\frac{1}{2}$		
?	£ 1,300		$? = \frac{\text{£ } 1,300 \times 100}{97\frac{1}{2}} = \text{£ } 1,333\frac{1}{2} \text{ stock.}$

From $4\frac{1}{4}\%$ stock

Cash	Income		
102	$4\frac{1}{4}$		
£ 1,330	?		$? = \frac{\text{£ } 1,300 \times 4\frac{1}{4}}{102} = \text{£ } \frac{325}{6}$ ✓

$$\therefore \text{income from former stock} = \text{£ } \frac{325}{6} - \text{£ } 8 \text{ 6s 8d}$$

$$= \text{£ } \frac{325}{6} - \text{£ } 8\frac{1}{3} = \text{£ } \frac{275}{6}$$

Former Stock	Income	
£1333 $\frac{1}{3}$	£ $\frac{275}{6}$	$?\text{=}\frac{100 \times 275}{6 \times 1333\frac{1}{3}} = \text{£}3\frac{7}{16}$
£100	?	

$\therefore$ dividend on former stock $= 3\frac{7}{16}\%$

Ex. 11. A person sells Rs. 3,500 stock in the 4 per cents at $102\frac{5}{8}$ and invests the proceeds in the 3 per cents at $74\frac{7}{8}$. Find the alteration in income, brokerage being $\frac{1}{8}\%$ in each case. (P.U. 1943)

Stock	Cash (sale)	
100	$102\frac{5}{8} - \frac{1}{8}$	$?\text{=}\frac{\text{Rs. } 3,500 \times 102\frac{1}{2}}{100}$
Rs. 3,500	?	$=\text{Rs. } \frac{35 \times 205}{2}$

Cash (cost)	Income	
$74\frac{7}{8} + \frac{1}{8}$	3	$?\text{=}\text{Rs. } \frac{35 \times 205 \times 3}{2 \times 75}$
Rs. $\frac{35 \times 205}{2}$	?	$=\text{Rs. } 143.50.$

Income from the 4% stock

Stock	Income	
100	4	$?\text{=}\text{Rs. } \frac{3,500 \times 4}{100}$
Rs. 3,500	?	$=\text{Rs. } 140.$

$\therefore$ gain in income $=\text{Rs. } (143.50 - 140) = \text{Rs. } 3.50$

Ex. 12. A man sells out £6,450 India $3\frac{1}{2}\%$ stock at $64\frac{1}{2}$ and with the proceeds bought as many Dunlop Rubber Co's 5%. Preference £1 share at 16s 3d as he could. What was his gain in income derived by this change of investment? (Brokerage $\frac{1}{4}\%$ for India stock and 3d for £1 share) (C.U. 1927)

Stock	Cash (sale)	
100	$64\frac{1}{2} - \frac{1}{4}$	$?\text{=}\frac{\text{£}6,450 \times 64\frac{1}{2}}{100}$
£6,450	?	$=\text{£}4,144\frac{1}{8}.$

The sale proceeds are invested in shares which can be purchased in complete units (whole numbers) only.

Now cost of each share = $16s\ 3d + 3d = 16\frac{1}{2}s = £\frac{3}{4}$

$$\therefore \text{number of shares} = \frac{4,144\frac{1}{8}}{\frac{3}{4}} = 5,023\frac{2}{11} = 5,023$$

(the fractional share, whatever it be, is *neglected*)

$$\therefore \text{face value of shares} = 5,023 \times £1 = £5,023.$$

Share	Income	
100	5	$? = \frac{£5,023 \times 5}{100} = £251\ 3s$
£5,023	?	

Income from stock

Stock	Income	
100	$3\frac{1}{2}$	$? = £\frac{6,450 \times 3\frac{1}{2}}{100}$
£6,450	?	

$$\therefore \text{gain in income} = £251\ 3s - £225\ 15s = £25\ 8s$$

✓ **Ex. 13.** A man had £2,100 stock in the 3 per cents which he transferred to the $3\frac{1}{2}\%$ stock at £87 $\frac{1}{2}$, thus increasing his income by £7. What was the price of the 3% stock?

$$\text{Income from 3\% stock} = £\frac{3}{100} \times 2,100 = £63,$$

$$\therefore \text{income from } 3\frac{1}{2}\% \text{ stock} = £63 + £7 = £70$$

Income	Cash	
$3\frac{1}{2}$	87 $\frac{1}{2}$	$£ = \frac{£70 \times 87\frac{1}{2}}{3\frac{1}{2}} = £1,750.$
£70	?	

This amount of £1,750 invested in $3\frac{1}{2}\%$ stock must have been realised by selling 3% stock.

$$\therefore \text{Sale proceeds of £2,100 stock} = £1,750.$$

$$\text{" " " £100 " } = £\frac{1,750}{2,100} \times 100 = £83\frac{1}{3}$$

which is the price of 3% stock.

Ex. 14. A man invests £2665 in $3\frac{1}{2}$ per cent stock at 97 $\frac{3}{4}$ and when it has risen to 102 $\frac{1}{4}$ he sells out and reinvests the proceeds in 3 per cent Railway shares at 91. What is the change in his income and what rate of interest

does he now get for the money originally invested? (Assume brokerage at $\frac{1}{8}$ per cent on each transaction). (C.U. 1944)

Cash (cost)	Stock	
$97\frac{3}{8} + \frac{1}{8}$	100	$? = \frac{£ 2665 \times 100}{97\frac{1}{2}}$
£ 2665	?	$= £ \frac{8200}{3}$ stock

Stock	Income	
100	$3\frac{1}{2}$	$= £ \frac{8200}{3} \times \frac{3\frac{1}{2}}{100}$
$£ \frac{8200}{3}$	?	$= £ 95\frac{1}{3}$

Again, he sells the stock at $102\frac{1}{8} - \frac{1}{8} = 102$

Cash (sale)	Stock	
102	100	$? = £ \frac{8200}{3} \times \frac{102}{100}$
?	$£ \frac{8200}{3}$	$= £ 82 \times 34$

Reinvesting the sale proceeds at $91 + \frac{1}{8}$ or $91\frac{1}{8}$, the number of Railway Shares he buys

$$= \frac{£ 82 \times 34}{£ 91\frac{1}{8}} = 30\frac{434}{729}$$

Neglecting fractional shares, he can buy 30 shares of £ 100 each

$$\therefore \text{total share value} = 30 \times £ 100 = £ 3,000.$$

Share	Income	
100	3	$? = \frac{£ 3000 \times 3}{100}$
£ 3,000	?	$= £ 90$

$$\therefore \text{loss in income} = £ (95\frac{1}{3} - 90) = £ 5 \text{ } 13\text{s } 4\text{d}.$$

Ex. 15. The Imperial Bank of India fully paid-up shares are selling at Rs 1,054.50 and 4% Calcutta Port Trust Debenture (1901-38) at Rs. 83.75. An investor has Rs. 9,307.75 at his disposal and he purchases a total number of 30 units of the above two securities. Calculate how many securities of each class he buys?

$$\text{Cost of 1 I.B. share} = \text{Rs } 1,054.50$$

$$\text{1 Port Trust Debenture} = \text{Rs } 83.75.$$

The shares and debentures are purchased in 30 complete units of Rs 100.

Let x = number of shares, then

$30 - x$ = number of units of Rs 100 debenture each.

By the question,

$$x \times 1,054.50 + (30 - x) \times 83.75 = 9307.75$$

$$\begin{aligned} \text{or } x(1,054.50 - 83.75) &= 9307.75 - 30 \times 83.75 \\ &= 9307.75 - 2512.50 \\ &= 6795.25 \end{aligned}$$

$$\therefore x = \frac{6795.25}{970.75} = 7.$$

Hence I.B. shares are 7 and Port Trust debentures are 23 units of Rs. 100 each.

Ex. 16. A man has Rs 2600 which he instructs his agent to invest in 3 per cent stock standing at Rs. 101. The agent buys in lots of Rs. 500 and as each deal is completed the price of the stock rises by $\frac{1}{18}$ th per cent. If the agent buys 5 lots and if his charge amounts to Rs 10, how much should he return to his employer?

Successive prices of Rs. 100 stock are

$$101, 101\frac{1}{18}, \boxed{101\frac{2}{18}}, 101\frac{3}{18}, 101\frac{4}{18}$$

$$\therefore \text{average price of Rs. 100 stock} = \text{Rs. } 101\frac{2}{18}$$

Amount of stock purchased in 5 lots

$$= \text{Rs. } 500 \times 5 = \text{Rs. } 2500 \text{ stock}$$

$$\therefore \text{the cost of stock} = \text{Rs. } 101\frac{2}{18} \times 25 = \text{Rs. } 2528\frac{1}{9}$$

$$\text{Add Agent's charge} = \text{Rs. } 10$$

$$\therefore \text{total cost} = \text{Rs. } 2538\frac{1}{9}$$

$$\therefore \text{amount to be refunded} = \text{Rs. } 2600 - 2538\frac{1}{9} = \text{Rs. } 61.87$$

Ex. 17. Having a sum to invest a man puts half of it in 5 p.c. stock at 105 and the other half in $4\frac{1}{2}$ p.c. stock at par, the price in each case includes brokerage. If he had invested $\frac{2}{3}$ of it in the 5 per cents and the rest in $4\frac{1}{2}$ per cents, his annual returns would have been £2 15s more. How much did he invest?

Let his total investment = £100.

First case. investment in 5% stock = £50,

" " 4½% " = £50

$$\text{Income from 5\% stock} = \pounds \frac{5}{105} \times 50 = \pounds \frac{50}{21}$$

$$\text{Income from 4½\% stock} = \pounds \frac{4\frac{1}{2}}{100} \times 50 = \pounds \frac{9}{4}$$

$$\therefore \text{total income} = \pounds \left(\frac{50}{21} + \frac{9}{4} \right) = \pounds \frac{389}{84}$$

Second case. investment in 5% stock = $\frac{3}{5} \times \pounds 100 = \pounds 60$

" " 4½% " = £40

$$\text{Income from 5\% stock} = \pounds \frac{5}{105} \times 60 = \pounds \frac{60}{21}$$

$$\text{Income from 4½\% stock} = \pounds \frac{4\frac{1}{2}}{100} \times 40 = \pounds \frac{9}{5}$$

$$\therefore \text{total income} = \pounds \left(\frac{60}{21} + \frac{9}{5} \right) = \pounds \frac{489}{105}$$

$\therefore$ gain in income in the second case

$$= \pounds \left(\frac{489}{105} - \frac{389}{84} \right) = \pounds \frac{11}{420}$$

Gain	Investment	
$\frac{11}{420}$	100	$\pounds \frac{2\frac{3}{4}}{4} \times 100$
$\pounds 2\frac{3}{4}$	?	$= \frac{11}{\frac{11}{420}}$
		$= \pounds 10,500.$

Ex. 18. An investor desired to put Rs. 50,000 in gilt-edged stocks, so that he might make an average income of 4%. He decided to invest in 4% Port Debentures of Rs. 10,000 at Rs. 115½ and the balance in 3% Government stock. What would be the maximum price he could afford to pay for the latter?

(Brokerage ½% for each transaction)

Total investment = Rs. 50,000.

Estimated income from investment = Rs. $\frac{4}{100} \times 50,000 = \text{Rs. } 2,000$

Cost of Rs. 10,000 stock = Rs. 116 × 100 = Rs. 11,600.

Income from this stock = Rs. 4×100 = Rs. 400

$\therefore$ balance investment in 3% stock
= Rs. $(50,000 - 11,600)$ = Rs. 38,400

and balance income from 3% stock
= Rs. $(2,000 - 400)$ = Rs. 1,600

Income	Cash (cost)	
1,600	38,400	$? = \frac{3 \times 38,400}{1,600}$
3	?	= 72.

Hence price of 3% stock = $72 - \frac{1}{8} = 71\frac{7}{8}$.

Ex. 19. X invests equal sums in 5% Preference shares and 6% Debenture stock of Rs. 100 each and obtains equal incomes. If the 6% Debentures are at a premium of 8%, find the price of Preference shares.

Let his investment = Rs. 100 in each.

The price of 6% stock = $100 + 8 = 108$

$\therefore$ by investing Rs. 108 in 6 per cents, income = Rs. 6

$\therefore$ " " Rs. 100 " " "

$$= \text{Rs. } \frac{6}{108} \times 100 = \text{Rs. } \frac{50}{9}$$

Hence Rs. $\frac{50}{9}$ is also the income from 5% Preference shares.

Cash	Income	
100	$\frac{50}{9}$	$? = \frac{5 \times 100}{\frac{50}{9}}$
?	5	= 90

Hence 5% Preference shares are at a discount of $(100 - 90)$ or 10%.

Ex. 20. A person bought $3\frac{1}{2}$ per cent G.P. Notes of the face value of Rs. 10,000 at $99\frac{1}{2}$. He sells Rs. 5,000 at $98\frac{1}{2}$ and purchases further Rs. 6,000 at $98\frac{1}{2}$. Show what profit or loss he made on the deal allowing brokerage at $\frac{1}{8}$ per cent.
(C.U. 1945)

Cost price of Rs 100 G. P. Notes = Rs $(99\frac{1}{4} + \frac{1}{8})$ = Rs $99\frac{1}{4}$

∴ " " Rs 10,000 " = Rs $99\frac{1}{4} \times 100$ = Rs 9,925

Sale proceeds of Rs 100 " = Rs $(98\frac{5}{8} - \frac{1}{8})$ = Rs $98\frac{1}{2}$

∴ " " Rs 5,000 " = Rs $98\frac{1}{2} \times 50$ = Rs 4,925

Hence the remaining stock = Rs 5,000.

His actual investment is now Rs (9,925—4,925) or Rs 5,000.

Cost price of Rs 100 G. P. Notes = Rs $98\frac{1}{8} + \frac{1}{8}$ = Rs $98\frac{1}{4}$

∴ " " Rs 6000 " = Rs $98\frac{1}{4} \times 60$ = Rs 5,895

Hence total cost of Rs 11,000 G. P. Notes

= Rs 5,000 + Rs 5,895 = Rs 10,895

But cost of Rs 11,000 G. P. Notes at the initial price

= Rs $99\frac{1}{4} \times 110$ = Rs 10,917.50

∴ his gain on the deal = Rs (10,917.50—10,895)

= Rs 22.50

Ex. 21. A company decides to pay off 6% debentures, one of Rs 1 lakh at 103 and another of Rs 2 lakhs at 102. To do this, new debentures are issued, one of Rs 2 lakhs at 99 and another of Rs 2 lakhs at par, the rates of interest being 4 and $4\frac{1}{2}$ per cents respectively. Find the annual saving in interest and the amount in hand after redemption of the debentures.

Sum required to pay off the debentures

= Rs 103×1000 + Rs 102×2000 = Rs 307,000,

Sum realised by issuing new debentures

= Rs 99×2000 + Rs 100×2000 = Rs 398,000.

∴ amount in hand after redemption

= Rs (398,000—307,000) = Rs 91,000

Annual interest on former debentures

= 6×1000 + 6×2000 = Rs 18,000

Annual interest on new debentures

= 4×2000 + $4\frac{1}{2} \times 2000$ = Rs 17,000

∴ annual saving in interest = Rs (18,000—17,000)

= Rs 1,000.

Ex. 22. Dividends on $4\frac{1}{4}\%$ Govt. stock are payable halfyearly on 1st April and 1st. October. On June 15 the cum-div. price was quoted at Rs $96\frac{3}{8}$. What price should be quoted ex-div. ?

$$\text{Halfyearly dividend} = \text{Rs } 2\frac{1}{8}$$

$$\begin{aligned}\text{Ex-div. price} &= \text{cum-div. price} - \text{halfyearly dividend} \\ &= \text{Rs } (96\frac{3}{8} - 2\frac{1}{8}) = \text{Rs } 94\frac{1}{4}\end{aligned}$$

Ex. 23. B. Banerjee purchased .2%, Rs 10,000 stock at 106 cum-div. Immediately after receiving the dividend he sold out the stock. At what minimum price should he sell them so as not to suffer any loss on the deal ?

$$\text{Cum-div. quoted price} = 106$$

$$\text{Add brokerage} \quad \frac{1}{8}$$

$$\text{Cum-div. price (cost)} = 106\frac{1}{8}$$

$$\text{Less dividend} \quad 2$$

$$\text{Ex-div. price (cost)} = 104\frac{1}{8}$$

$\therefore$ Ex-div. price (sale) must be quoted at $104\frac{1}{8} + \frac{1}{8}$ or $104\frac{1}{4}$, so that he may recover his purchase price $104\frac{1}{8}$ after deducting brokerage.

Ex. 24. A person desires to earn a minimum income of Rs 380 a year by investing Rs 10,200 in the following

3% Govt. paper at $79\frac{3}{4}$

4% Preference stock at $109\frac{3}{4}$

Distribute the investments, brokerage being $\frac{1}{4}\%$.

$$\text{Yield from 3\% Govt. paper} = \frac{3}{80} \times 100 = \frac{15}{4}$$

$$\text{" " 4\% Pref. stock} = \frac{4}{110} \times 100 = \frac{40}{11}$$

$$\begin{aligned}\text{Average yield} &= \frac{380}{10,200} \times 100 = \frac{190}{51}\end{aligned}$$

by the method of mixture

	yield	difference	proportion
3% Stock	$\frac{15}{4}$	$\frac{50}{11 \times 51}$	40
Average	190	51	
4% Stock	$\frac{40}{11}$	$\frac{5}{4 \times 51}$	11

$\therefore$ investments in 3% and 4% stocks are distributed in the ratio 40 : 11

Hence investment in 3% paper = $\frac{40}{40+11} \times \text{Rs } 10,200 = \text{Rs } 8,000$

" " 4% stock = $\frac{11}{40+11} \times \text{Rs } 10,200 = \text{Rs } 2,200$

Ex. 25. *X* invests £ 27,900 partly in 4% stock at $114\frac{7}{8}$ and partly in 3% stock at $96\frac{3}{8}$. If the resulting income is £ 873, find how much is invested in each stock allowing as brokerage 2s 6d for each £ 100 stock and £ 3 for stamp on the total amount invested. (C.U. 1943)

Total investment in stock = £ (27,900 - 3) = £ 27,897

Brokerage = 2s 6d = £ $\frac{1}{8}$ = $\frac{1}{8}\%$

(In the previous example the method of mixture was employed. Here we propose to use algebraical method.)

Let £ x = investment in 4% stock at $114\frac{7}{8} + \frac{1}{8}$ or 115
Then £ (27,897 - x) = " " 3% " " $96\frac{3}{8} + \frac{1}{8}$ or $96\frac{1}{2}$

Income from 4% stock

Cash	Income	
115	4	? = $\frac{4x}{115}$
x	?	

Income from 3% stock

Cash	Income	
$96\frac{1}{2}$	3	? = $\frac{3(27,897-x)}{96\frac{1}{2}} = \frac{6(27,897-x)}{193}$
27,897 - x	?	

∴ by the question,

$$\frac{4x}{115} + \frac{6(27,897 - x)}{193} = 873$$

or
$$\frac{4x}{115} - \frac{6x}{193} = 873 - \frac{6 \times 27,897}{193}$$

or
$$\frac{82x}{115 \times 193} = \frac{1107}{193}$$

∴
$$x = \frac{1107 \times 115}{82} = \frac{3105}{2} = 1552\frac{1}{2}$$

∴ investment in 4 p.c stock = £ 1,552 10s
 " " 3 " = £ 26,344 10s.

Exercise 16

- Find the cost of
 - Rs 5,000 Govt. 5% stock at 98½
 - £ 2,500 Debentures at 99⅞, brokerage ⅛%
 - Rs 5,200 Municipal stock at a premium of 2%, brokerage ⅛%
- Calculate income from following investments
 - Rs 9,000 stock in the 3½ per cents
 - £ 5,210 stock in 4 per cents
- Compare the yields of
 - 4% Debentures at 110
 - 3% Govt. stock at a discount of 3%
- Which is the better investment
 - 4% stock at 120 or 3% stock at 93 ?
 - ✓(ii) 5% Govt. stock at a discount of 5% or 6% Port Trust Debentures at a premium of 8% ?
- How much stock can be bought by investing
 - Rs 5,390 in Railway 5% stock at 98 ?
 - ✓(ii) Rs 2045 in 3½% stock at 102⅞, brokerage ⅛% ?

6. How much

(i) stock in the 3 per cents at 98 should be sold to realise Rs. 4900/-

(ii) India $4\frac{1}{2}\%$ stock at $64\frac{1}{2}$, brokerage $\frac{1}{4}\%$ to realise Rs. 1290/- ?

7. How much will be realised by selling

(i) Rs 12,000 Railway stock in $2\frac{1}{2}\%$ at 105 ?

(ii) Rs 3,500, 3% consols at discount of 3, brokerage $\frac{1}{4}\%$?

8. If the 4% stock is quoted at 99, what should be the price of 3% stock ?

9. If 4 per cent stock pay interest at the rate of 3 per cent. on investment, what is the price of the stock ?

10. A man invests Rs 21,000 in 4% G. P. Notes at 105. Find the rate of his yield. Ignore Income Tax.

Alternatively, he can invest in 8% Preference shares at 112.50 for every Rs 100 shares. What sum must he invest in these shares in order to obtain a yield equal to what he gets on the G. P. Notes ? Ignore Income Tax. (C.U. 1962)

11. (i) X buys 1000 shares of Rs 25 each at a premium of Rs 5 and was forced by circumstances to sell them when they were at a discount of Rs 2.50. Find out the loss of X. (C.U. 1949)

(ii) What must be the price of 5% stock to be as good an investment as 3% at $95\frac{1}{8}$? (C.U. 1949)

12. A man derived an annual income of £ 253 6s 8d from money invested in $2\frac{1}{2}\%$ consols. He sold out at $77\frac{1}{2}$ and with the proceeds bought $3\frac{1}{2}\%$ stock, thereby increasing his annual income by £ 36. At what price did he buy the $3\frac{1}{2}\%$ stock ? (C.U. 1948)

13. A person has £ 10,000 in the 3%, which he sells and reinvests the proceeds in the $3\frac{1}{2}\%$ at $87\frac{1}{2}$ and increases his income by £ 10. Find the price of the 3 per cents. Brokerage is $\frac{1}{4}\%$ on each transaction.

14. (i) A person invests equal sums in the 4 per cents and in the $4\frac{1}{2}$ per cents and obtains equal incomes. If the 4 per cents are at a discount of 4, find the quoted price of the $4\frac{1}{2}$ per cents ? (C.U. 1947)

(ii) Y invests equal sums in 5% Debentures and 6% Preference shares of a company and obtains equal income. If the 5% debentures are issued at a discount of $2\frac{1}{2}$ %, find the issue price of the Preference shares. Ignore Income Tax, Stamp charges etc. (C.U. 1958)

15. A person invests a certain sum in the $2\frac{1}{2}$ % at 90 and an equal sum in the 4% at 120. If his income from the latter source exceeds the income from the former by Rs 124, how much does he invest ? (C.U. 1947)

16. By selling out Railway $3\frac{1}{2}$ % stock at 96 and buying Tramway $4\frac{1}{2}$ % stock at 108 with the money obtained, a man increased his annual income by Rs 25. What amount of stock did he sell ?

17. A person finds that if he invests his money in the 6 per cents at 126, his income will be £ 22 10s greater than if he invests it in the 9 per cents at 210. Find the sum invested ?

18. Having a certain sum to invest, a man puts half of it in 4% at par and the other half in 3% at $93\frac{3}{4}$. Had he invested $\frac{2}{3}$ ths of the sum in the first stock and $\frac{1}{3}$ ths in the second, his income would have been 80 nP. less. What sum did he invest ?

19. On retirement from service, A received Rs 50,000 as his provident fund money. He wanted to invest this amount so that he might make an average income of $3\frac{1}{2}$ %. If it was decided to invest in 4% Improvement Trust Loan of Rs 10,000 at Rs 115 and the balance in 3% Govt. stock, calculate at what price he could afford to buy the Govt. stock. (C.U. 1955)

20. How much stock must I sell at Rs 96 4 as 0 p in order to collect Rs 15400 ? If this amount was invested

in $3\frac{1}{2}\%$ stock at Rs 101—2—0 my income increased by Rs 120 per annum. What was the rate of interest on my first stock ? (C.U. 1948)

21. Money is invested in buying $3\frac{1}{2}\%$ debentures of Rs 6,400 at 102. Assuming brokerage to be $\frac{1}{8}\%$ and income tax at 5as in the Re., calculate the annual net amount received. What percentage is it of the money invested ? (C.U. 1950)

22. Bimalendu Banerjee, on retirement from service, drew his provident fund accumulations and invested them in Rs 50,000 (face value) 3% conversion loans of 1946. If he now sells them at Rs 74/40 nP. and invests the amount in Rs 100 (face value) equity shares of Bengal Investments Ltd at Rs 124, he estimates that his income would increase by Rs 450 annually. What is the rate of annual dividend on the shares of Bengal Investments Ltd ? (C.U. 1961)

23. A Sarker sold Rs 10,000 $2\frac{1}{2}\%$ G. P. Notes at $99\frac{1}{4}$, and with the proceeds bought as many 4% cumulative preference shares of Rs 100 each in Bharat Industry Limited at 102 as he could. What is the gain in income derived by this change of investment ? Ignore brokerage and income tax. Also, ignore the balance of the sale proceeds that cannot be invested in the above shares. (C.U. 1951)

24. B. Mukherjee sold Rs 6500, $3\frac{1}{4}\%$ Govt. stock at Rs 64.75 and with the proceeds he bought as many 6% Preference shares of Rs 15 each of a public limited company as he could. What was the gain or loss of income derived by this change of investment ?

Brokerage was charged for the Govt. stock at $\frac{1}{8}\%$ and for Preference shares at 25 nP. per share. (C.U. 1950)
(Preference shares are to be taken at par)

25. H. Bose sold Rs 10,000, $3\frac{1}{2}\%$ Govt. Loan at Rs 81 and with the proceeds bought as many Indian minerals Co. shares of Rs 10 each as was possible @ Rs 7/25 nP. each. If the brokerage on Govt. paper is $\frac{1}{4}\%$ and that

on company shares is 12 nP. each; find the 'change in his income if the company declares a dividend of 5%. Ignore Income Tax in either case. (C.U. 1958)

26. A company decides to pay of 6% debentures, one of Rs 6 lakhs at 103 and another of Rs 5 lakhs at 102. To do this new debentures are issued, one of Rs 6 lakhs at 99 and another of Rs 6 lakhs at par, the rate of interest being 4% and $4\frac{1}{2}\%$ respectively. Find the annual saving in interest and the amount in hand after redemption of the debentures. No brokerage. (C.U. 1949)

27. A man has £ 5,000 which he instructs his agent to invest in 3 per cent stock standing at £ 102. The agent buys in lots of £ 500 and as each deal is completed the price of the stock rises by $\frac{1}{18}$ th per cent. The agent buys 9 lots, when he is asked to cease buying. If his charge amounts to £ 10, how much money should he return to his employer? (C.U. 1941)

28. (i) A man wishes to sell 6% Preference shares just before the company is about to pay dividend. The shares are quoted at 109 'ex-div'. What should be their price cum-div? (C.U. 1943)

(ii) X wishes to sell 7% Preference shares of Rs 100/- in a company before the declaration of a dividend by the company. These shares are quoted at Rs 102/25 nP. 'ex-div'. What should be their price 'cum-div'? (C.U. 1958)

29. S. Ray purchased some 4% stock at 105 cum-div. Immediately after receiving this dividend he was forced to sell them on account of financial difficulties. At what minimum price should he sell them so that he might not suffer any loss on the whole? The brokerage in both the cases was charged at $\frac{1}{8}\%$. Calculate also the amount of dividend received by him in case he originally invested Rs 50,000. Ignore income-tax. (C.U. 1960)

30. A shareholder in a mining company obtains from his dividends an income of £ 200 per annum after paying

income tax at the rate of 4s in the £. The shares are 5s shares, upon which were paid two dividends in the year of 1s and 1s 6d respectively. How many shares does he hold in the company, and what money would he realise by their sale if the present market quotations is 17s 6d?

(C.U. 1938, 42)

✓ 31. A person invested Rs 10,200 partly in 3% stock at 79 $\frac{3}{4}$ and partly in 4% stock at 109 $\frac{3}{4}$, the brokerage being $\frac{1}{4}$ % in each case. The aggregate income was Rs 380. How much did he invest in 3% and how much in 4%? (C.U. 1953)

32. A person desires to invest Rs 1,26,000. Find the income he might obtain if he purchased (i) 3% Govt. stock at 81, or (ii) 6% Jute Debentures at 150. It should be observed that these stocks are available in units of Rs. 100 only.

If he desires an annual income of Rs 4,800, how should he distribute his investments in the two kinds of stock?

(C.U. 1952)

33. A company's capital is divided into 10,000 6% Preference shares of Rs 10 each, and 20,000 ordinary shares of Rs 10 each, all fully paid up. The profits for the three consecutive years were Rs 9,000, Rs 18,000 and Rs 36,000 distributed to share holders. How much dividend was paid to the ordinary shares each year?

34. The capital of a company is made up as follows :—
Share capital

5% Preference shares.....	Rs 5,00,000
Ordinary shares.....	Rs 25,00,000

Debenture Capital

4% First Debenture	Rs 3,75,000
4 $\frac{1}{2}$ % Second Debenture	Rs 3,25,000

The net trading profit for the year before charging debenture interest was Rs 1,20,000. If all is distributed, find the rate of dividend for ordinary shares.

Chapter XV

Simple Mensuration

Plane Figure

1. Rectangle

$$\begin{aligned}\text{Area} &= \text{base} \times \text{height} \\ \text{or} &= (\text{length} \times \text{breadth}) \\ &= a \times b.\end{aligned}$$

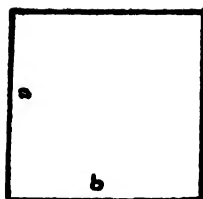


Fig. 1

$$\begin{aligned}\text{Perimeter} &= \text{sum of the bounding sides} \\ &= 2 (\text{length} + \text{breadth}) \\ &= 2(a + b)\end{aligned}$$

2. Parallelogram

$$\begin{aligned}\text{Area} &= \text{base} \times \text{height} \\ &= a \times h.\end{aligned}$$

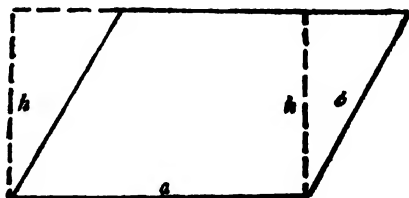


Fig. 2

$$\begin{aligned}\text{Perimeter} &= \text{sum of the bounding sides} \\ &= 2 (\text{length} + \text{breadth}) \\ &= 2(a + b).\end{aligned}$$

3. Square is a rectangle whose sides are equal i.e., $a = b$.

$$\begin{aligned}\text{Area} &= (\text{side})^2 = a^2. \\ \text{Perimeter} &= 4 \times \text{side} = 4a.\end{aligned}$$

4. Triangle

$$\begin{aligned}\text{Area} &= \frac{1}{2} \text{ base} \times \text{height} \\ &= \frac{1}{2} a.h. \\ \text{or} &= \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$

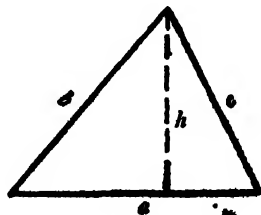


Fig. 3

where a, b, c are its sides and $s = \frac{1}{2}(a + b + c)$

$$\text{Perimeter} = 2s = a + b + c.$$

Particular cases (a) for equilateral triangle of side a

$$\text{Area} = \frac{1}{2}a \cdot \frac{a}{2}\sqrt{3} = \sqrt{\frac{3}{4}} a^2.$$

$$\text{Perimeter} = 3a.$$

(b) for a right angled triangle whose sides are a and b subtending the right angle.

$$\text{Area} = \frac{1}{2}ab.$$

$$\text{Perimeter} = a + b + \sqrt{a^2 + b^2}.$$

5. Trapezium.

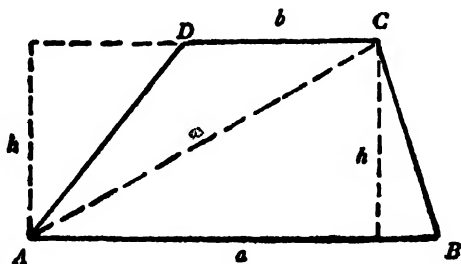


Fig. 4

$$\begin{aligned} \text{Area} &= \triangle ABC + \triangle ACD \\ &= \frac{1}{2}AB \cdot h + \frac{1}{2}CD \cdot h \\ &= \frac{1}{2}ah + \frac{1}{2}bh \\ &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2}(\text{sum of parallel sides}) \times \text{height}. \end{aligned}$$

6. Circle

$$\text{Area} = \pi \times (\text{radius})^2 = \pi r^2$$

$$\text{Circumference} = 2\pi \times (\text{radius}) = 2\pi r$$

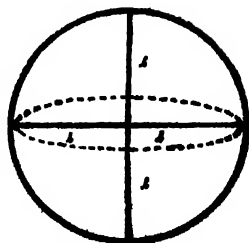


Fig 5

7. **Annulus** (Circular Ring) is the portion between two concentric circles.

$$\text{Area} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)(R - r)$$

8. Path round a rectangle (inside or outside)

$\text{Area} = \text{greater area} - \text{smaller area}$

or $= (\text{length of the path}) \times \text{width}$.

9. Path round a circular plot.

$\text{Area} = \text{greater area} - \text{smaller area}$.

10. Four walls of a room.

$\text{Area} = 2(\text{length} + \text{breadth}) \times \text{height}$
 $= (\text{perimeter of floor}) \times \text{height}.$

Ex. 1. A room is 19' 6" by 15' 7". It is 11' 5" high. Ascertain the cost of painting it at Rs. 2-11 as. per sq. yd., assuming that there are four windows each measuring 5' by 4'.

(C. U. 1946)

$$\begin{aligned}\text{Area of 4 walls} &= 2 \times (\text{length} + \text{breadth}) \times \text{height} \\ &= 2 \times (19' 6'' + 15' 7'') \times 11' 5'' \\ &= 2 \times 35' 3'' \times 11' 5'' \\ &= 2 \times 35\frac{1}{2} \times 11\frac{5}{12} \text{ sq. ft.} = \frac{8432}{3} \text{ sq. yds.}\end{aligned}$$

$$\text{Area of 4 windows} = 4 \times 5' \times 4' = \frac{80}{3} \text{ sq. yds.}$$

$$\therefore \text{area to be painted} = \frac{8432}{3} - \frac{80}{3} = \frac{8352}{3} \text{ sq. yds.}$$

$$\therefore \text{required cost} = \frac{8352}{3} \times \text{Rs. } 2\frac{11}{12} = \text{Rs. } 216'7 \text{ as. } 3 \text{ p.}$$

Ex. 2. The correct measurements of a rectangular area are 4'3 yds. and 3'2 yds. But unfortunately these were recorded in the register as 4'3 metres by 3'2 metre. Find the error in recording those measurements in square yds., and also the percentage error (given 1 metre = 39'37 in.). (C. U. 1951)

$$\text{Correct area} = 4'3 \text{ yd.} \times 3'2 \text{ yd.} = 13'76 \text{ sq. yd.}$$

$$\text{Recorded area} = 4'3 \text{ m} \times 3'2 \text{ m} = 13'76 \text{ sq. m.}$$

$$= 13'76 \times \left(\frac{39'37}{36}\right)^2 \text{ sq. yds.}$$

$$= 13'76 \times (1'094)^2 \text{ sq. yds.}$$

$$= 13'76 \times 1'197 \text{ sq. yds.}$$

$$\begin{aligned}
 \therefore \text{required error} &= 13'76 \times 1'197 - 13'76 \text{ sq. yd.} \\
 &= 13'76(1'197 - 1) \text{ sq. yd.} \\
 &= 13'76 \times '197 \text{ sq. yd.} \\
 &= 2'711 \text{ sq. yd.}
 \end{aligned}$$

$$\therefore \text{percentage error} = \frac{2'711}{13'75} \times 100 = 19'7$$

Ex. 3. A rectangular grass plot, the length of whose sides are in the ratio 5 : 6 costs £ 16 11 s. 6½ d. for turfing at the rate of 5 d. per square yard. Find the length of its sides.

$$£ 16 \ 11 \text{ s. } 6\frac{1}{2} \text{ d.} = 3978\frac{1}{2} \text{ d.}$$

$$\text{The area of the grass plot} = \frac{3978\frac{1}{2}}{5} \text{ sq. yd.} = \frac{3183}{4} \text{ sq. yd.}$$

Clearly the plot can be divided into 5×6 or 30 equal squares.

$$\therefore \text{area of each square} = \frac{3183}{4} \div 30 = 26'525 \text{ sq. yd.}$$

$$\therefore \text{side of each square} = \sqrt{26'525} \text{ yd} = 5'15 \text{ yds.}$$

Hence length of the plot = $5'15 \text{ yds.} \times 6 = 30'90 \text{ yds.}$
and breadth of the plot = $5'15 \text{ yds.} \times 5 = 25'75 \text{ yds.}$

Ex. 4. The area of a circular pond is one acre. Find the cost of gravelling a path 4 ft wide around it at Re. 1 per square foot. (C. U. 1953)

$$1 \text{ acre} = 4840 \text{ sq. yds.}$$

$$\therefore \text{area of circular pond} = 4840 \text{ sq. yds.}$$

Let r yds. be the radius of the water surface.

$$\text{Then } \pi r^2 = 4840$$

$$\text{or, } r^2 = 1540. \quad \text{Hence } r = 39'243$$

$$\therefore r = \sqrt{1540} \text{ yds.} = 39'243 \text{ yds.} = 117'73 \text{ ft.}$$

$$\begin{aligned}
 \therefore \text{the area of the path} &= \text{area of pond \& path} - \text{area of pond} \\
 &= \pi(121'73)^2 - \pi(117'73)^2 \text{ sq. ft.} \\
 &= \pi(121'73^2 - 117'73^2) \text{ sq. ft.} \\
 &= \pi(121'73 + 117'73) \\
 &\quad \times (121'73 - 117'73) \text{ sq. ft.} \\
 &= \pi \times 239'46 \times 4 \text{ sq. ft.} \\
 &= 3010'55 \text{ sq. ft.}
 \end{aligned}$$

Hence the required cost = $3010'35 \times \text{Rs. } 1 = \text{Rs. } 3010'35$.

Ex. 5. Find the cost of covering with carpet 32 inches wide costing Rs. 6 per yd. a room 22 ft. long and 18 ft. wide, reckoning the cost of any fraction of a width of carpet as of whole width.

Let The carpet be laid lengthwise.

Length of each strip = 22 ft.

$$\text{Number of such strips} = \frac{18 \text{ ft.}}{32 \text{ in.}} = \frac{18 \times 12}{32} = 6\frac{3}{4} \text{ or } 7.$$

$$\therefore \text{ total length of carpet required} = 7 \times 22 \text{ ft.} = 154 \text{ ft.}$$

Let the carpet be laid breadthwise

Length of each strip = 18 ft.

$$\text{Number of such strips} = \frac{22 \text{ ft.}}{32 \text{ in.}} = \frac{22 \times 12}{32} = 8\frac{1}{4} \text{ or } 9$$

$$\therefore \text{ total length of carpet required} = 9 \times 18 \text{ ft.} = 162 \text{ ft.}$$

$$\therefore \text{ minimum requirement of carpet is } 154 \text{ ft.}$$

$$\text{Its cost} = 154 \times \text{Rs. } 6 = \text{Rs. } 924.$$

Ex. 6. A plot of land has the shape formed by a square having a semi-circle described on the outside about each of the four sides as diameter. If the length of the side of the square is 60 ft., calculate the amount of rent realisable when the plot is let out at Rs. 1, 100 an acre per annum.

(C. U. 1955)

$$\text{The area of the square} = 60 \text{ ft.} \times 60 \text{ ft.} = 3600 \text{ sq. ft.}$$

$$\text{The area of 4 semi-circles} = 4 \times \frac{1}{2} \pi \cdot 30^2 \text{ sq. ft.}$$

$$= 2^2 \times 1800 \text{ sq. ft.}$$

$$\therefore \text{ total area} = 3600 + 2^2 \times 1800 = 8400 \text{ sq. ft.}$$

$$= \frac{8400}{7} \times \frac{1}{9 \times 4840} \text{ acres.}$$

$$= \frac{180}{847} \text{ acres.}$$

$$\text{Hence the required rent} = \text{Rs. } \frac{180}{847} \times 1, 100 = \text{Rs. } 233'77.$$

Solid Figures

11. Rectangular Solid or

Rectangular Parallelopiped or Cuboid—a body

bounded by six rectangular faces, opposite faces being equal and parallel.

$$\text{Volume} = \text{length} \times \text{breadth} \times \text{height}$$

$$= a \times b \times c$$

$$\text{or, } = \text{area of base} \times \text{height}$$

$$= (a \times b) \times c$$

Surface area

$$= \text{area of six faces}$$

$$= 2(ab + bc + ca)$$

Length of a diagonal

$$= \sqrt{a^2 + b^2 + c^2}$$

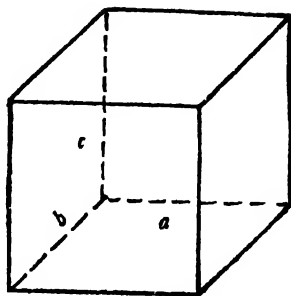


Fig. 6

1. **Cube**—cuboid whose sides are equal, i.e., $a = b = c$.

$$\text{Volume} = a^3$$

$$\text{Surface} = 6a^2$$

$$\text{Length of a diagonal} = a\sqrt{3}.$$

13. **Right Prism**—a solid bounded by plane faces. The side of faces are rectangles. The two end faces are similar, equal and parallel figures.

$$\text{Volume} = (\text{area of base}) \times \text{height}$$

Lateral surface

$$= (\text{perimeter of base}) \times \text{height}$$

Whole surface

$$= \text{lateral surface} + \text{area of two ends.}$$

15. **Pyramid**—a solid bounded by plane faces, of which the base is a rectilinear figure and the rest are rectangles having a common vertex.

14. **Right Circular Cylinder**—a solid described by the revolution of a rectangle about one of its sides which remains fixed. It is a prism with circular base.

$$\text{Volume} = (\text{area of base}) \times \text{height} = \pi r^2 . h.$$

$$\text{Curved surface} = (\text{perimeter of base}) \times \text{height}$$

$$= 2\pi r . h.$$

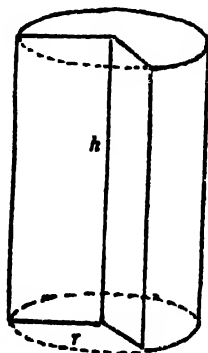


Fig. 8

Whole surface = curved surface + area of two ends

$$= 2\pi rh + 2\pi r^2$$

$$= 2\pi r(h + r)$$

Volume = $\frac{1}{3}$ (area of base) $\times$ height

Slant Surface = $\frac{1}{2}$ (perimeter of base) $\times$ (slant height)

16. Tetrahedon—a pyramid on a triangular base.

17. Right Circular Cone—a solid described by the revolution of a right angled triangle about one of its sides, containing the right angle, which remains fixed.

Volume = $\frac{1}{3}$ (area of base) $\times$ height

$$= \frac{1}{3} \pi r^2 h.$$

Curved surface = $\frac{1}{2}$ (circumference of base) $\times$ (slant height)

$$= \frac{1}{2} \cdot 2\pi r \times l = \pi rl.$$

Whole surface = curved surface + area of base

$$= \pi rl + \pi r^2 = \pi r(l + r)$$

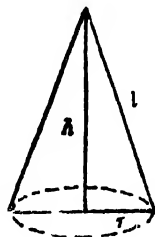


Fig 9

18. Sphere—a solid described by the revolution of a semi-circle about its diameter, which remains fixed.

$$\text{Volume} = \frac{4}{3} \pi r^3$$

$$\text{Surface} = 4\pi r^2.$$

Ex 7. A hollow rectangular vessel formed of material 1 inch thick has external dimensions 13 ft 6 in, 6 ft 8 in, and 6 ft 11 in. respectively. It weighs 1 cwt. 1 qr. 10 lb. Find the weight of a solid mass of the same material and of the same dimensions.

External dimensions are 162 in, 10 in, and 83 in.

$$\therefore \text{External volume} = \frac{162 \times 80 \times 83}{12 \times 12 \times 12} \text{ cu. ft} = \frac{15 \times 83}{2} \text{ cu. ft.}$$

Internal dimensions are 162 - 2 in, 80 - 2 in. and 83 - 2 in.

$$\therefore \text{Internal volume} = \frac{160 \times 78 \times 81}{12 \times 12 \times 12} \text{ cu. ft.} = 15 \times 39 \text{ cu. ft.}$$

$\therefore$ Volume of material

$$= \text{external volume} - \text{internal volume}$$

$$= \left(\frac{15 \times 83}{2} - 15 \times 39 \right) \text{ cu. ft.}$$

$$= \frac{75}{2} \text{ cu. ft.}$$

$$\text{Hence weight of } \frac{75}{2} \text{ cu. ft.} = 1 \text{ cwt. 1 qr. 10 lbs} = 150 \text{ lbs.}$$

$$\therefore \text{weight of } \frac{15 \times 83}{2} \text{ cu. ft.} = 150 \times \frac{2}{75} \times \frac{15 \times 83}{2} \text{ lbs}$$

$$= 2490 \text{ lbs.}$$

$$= 1 \text{ ton 2 cwt 0 qr 26 lbs.}$$

Ex 8. A wall 120 ft. long 6 ft high and 9 inch thick cost Rs. 1,638-12-0. Find the cost at the same rate of a wall of the same thickness and 8 ft. high, which encloses a garden 50 ft. long and 39 ft 6 inch wide, if a gap of 8 ft wide is left for the gate. (C.U. 1951)

Length of the wall measured outside the garden

$$= [2(39'6'' + 9'' + 9'') + 2 \times 50'] - 8'$$

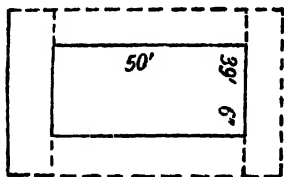
$$= [2 \times 41' + 2 \times 50'] - 8'$$

$$= 182' - 8' = 174'$$

$\therefore$ volume of wall = length $\times$ breadth $\times$ height. (a rectangular solid)

$$= 174' \times \frac{9}{12} \times 8'$$

$$= 1044 \text{ cu. ft.}$$



$$\text{Volume of the given wall} = 120 \times 6 \times \frac{3}{4} \text{ cu. ft.} = 540 \text{ cu. ft.}$$

$$\therefore 540 \text{ cu. ft. cost Rs. 1698}^{\cdot}75$$

$$\therefore \text{Hence cost of 1044 cu. ft.} = \text{Rs. } 1698^{\cdot}75 \times \frac{1044}{540}$$

Exercise

Plane Figures

1. A rectangular garden 4 times as long as it is broad contains 2 acres. Find its length and breadth.

2. A room is 19' 2" by 15' 7". It is 11' 5" high. Ascertain the cost of painting it at Rs. 2-11as per sq. yd, assuming that there are four windows measuring 5 ft. by 4 ft. (C U. 1946)

3. A court-yard 50 ft. long by 42 ft broad contains a rectangular lawn surrounded by a gravel-path of uniform width. If the width of the path is 6', find the area of the lawn and area of the path.

4. A rectangular grass plot, the length of whose sides are in the ratio of 5 : 6 cost £ 16 11 s 6 $\frac{3}{8}$ d, for turfing at the rate of 5d. per square yard. Find the length of its sides.

5. Find the cost of covering with carpet, 32 in. wide, costing 4 s. 6 d. a yard, a room 20 ft. long, 18 ft. wide, reckoning the cost of any fraction of width of carpet as of a whole width.

6. A triangular plot of land with a perpendicular height of 60 yds. 1 ft. and the base measuring 135 yds. 2 ft. is to be paved. Given the cost of Rs. 2. 4as. 6p per sq. yard, find the total expense for paving the entire area. C. U. 1944

7. A circular pond 70 ft. across has a path 7 ft, wide three quarters of the way round. Find the cost to gravel the path at 7 $\frac{1}{2}$ d, per sq. yd.

8. A wall 50 yds. long, 8 ft. high and 9 inches thick cost Rs. 134'75. Find the cost, at the same rate, of a wall of the same thickness and 6 ft. height which encloses a garden 45 ft. long, 33 ft, 6in, wide with a gap of 9 ft, wide left.

9. A plot of land has the shape formed by a square having a semi-circle described on each side as diameter. If the length of the sides of the square is 54 ft., what ought the rent of the land to be at Rs. 750 per acre ?

10. Calculate the area of a figure consisting of an equilateral triangle (side 4 ft.) with a square superimposed on each side.

(C. U. 1955)

11. A circular track 10 ft. wide has an outer radius of 61 ft. Find the cost of paving it at $3\frac{3}{4}$ as per square foot. What is the average length of the track ?

(C. U. 1952)

Solid Figures

12. Find as accurately as possible the weight of a cubical piece of metal, the edge of which is 3 yds. 2 ft. 3'1 inches long, while a cubic foot of metal weighs 8'8764 cwt.

(C. U. 1943)

13. Find the number of gallons of water a cistern measuring inside 3 ft. by 2 ft. 6 inch. by 7 ft. in height will hold when full ; find also the weight of the contents, having given 1 pint equals 34'7 cubic inches and that the weight of 1 gallon of water is 10 lbs.

(C. U. 1944)

[Hints : The cross-section of the wall is a trapezium.

∴ volume = area of cross-section $\times$ length.]

14. The great wall of China is said to be 2400 k.m. long, 61 d.m. high and 457'5 c.m. thick at the top and 7625 m.m. thick at the bottom. How many cubic metres of building materials does it contain ?

(C. U. 1940)

15. A cistern 12 ft. long, 8 ft. deep and 5 ft. 6 inches broad is filled with water. If the waste pipe has a cross-section of three-fourths of square inch and water flows out through it at the rate of 14 ft. 6 in per minute, in what time will it be emptied ?

(C. U. 1938)

[Hints : Vol of water flowing out per minute = volume of the waste pipe 14 $\frac{1}{2}$ ft. long = area of cross-section $\times$ length = $\frac{3}{4}$ sq. in $\times$ 174 in = 8 $\frac{3}{8}$ cu. ft.]

16. A cubical box of external dimensions 17 inch each way, would contain crushed ore of the value of £ 421 17s 6d, if it was made of material 1 inch thick. But by mistake it has been made of thicker material and the difference in the value of ore

which it will hold is consequently £ 78 17s 6d less. What is the real thickness of the material ? (C. U. 1934, 40)

[Hints : Since the ore is powdered, the volume of ore = internal volume of the box.

$$\begin{aligned}\text{Value of } 15 \times 15 \times 15 \text{ cu. ft. of ore} &= \text{£ } 421\frac{7}{8} \\ &= \text{£ } 421\frac{1}{8} - \text{£ } 78\frac{7}{8} = \text{£ } 343\end{aligned}$$

$$\therefore \text{Real volume of ore} = \frac{15^3 \times 343}{421\frac{7}{8}} = 14^3]$$

17. A cylindrical shell of height 12 ft. is open at both ends, and its external and internal radii are 1 ft. and 10 inches respectively. Find the surface of the shell and its weight if the material composing it weighs $3\frac{1}{2}$ lbs per cu. ft. (C. U. 1953)

18. Two cylindrical tanks of diameters 8 ft. and 10 ft. are partly filled with water, the former to a depth of 5 ft. and the latter to a depth of 9 ft. If they are connected so that water comes to the same depth each, what will that depth be ?

(C. U. 1962)

19. One Polrow metal pipe $\frac{1}{4}$ inch thick and 3 ft. long has an outer diameter of 4.5 inches. Calculate the cost of the pipe at Rs. 3.20 nP. per lb., if one cubic foot of metal weighs 756 lb.

(C. U. 1961)

20. A weir is 1600 yds. wide. If the water rises 3 inches per minute while the stream is flowing at 5 miles per hour how many additional tons of water will pass over the weir per minute ? Given one cubic yard of water weighs $\frac{3}{4}$ ton.

(C. U. 1962)

21. Find the whole surface and volume of the cone whose height is 1 foot, and diameter of whose base is 5 ft. 10 inch.

22. An electric pump has to fill a petrol tank. The tank is cylindrical with hemispherical ends. The height of the cylindrical portion is 6 ft. and the hemispheres have radius of 3 ft. The pump has plunger of diameter 3 inches making 90 strokes each

of length 9 inches every minute. How long the pump will take to fill the tank. (C. U. 1958)

[Hints: Volume of tank $= \pi 3^2 \cdot 6 + \frac{\pi}{4}$, $\pi \cdot 3^2 = 90\pi$ c. ft.]

The volume of plunger $= \pi \left(\frac{3}{24} \right)^2 \times \frac{3}{4}$ c. ft. $= \frac{3\pi}{256}$ c. ft.]

23. Three solid balls of radii 3 in, 4 in, and 5 in are melted and moulded into a single solid ball. Find the diameter of this ball.

24. A swimming bath is 30 yds long, 12 yds. wide and its depth ranges from 3'4" to 6'6". Find the cost of filling the bath if 100 gallons cost Rs. 1'50 nP. (1 cu. ft. of water weighs 6 $\frac{1}{4}$ gallons). (C. U. 1959)

Exerciso 1

Pages 5—6

1. (i) 8 (ii) $\frac{1}{25}$
2. (i) $1 - a^{-3}$ (ii) 1 (iii) 1
3. (i) x^{-m} (ii) 8 (iii) 4
4. (i) 1 (ii) 1 (iii) 1 (iv) 1
9. $\left(\frac{p}{q}\right)^{2m}$
13. (i) $2\frac{1}{3}$ (ii) 1
14. (i) $x = -1, y = 1$ (ii) $x = -1, y = -2$

Exorcise 2

Pages 15—16

1. (i) 8 (ii) 3 (iii) 1 (iv) $\frac{a}{7}$ (v) $ab + bc + ca$
2. (i) $\pm \frac{3}{2}$ (ii) ± 7 (iii) ± 3
3. (i) 1, 9 (ii) $1, 3\frac{2}{3}$ (iii) a, b (iv) $3, \frac{1}{8}$ (v) $b, \frac{a^2}{b}$
4. (i) $\frac{5 - 3 \pm \sqrt{57}}{4}$ (ii) $\frac{7 \pm \sqrt{13}}{6}$
5. (i) $\sqrt{3} \pm \sqrt{2}$ (ii) $\sqrt{3 + 2}, 1$
6. (i) $0, \pm \sqrt{6}$ (ii) $-11 \pm \sqrt{13}$
7. (i) $-3^9, 5^9$ (ii) $3^{13}, 4^{13}$ (ii) $2^{12}, 8^{12}$
8. $\pm 1, \pm \frac{1}{2}$ 9. $\frac{9}{2}, \frac{1}{9}$ 6¹²
10. (i) $-7, -2, 3, 8$ (ii) $3, -10, \frac{1}{2}(-7 \pm \sqrt{-79})$
11. 1, 2, $-2, 5$ 12. (i) 9 (ii) 2
13. ± 4 14. -2 15. 4
16. $-2, -18, -8 \pm 4\sqrt{5}$
17. $3, -1\frac{2}{3}, \frac{1}{3}(2 \pm \sqrt{70})$
18. (i) $\frac{1}{2}, -2, \frac{1}{2} - 5$ (ii) $2, -\frac{1}{2}, 3, -\frac{1}{2}$
19. (i) $1, 1, -2 \pm \sqrt{3}$ (ii) $3 \pm \sqrt{10}, 2 \pm \sqrt{5}$
20. $5, \frac{1}{5}$ 21. 0, 2
22. ± 3 22. ± 1 .

Exercise 3 Pages 23—25

1. (i) $x = 3, 4$ (ii) $x = 11, -8$ (iii) $x = \frac{4}{5}, \frac{5}{7}$
 $y = 4, 3$ $y = 8, -11$ $y = \frac{3}{6}, \frac{5}{7}$
2. (i) $x = 2, 3$ (ii) $x = 4, 3$
 $y = 3, 2$ $y = 3, 4$
3. (i) $x = 5, 1$ (ii) $x = 6, 4$
 $y = 1, 5$ $y = 10, 15$
4. (i) $x = 1, 4$ (ii) $x = 4, 9$ (iii) $x = 6$
 $y = 4, 1$ $y = , 4$ $y = \frac{3}{2}$
5. $x = 8, 2$ 6. $x = 7$ 7. $x = -1, -4$
 $y = 2, 8$ $y = 1$ $y = 1, 2$
8. $x = \frac{1}{5}, \frac{4}{5}$ 9. $x = 1, 2$ 10. $x = 3, 6$
 $y = 5, 20$ $y = 2, 1$ $y = 6, 3$
11. (i) $x = a, \frac{1}{2}(a + b)$ (ii) $x = a$
 $y = b, \frac{1}{2}(a + b)$ $y = b$
12. (i) $x = 1, -1, 3, -3$ (ii) $x = 8, 4$
 $y = 2, -2, 1, -1$ $y = 4, 8$
13. (i) $x = 1$ (ii) $x = 4$ (iii) $x = y = 6$
 $y = 2$ $y = 1$
14. (i) $x = 2, -2$ (ii) $x = 3$ (iii) $x = 3$
 $y = \pm 2, \pm \frac{1}{2}$ $y = 3$ $y = 2$
15. $x = 2, y = 6, z = 6.$
16. $x = \pm 3, y = \pm 5, z = \pm 4.$
17. $x = \pm 3, y = \pm 1, z = \pm 2.$
18. $x = 2, 3$ 19. $x = 2, 8$ 20. $x = 2, -2$
 $y = 3, 2$ $y = 4, 4$ $y = 0, 0$
 $z = 4, 4$ $z = 8, 2$ $z = -1, 1$
21. (i) $x = \pm 6, y = \pm 12, z = \pm 5.$
- (ii) $x = 5, -7$ 22. $x = y = z = \pm 1.$
 $y = 3, -5$
 $z = 6, -8$

$$23. \quad x = \pm \frac{a^2}{\sqrt{a^2 + b^2 + c^2}}, \quad y = \pm \frac{b^2}{\sqrt{a^2 + b^2 + c^2}},$$

$$z = \pm \frac{c^2}{\sqrt{a^2 + b^2 + c^2}}$$

$$24. \quad x = 2, -2$$

$$y = 1, -1$$

$$z = -1, 1.$$

Equational Problems Pages 25—27.

1. 24 2. 68 miles 3. 8 days
 4. $33\frac{1}{3}$ m.p.h. ; 200 miles 6. Rs 600, Rs 1000
 7. 45 miles 8. 240 9. 10, 8 10. 14, 10, 2
 11. $\frac{(x-y)^4}{x^3}$.

PART II

Exercise 21. Pages 33—38

1. (i) 23, 130 (ii) 35, 324 (iii) 20, 609
 (iv) -7, -67 $\frac{1}{2}$ (v) $39 - \frac{1}{2}$, 390
 (vi) $\frac{a+29b}{a+b}$, $\frac{15(a+15b)}{a+b}$ (vii) 197, 2040
2. (i) $2n, n(n+1)$ (ii) $4n-3, n(2n-1)$,
 (iii) $13-3n, \frac{n}{2}(23-3n)$
 (iv) $n(2x-y)-x, n^2x - \frac{1}{2}n(n+1)y$
 (v) $n(a-b)+2b, \frac{1}{2}n(a+3b) + \frac{1}{2}n^2(a-b)$
 (vi) $(x+y)^2 - 2(n-1)xy, n(x+y)^2 - n'(n-1)xy$
 (vii) $0, \frac{1}{2}(n-1)$
3. (i) 952 (ii) 125320 4. 24, 23, 22.....
6. 24, 55 7. 2815
9. (i) 2, 4, 6, 8 (ii) 3, 5, 7, 9
11. (a) 18 (b) $4n+1$; 5, 9, 13.....
13. (i) $-p-q$ 14. (i) -1950 (ii) -160 (iii) 0
15. 47, 86 18. (a) $n(n+1)^2$, (b) $\frac{1}{2}n(n+1)(n+2)$

19. (i) $\frac{n}{2(n+2)}$, (ii) $\frac{n}{n+1}$, (iii) $\frac{n}{4(3n+4)}$, (iv) $\frac{n}{x(x+ny)}$
 21. Rs. 51'75 22. Rs 150
 23. 15 months 24. 1140 yds.
 25. 34 min. 26. Rs 82'50, Rs 87'50, Rs 92'50

Exercise 5 Pages 45 - 49

1. (i) 4096, 8191 (ii) $\frac{1}{18}$, 2047 $\frac{1}{18}$
 (iii) -16384, -10920 (iv) 729/16384, 58975/196608
 (v) $81\sqrt{3}$, $131(\sqrt{3}+1)$ (vi) -16, $-\frac{1}{2} \cdot 31(2-\sqrt{2})$
 (vii) 6361 , $\frac{1}{3}9841(3+\sqrt{3})$.
2. (i) $2^n - 1$ (ii) $\frac{3}{2}\left(1 - \frac{1}{3^n}\right)$ (iii) $\frac{2}{3}\left\{1 + (-1)^{n+1} \cdot \frac{1}{2^n}\right\}$
 (iv) $\frac{1}{14} \cdot \frac{5^n \pm 2^n}{5^{n-2}}$ as n is odd or even.
 (v) $\frac{1}{2y(x-y)} \left\{ \frac{(x+y)^n - (x-y)^n}{(x+y)^{n-2}} \right\}$
3. (a) $54\left(3^{\frac{2}{3}} + 3^{\frac{1}{3}} + 1\right)$ (b) 6 4. (a) 7448/3125
5. (i) 16/81 (ii) $\frac{1}{8}$
6. (i) 4, 6, 9 (ii) 9, 6, 4 (iii) 1, 5, 25.
7. (i) 2^{n+2} , $16(2^n - 1)$ (ii) $3 \cdot 2^n / 2^2$; $3(2^{n/2} - 1)(2 + \sqrt{2})$.
8. (i) $\frac{1}{8\frac{1}{2}}$ (ii) $\frac{1}{2^{\frac{1}{2}}}$.
9. (i) 20, 50 (ii) $3^{5/4}$, $3^{3/2}$, $3^{7/4}$ (iii) $r = \left(\frac{b}{a}\right)^{1/n+1}$
10. 16, 8, 4,
11. (i) 1 (ii) $3/10$ (iii) $\frac{1}{2}(4 + 3\sqrt{2})$ (iv) $3\sqrt{3}/2$.
12. (i) $1/1-x$ (ii) $(1+x)/(1-x)^2$ (iii) $1/(1-x)(1-bx)$
13. (i) $\frac{10}{27}(10^n - 1) - \frac{n}{3}$ (ii) $\frac{50}{81}(10^n - 1) - \frac{5n}{9}$
 (iii) $\frac{5}{81}(10^n - 1) - \frac{n}{18}$ (iv) $n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)$.
14. (i) $10^2 \{1.03\}^n - 1$ (ii) $20[1 - (1.05)^{-n}]$
 (iii) $\frac{a}{2} \{1 - (1+i)^{-n}\}$.
16. (nth term = $2^n - 1$), $2^{n+1} - n - 2$.
17. (i) 4, 9, 14 or 19, 1, 9 - 1 (ii) 1, 3, 9.
22. 28, 25, 761; 25, 60000.
24. 400 ft. 25. Rs. 26,84,353'56
26. 5 ft. $2\frac{1}{2}$ in.

Exercise 6 Page 50

1. $15\frac{3}{11}$.

3. 24, 54.

Exercise 7 Pages 61—64

1. 20. 2. 6720. 3. 576. 4. 2450.
5. 3024. 6. 60. 7. 96.
8. 720, 120, 24. 9. 576.
10. 8^4 . 11. 1023.
12. (a) 5^{10} . (b) 2^8 . 13. 3600
14. 2903040. 15. $(n-2) \lfloor n-1$.
16. 48. 17. 1440.
18. $\lfloor 39 / \lfloor 5(\lfloor 4)^2(\lfloor 6)^3$.
19. (i) 5040 (ii) 840 (ii) $\frac{1}{2} \lfloor 11$. (iv) 1260.
(v) 4989600. (vii) 34650.
20. 144. 21. $\lfloor n / \lfloor n-m$.
22. (i) 2016 (ii) 3024. 23. 154.
24. 360. 25. 2520.
26. 5040. 27. 60.
28. (1) 4320. (2) 720. 29. $(\lfloor 7)^3$.

Exercise 8 Page 74—77

1. 56, 210, $n(n-1)(n-2)$ 2 (i) 13. (ii) 120.
4. 84, 28, 35. 5. 220. 6. 560.
7. (i) 20 (ii) 10 8. 20
9. (i) 12, 35 (ii) 163, 1084
11. 868224 12. 441
13. (a)(i) 196 (i) 252 (b) 186
14. 180 15. 3090
16. 12 17. 255
18. 7 19. 59
20. 816000 22. 127 23. 63
25. 68 27. $\lfloor 22 / \lfloor 2(\lfloor 11)^3$
28. $\lfloor 18 / (\lfloor 6)^5$. 29. 2047. 30. $\lfloor 52 / (\lfloor 13)^4$.

Exercise 9 Pages 91—93

1. 3, 1, 0, -3. 2. 2'9523, 1.8756, 0'4879, 0'0447, 1.8957.
3. 74470, 0'04831, 2'389 4. 1'6532126.
5. (i) 4, (ii) 6, (iii) 3. 9. (i) 2, (ii) 3 ✓
10. (i) $\frac{1}{2}$. (ii) 3. 11. (i) 1386, (ii) 2'083
12. (a) 0'5988 (b) 20'19.2720 (iii) 1'37.
14. (i) '0042 (ii) 2'063. (iii) 1'37.
15. (i) 8. (ii) 1. 16. (i) 5'427
17. (i) 14'22, (ii) 1'03. 18. (i) 0'056 (ii) 0'057.
19. 12640.

PART II**Exercise 10**

1. (a) Rs. 215.51 (b) Rs. 200 2. 78
3. Rs. 1'20 4. (i) £ 900 (ii) Rs 2800

- | | |
|----------------------------------|------------------------------|
| 5. £ 19 12 s | 6. Rs. 581'99 |
| 7. Rs. 1592'12 | 8. £ 3769 |
| 9. £ 640 ; £ 643 38 | 10. (i) £ 400 (ii) £ 625 |
| 11. £ 6711, 14 s | 12. Rs. 9341 |
| 13. (a) Rs. 1891'30 (b) Rs. 1382 | 14. 14'2 years |
| 15. 18'6 years | 16. 4'16 |
| 17. 5'063 | 18. 5'7 |
| 19. (i) 4'04% (ii) 4'9604% | 20. 3'94 |
| 21. (i) 15 years (ii) 17 years | 22. 23. |
| 23. Rs. 3483 | 24. 39 years |
| 25. 4'52 years ; 11'6 years | 26. 24'7% of original value. |

Exercise 11.

- | | |
|---------------------|-------------------------|
| 1. Rs. 9839, | 2. Rs. 62183'53, 33070. |
| 3. 1'9524. | 4. Rs. 5'3'19. |
| 5. £92 8s. | 6. Rs. £582 14s 3d. |
| 7. £ 130 16s. | 8. Rs. 266'5 |
| 9. Rs. 1735. | 10. Rs. 12075 |
| 11. Rs. 1166'66. | 12. Rs. 2'5%. |
| 13. Rs. 31, 203'32. | 14. Rs. 20,000. |
| 15. Rs. 39'46. | 16. 40 years. |
| 17. £ 220. | 18. £ 3580. |
| 19. Rs. 1234'88 | 20. 11 $\frac{2}{3}$ %. |
| 21. Rs. 2755. | 22. Rs. 682'50 |
| 23. Rs. 5960. | 24. Rs. 5176. |

Exercise 12

- | | |
|-------------------------------------|--------------------------|
| 1. (i) Rs. 255, Rs. 12495. | (ii) £ 25 5s £ 1994 15s |
| (ii) £ 13 12s 7d, £ 1413 17s 5d | |
| (iv) £ 1 0s 3d, £ 290 19s 9d. | |
| 2. (i) £ 71. £ 1400 | (ii) Rs. 13'75, Rs. 2500 |
| (iii) Rs. 10'47, Rs. 198'53 | |
| 3. (i) Rs. 225, Rs. 219.50. | (ii) £ 12 3s 7'2d, £ 12. |
| 4. £ 5 17s 4d. | 5 4s. |
| 6. Rs. 2'75. | 7. 81% of bill value. |
| 8. (a) Rs. 101,000 | (b) Rs. 9, 504. |
| 9. Rs. 109'45, Rs 112'50, Rs 4,500. | |
| 10. £ 292. | 11. Rs 11,925 83 nP. |
| 12. (a) 2 27% | (b) 4'08% |
| 13. Rs 8 67, Rs 8'55, Rs 0'12. | |
| 14. 5%, Rs 3,240. | 15. 73 days after date. |
| 16. 15 months. | 17. Sept. 19, 1935. |
| 18. Rs 1575'23. | 19. Rs 2,500. |
| 20. Rs 180'05. | 21. Rs £ 618 11s. |
| 22. Rs 1257'37. | |

Exercise 13

- | | | |
|--------------------|---------------|------------------------|
| 1. 6th Apr. | 2. 27th June. | 3. 22nd Dec, 1956. |
| 4. 14th Feb, 1944. | 5. 6th Nov. | 6. 9th Apr. |
| 7. 6th May. | 8. 15th Apr. | 9. Rs. 2393 42 |
| 10. 21st June. | | 11. 46 $\frac{2}{3}$ % |

Exercise 14

1. 19'07 lire, '052 dollar.
2. Rs $\frac{2}{3}$.
3. Rs 3200.
4. £ 1 = 20'45 mark.
5. Rs 96,000
6. 8'90 fr.
7. Rs 3.
8. 3s 6 $\frac{3}{4}$ d, 9%
9. Rs 8333 5as 4p.
10. Rs 2 25
11. Rs 651'64.
12. 3661 17s 6d.
13. 122967'9 fr.
14. \$ 764'5
15. Gain Rs 340'52.
16. £ 197 14s 7d.
17. Rs 9,600.

Exercise 15

1. $\frac{1}{8}$.
2. $\frac{1}{4}, \frac{1}{32}, \frac{4}{18}, \frac{3}{32}$.
3. $\frac{13}{102}$.
4. $\frac{38}{66}$.
5. $\frac{1}{9}, \frac{1}{12}, \frac{1}{18}, \frac{1}{30}; \frac{5}{18}$.
6. $\frac{1}{30}, \frac{1}{18}$.
7. $\frac{11}{24}$.
8. $\frac{1}{72}$.
9. $\frac{21}{16}$.
10. $\frac{5}{6}$.
11. $\frac{3}{18}$.
12. $\frac{7}{18}$.
13. $\frac{16}{8820}, \frac{38}{190}$.
14. $\frac{68}{328}$.
15. $\frac{68}{288}$.
17. $\frac{91}{216}$.
19. $\frac{25}{81}$.
20. $\frac{847}{70}$.
21. $\frac{8}{25}$.
22. $\frac{30}{121}, \frac{5}{11}$.
24. $\frac{4}{9}$.

Exercise 16

1. (i) Rs. 4,925. (ii) £ 2,500. (iii) Rs. 5310.50.
2. (i) Rs. 315. (ii) £ 208 8s.
3. (i) 3'64. (ii) 3'09.
4. (i) 4% stock. (ii) 6% debenture.
5. (i) Rs. 5,500 stock. (ii) 1,990'26 stock.
6. (i) Rs. 5,000 stock. (ii) Rs. 1004 stock.
7. (i) Rs. 12,600. (ii) Rs. 3386'25.
8. 74 $\frac{1}{4}$. (9) 133 $\frac{3}{4}$.
11. (i) Rs. 7500. (ii) 158 $\frac{3}{4}$.
12. 95. 13. 77 $\frac{3}{4}$. 14. (i) 108. (ii) 117.
15. Rs. 44640. 16. Rs. 9000 stock.
17. £ 4,725. 18. Rs. 1000.
19. Rs. 85 $\frac{1}{2}$. 20. 2 $\frac{291}{328}$.
21. Rs. 154, 2'36%. 22. 6'5%.
23. Rs. 138. 24. gain Rs. 20.
25. gain Rs. 196. 26. Rs. 15,000 ; Rs. 66,000.
27. £ 388 15s. 28. (i) 115. (ii) Rs. 109/25.
29. Rs. 101/25, Rs. 1920. 30. 2000, £ 1750.
31. Rs. 8000, Rs. 2200.
32. Rs 4665, Rs. 5040 ; Rs. 81000, Rs. 45000.
33. 1 $\frac{1}{2}$ %, 6%, 15%. 34. 2 615%.

Exercise 17

1. 196'76 yds, 49'19 yds.
2. Rs. 216—7—3p.
3. 1140 sq. ft, 960 sq. ft.
4. 30'9 yds, 25'75 yds.
5. £ 11. 11s.
6. Rs. 9336'23
7. £ 4 8s. 2 $\frac{1}{2}$ d.
8. Rs. 103.
9. Rs 129 2 as.
10. 54 92 sq ft.
11. Rs. 825. 325 ft.
12. 1266 $\frac{1}{2}$ 2 cwt.
13. 3268 lbs
14. 8,93,04,000.

Commercial Mathematics

Part III

ELEMENTARY STATISTICAL CONCEPTS AND GRAPHICAL REPRESENTATIONS.

Chapter I

STATISTICS — its meaning and use.

1. The word '*statistics*' in the plural sense is now used to mean (a) numerical data or figures and (b) the statistical methods for handling these numerical data. But the word "*statistic*" in the singular sense is used to mean the statistical methods only.

“Ancient states used to collect data of population and resources to assess their manpower and wealth. Gradually more and more data relating to birth, death, disease etc. were collected. To-day such data are numerous and easily available to research workers.

Modern statistics is not merely a collection of data. It is a scientific method and widely applied to every branch of human knowledge—sociology, agriculture, trade and commerce and even to physical sciences, *i. e.* to all the ways of man and nature.

Statistics does not deal with individual or isolated things. It deals with aggregates—that is, a large number of similar things taken collectively and attempts to bring out certain uniformities in group behaviour. The inference or relationship in mass behaviour is not exact but only approximate and holds only for a large group of observations.

By applying statistical principles it is now possible to obtain necessary information regarding a group of data to make correct estimates or forecasts. The policy decisions of Commercial Firms, Manufacturing Industries and Govern-

ments rest on accurate and detailed statistical information. Thorough statistical study makes budgetary and quality control possible at every stage of production of a modern industry.

Statistics is now defined as the (a) collection, (b) classification and tabulation, (c) analysis and (d) interpretation of numerical data.

2. Statistical collection

The statistician may be regarded as an architect with quantitative data as raw material and statistical methods as tools to deal with;

These data may be obtained by himself or from Government and private sources, such as various business firms, chambers of commerce, newspapers, journals etc. Majority of these collections has been obtained by non-random observations. These data are not truly representative, but yet they furnish much information. However, data so obtained, when interpreted and generalised, may lead to quite wrong conclusions, and they must be taken with their limitations.

To yield any satisfactory result, the data must be *trustworthy* and *accurate*. Reliability and accuracy are the two essential requirements in statistical collection.

Before the data are collected, *the aim and object* must be clearly understood. Unsuitable data are worse than useless.

Further, the data must be *sufficient*. A few data are useless. The sample must be *representative* of the population from which it is drawn.

3. Unit.

Statistical data must be capable of numerical expression. Units are quantitative measures.

The units must be clearly mentioned, otherwise the data will convey no meaning. The rainfall data may be expressed in inches, or centimetres or millimetres. If the rainfall at a place is entered as 200, we can not say if it is 200 inches, 200 cm. or 200 mm.

Secondly, the same data may be expressed in different units. A rainfall of 200 inches may be expressed as 200, 20, 2 or 1 according as the unit chosen is 1, 10, 100 or 200 in.

Thirdly, the same unit may have different values. Thus a ton is supposed to have many values. A *long ton* is equal to 2240 lbs., a *short ton* is equal to 2000 lbs. In shipping business, *tonnage* means space available for cargo and its value varies from port to port and from one cargo to another.

Fourthly, the monetary units—such as rupees, francs, dollars are subject to fluctuations and are unsatisfactory. As a measure of value they change with the rise or fall of price level.

4. Classification and Tabulation

The collected data are without any form or structure. They are called raw data. The *raw data* are next re-arranged or classified in groups on the basis of some common characteristics.

There are four bases of classification—quality, quantity, time and location. The four types of classification are :

1. **Qualitative**—Qualitative classification is on the basis of kind or quality. When inhabitants are classified according to sex—into male or female, or according to religion into Hindus, Moslems, Christians, we have a qualitative classification.

Rice may be classified as hill rice or swamp rice ; silk may be classified as natural or artificial.

2. **Quantitative**—It is on the basis of quantity. Inhabitants may be classified according to height, weight, age or income. Colleges may be classified according to the number of students. Families may be classified according to the number of members.

3. **Chronological**—Here the basis is time. When data are classified in relation to time, we have a chronological classification. Chronological data form a *time series*. Population of a country, rainfall, crop production of an area, when classified according to years form time series. The time series exhibits certain patterns—cyclic, seasonal or irregular. The incidence of small-pox in Calcutta shown by monthly data forms a monthly time series.

4. Geographical. When the basis is regional or location, for example, when population of West Bengal is shown by districts or by subdivisions or by smaller village units, we have a geographical classification.

Quantitative, chronological and geographical classifications are sometimes put into the form of frequency distribution which will be discussed in the next chapter.

5. Presentation. Statistical data are presented in any of the following three methods.

(1) Text presentation—i.e., by means of figures and description (called text). This method is less effective. It is impossible to grasp the matter without going through the paragraph.

(2) Tabular presentation—It is a shorter but more effective method than the text.

High Density Sub-Regions of India, 1951 (TITLE)

CAPTION

STUB	Name of Sub-regions	Population (lakhs)	Density (Number per sq. mile)	Land area (Lakhs of acres)	BODY
	Lower Gangetic Plains	700	832	538	
	Upper Gangetic Plains	389	681	366	
	Malabar-Konkan	238	638	239	
	South Madras	307	554	355	
	North Madras and Orissa Coastal	211	461	293	
	Total 5 Sub-regions	1,845	660	1,791	

(grand total)

Foot note—

Source note—Data from Census Report of India, 1951, Vol. 1.

A table consists of four parts—the title, the stub, the caption or box-head, and the body.

The title is self-descriptive and is at the head of the table. It shows what the data are, where they taken, how they are classified and when.

The **stub** is the extreme left hand column with its heading.

The **caption** is the row of headings of all the columns excepting that of the stub.

The **body** contains the other rows and columns. Often there are one or more footnotes at the bottom of the table to explain some points.

A **source note** if required is appended at the end indicating the source from which the table is taken.

The advantage of tabular presentation is that use of rows and columns makes comparison of data easier.

Types of table : There are two types of tables—the general and summary. General tables are very large and contain much more information than summary tables. The summary tables, as the name implies, are short and contain grouped data.

When data are large, ratios, percentages, averages are used.

A **simple table** contains only a few data.

If the figures are only roughly accurate, they are rounded. In rounding they are taken in hundreds, thousands, or millions etc. Thus Rs. 4896 would be Rs. 4900, or Rs. 5000 when rounded to the nearest hundred or thousand.

Size : Usually a table is of the size of a page or pamphlet. The length and breadth of a table should be proportionate. It should neither be too short and broad, nor too long and narrow.

Totals : Sometimes cross totals and long totals are made to show the grand total. Although it is not a positive check, it affords a check to the correctness of the row and column totals.

(3) **Charts and graphs** are more effective than the text and table. They present a quick picture even to a casual observer. But these are not without limitations. Firstly, they lack in details and exhibit only a limited information. Secondly, such limited information is only approximate,

for even accurate figures such as 5231, 1390, can only be shown in hundreds as 52, 14 or in thousands as 5, 1.

6. Charts are of different types—line chart, barchart, piechart, pictograph etc.

7. Analysis and Interpretation : After presentation the data are analysed and interpreted. The analysis may discover certain common characteristics and explain their significance by which a fair estimate or forecast can be made.

Chapter II

METHODS OF COLLECTING STATISTICAL DATA

(including elementary notions of sampling)

8. As already pointed out statistical data are obtainable from various sources. They may be divided into two broad groups (a) Primary and (b) Secondary.

Primary Data are collected and compiled by original inquiry and published by the agency responsible for collection. The **census of India** is published by the Government of India who are also responsible for collecting the data. Such data collected by the issuing authority are primary.

Secondary Data—Data originally collected by other than the Publishing authority are called secondary. Secondary data must have some purpose for their original collection. They are reprinted from the primary data.

It is preferable to use primary data, because (a) primary data are given in greater details, (b) they are more reliable than the secondary, for the latter may contain mistakes and misprints due to transcription, (c) usually primary data contain a schedule and method of collection, and (d) the primary source defines terms and units used.

9. Principal Sources of Primary Data

Official Publications—

- (a) The Census of India (Decennial)
- (b) Agricultural Situation in India (Monthly)
- (c) Reserve Bank of India Bulletin (Monthly) etc.

Principal Sources of Secondary Data

- (a) Official Publications *e.g.* Statistical Abstract of India (Annual), The Indian Trade Journal, (weekly), Monthly Statistical Digest.
- (b) Publications by Stock Exchanges, Trade Associations, Chambers of Commerce, etc.
- (c) Publications by Research Associations
- (d) Individual collections.

10. Methods of Collection

- (1) The complete enumeration method. (2) Postal method. (3) Method of sample survey.

The first two methods are also known as **questionnaire method**.

(1) The Complete Enumeration Method.

By this method, statistics are collected by complete coverage of the entire field of inquiry.

For such a study the layout is made in advance. A set of questions—called a **questionnaire**—is framed and a schedule is prepared so that the replies received by the enumerators may be entered. The schedule is edited and the data are classified. The method is popularly known as *census*.

Preparation of Questionnaire

- 1. The questions should be simple and clear.
- 2. Offending, lengthy and leading questions should be avoided.
- 3. Questions should be such that the answers may be given in a short form: yes or no or in round figures as 15 or 20 etc.

This method has several drawbacks

- 1. It is very costly. 2. It requires much time for planning and execution. 3. It is not always feasible without a huge organisation. But it has one particular advantage that the trained enumerators interview the informants and explain the questions.

(2) Postal Method

A list of questions called the questionnaire is sent to individuals either by post or by person to be filled up at convenience and sent back by post. There are no enumerators meant for any personal contact with the individuals.

The drawbacks of the system are

1. Some individuals may not understand the questions and so their entries are misleading.
2. Some may not return the forms at all. But this method spares time and money.

Complete or quasi-complete census is very expensive and time-absorbing. Moreover, for a statistical study of any assigned problem such complete coverage of population is not always necessary. Modern statisticians have developed techniques of drawing a representative sample from a large population.

One of the most important statistical laws is the **Law of Statistical Regularity** which states *that if out of a larger population a small group is drawn at random, the random sample will exhibit the characteristics of the larger group.*

Hence, the essential point in statistical enumeration is to collect a sample as representative as possible.

(3) Method of Sample Survey

Definition : Population (or Universe)—This word is used to mean the aggregate or totality of all persons, or objects under study.

When we study the problem of marks received by candidates in an examination, our population consists of marks and not of candidates. Thus population may mean population of heights, population of age, population of dwellings, population of bricks or factories, etc.

Again, population may be finite or infinite.

A finite population has a finite number of values between any two fixed values. Thus, population of

students is finite for, between any two limits, say 50 and 100, it can take up only limited integral values. The heights, age, weights of the students each from an infinite population.

Sample—A portion of the population under study is called a sample of the parent population.

Random Sample—A sample is called random, if every item in the parent population has an equal chance of being included in it.

There are several types of Sampling.

(a) **Random Sampling**—In a random sample, a group of specified number of items will have the same probability of selection.

The principle of random sampling is applied in drawing lotteries. Tickets alike in all respects—size, shape, weight and colour—are each given a number of sold tickets. These are mixed together and one is drawn from them. After the first ticket is drawn, all other tickets are again mixed together and again one ticket is drawn. This process is continued till the requisite tickets are drawn*.

(b) **Systematic Sampling**—If a sample is obtained by drawing every 10th or any other fixed item in an ordered arrangement the sample is called systematic. In such a sample the first item is selected at random. The list must represent the entire population. It is practically a simple random sample. Such a sample may be drawn from a list of names arranged alphabetically.

(c) **Stratified Sampling**—In stratified sampling, the parent population is divided into distinctive classes—called strata before any sample is drawn. Each stratum is then subdivided into sampling units and from them a sample of desired size is selected at random.

* A random sample may be conveniently and most reliably drawn by means of random numbers. Tables of random numbers are available for ready use. (See Appendix)

Other types of sampling are

- (d) Area or Cluster Sampling (e) Quota-Sampling
(f) Purposive Sampling (g) Sequential Sampling etc.

Samples must not be *biased*. Bias may be introduced in different ways. Biased samples lead to misleading results.

Chapter III

FREQUENCY DISTRIBUTION

11. Raw Data—After the filled up schedules are received, a huge mass of data is at hand. These are in a raw state—without any shape or form and form an un-classified mass.

The raw data are then rearranged in groups. This process is called classification.

12 Variable—A characteristic of the data such as height, weight, wages, income which can be expressed numerically is called a variable or variate.

A variable may be *continuous* or *discrete*. A continuous variable is that which is capable of assuming any numerical value within a given range. Age, height, weight are continuous variables. A discrete variable can take only a limited number of values. Numbers of boys, houses, books are discrete variables, for they can assume only positive integral values.

13. Frequency—The number of times a particular value of a variable occurs is called frequency of the value and the table showing different values of the variable with their respective frequencies is called *frequency table*. Classification or distribution of the frequency of the variable is called *frequency distribution*.

The *variate* is usually denoted by x and *frequency* by f .

14. Array—When the raw data are arranged in ascending or descending order of magnitude, we get an *array tabulation*.

15. Class-interval—The difference between the maximum and minimum value of a variable is called the **range**. Each sub-range into which a large range of the

variable is divided is called a class-interval. The class-interval may be made narrower or wider according to requirement. Too many or too few classes are unsuitable. The end limits of each class interval are called *class limits*. The number of classes should be 8 to 20.

Usually the mid-value or central-value or class-mark is taken to represent the value of the class. The *mid-value* of any class interval is the arithmetic mean of its lower and upper limits.

Thus if the classes be 10–15, 15–20, 20–25 etc. the class limits are 10, 15, 20, 25..... The mid-value of the class 10–15 is $\frac{10+15}{2}=12.5$, that of 15–20 is $\frac{15+20}{2}=17.5$ etc.

The class-interval is 5 (being 10 and under 15 i.e. from 10 to 14 only), similarly from 15 to 19 etc.

The mid-value of any class can also be found thus:—

The mid-value of any class

$= \text{lower limit} + \frac{1}{2} \times \text{class-interval}$

Thus the mid-value of class 10–15 $= 10 + \frac{1}{2} \times 5$
 $= 10 + 2.5 = 12.5$, that of 15–20 $= 15 + \frac{1}{2} \times 5 = 15 + 2.5 = 17.5$ etc.

The number of items falling in any class-interval is called *class frequency*.

Class-interval when equal, can easily be found by mid-values of classes. Thus 12.5, 17.5, 22.5 being the mid values of classes, the size of the class interval $= 17.5 - 12.5 = 5$ or $22.5 - 17.5 = 5$.

Raw Data

TABLE 1-A

Marks obtained by 50 students in a Test Examination

February 21, 1962.

80	25	25	40	30	29	34	44	42	45
44	20	48	44	44	48	36	46	46	46
36	60	32	48	32	60	65	75	35	10
60	20	46	48	50	38	48	34	50	60
34	80	75	20	15	70	68	38	62	48

Array**TABLE 1-B**

10	32	42	48	60
15	34	44	48	60
20	34	44	48	62
20	34	44	48	65
20	35	44	48	68
25	36	45	48	70
25	36	46	50	75
29	38	46	50	75
30	38	46	60	80
32	40	46	60	80

Table 1-B shows the marks in ascending order. It is called an array. An array has troublesome form. It, however, shows the maximum and minimum marks and the frequency of the data. But some characteristics of the data are not yet discernable so easily and something yet remains to be done.

FREQUENCY DISTRIBUTION OF STUDENTS*(Classified on the basis of marks)***TABLE 1-C**

(Class-interval=5)

Marks	Frequency
10—14	1
15—19	1
20—24	3
25—29	3
30—34	6
35—39	5
40—44	6
45—49	11
50—54	2
55—59	0
60—64	5
65—69	2
70—74	1
75—79	2
80—84	2
	50

TABLE 1-D

(Class-interval=10)

Marks	Frequency
10—19	2
20—29	6
30—39	11
40—49	17
50—59	2
60—69	7
70—79	3
80—89	2
	50

Tables 1-C and 1-D represent two different degrees of condensation. Table 1-C presents a condensed summary of 1-B and shows a classification with class-intervals of 5 marks. There are 15 classes in all. Table 1-D shows a decrease in classes and an increase in the size of class-interval. There is a further loss of details.

In the case of continuous variables such as heights, weights, wages, the class-limits sometimes appear to be overlapping. In such a case, we may specify the class-limits as follows :—

TABLE 2

*Frequency distribution of workers in a Machine
Tools workshop at Calcutta
(classified according to daily wages).*

Daily wages	Frequency
Rs 5·00 but less than Rs. 5·50	4
5·50 „ „ 6·00	7
6·00 „ „ 6·50	13
6·50 „ „ 7·00	22
7·00 „ „ 7·50	30
7·50 „ „ 8·00	18
8·00 „ „ 8·50	10
8·50 or over	6
Total	110

16. Tally or Score Sheet—It is hazardous to construct an array table. The table of frequency distribution can be made directly from *Raw Data* by the following device.

Let us take the Table 1-A. From this we can directly construct Table 1-C without forming the array Table 1-B.

The classes are written down vertically.

In the first class-interval 10—14, there is 1 item. Put 1 small vertical line against this class. Similarly for the second class-interval.

There will be as many small vertical lines against each class as there are items falling within that class.

For convenience, put four such lines side by side and for the fifth strike a diagonal across them. Thus the counting is arranged in groups of five.

a group of 5 is shown by 
 a „ „ 12 „ „ 

Similarly, we can construct Table 1-D with a class-interval of 10 marks as follows:

*Frequency table 1-D compiled from
Raw Data Table 1-A*

Class-intervals		Frequency
10—19	//	2
20—29		6
30—39		11
40—49		17
50—59	//	2
60—69		7
70—79		3
80—89	//	2
Total 50		

17. Cumulative Frequency

For certain purposes cumulative frequency distribution is useful—particularly when we want to know how many of the items are above or below a certain level. It is prepared by adding up frequencies from either end. Thus it is possible to cumulate a frequency series in two ways—from above or from below.

The following is the cumulative frequency distribution from Table 1-D.

TABLE 1-E

*Cumulative Distribution of 50 candidates,
(classified according to marks.)*

Marks class-interval =10	Cumulative frequency number of students	
	Below the upper limit of each class	Equal or greater than the lower limit of each class
10—19	2	50
20—29	8	48
30—39	19	42
40—49	36	31
50—59	38	14
60—69	45	12
70—79	48	5
80—89	50	2

From the above table we can readily say that number of students getting 50 marks and above is 14, number of those getting below 40 marks is 19 etc.

Exercise I

1. Define statistics and state its use.
2. How are statistical data classified? Give suitable examples.
3. Distinguish between Primary and Secondary data and name their principal sources.
4. What are the different methods of collection of statistical data? Enumerate their relative advantages and disadvantages.
5. What do you mean by sample survey? What are the different types of sampling? Give suitable examples.
6. Distinguish between continuous and discrete variables. How do their graphs differ? Illustrate your answer.
7. Attempt a tabular representation of the information contained in the following extract relating to industrial accidents :—

In 1929 there were 19,077 non-fatal compensated cases which might be classed as permanent injury whereas there were 80,107 which might be classed as temporary injury. For 1930, these figures respectively were 22,434 and 86,106; and for 1931 they were 21,761 and 70,000. The total accidents for the three years were 523,604; 471,510; 419,072 respectively. The total deaths (included in the former accident totals) were 2,093; 2,006; 1,793 respectively of which totals compensations were paid for 1,217, 1,308, 1,241 respectively. The number of total accidents for which compensation was paid was 100,401 in 1929; 109,848 in 1930 and 102,985 in 1931.

(CU. B. Com. 1955).

8. Make a cumulative frequency distribution from the following series of values in rupees.

(a) 40, 20, 5, 7, 3, 8, 9, 15, 20, 25, 17, 28, 40, 20, 18, 10, 15, 30, 35, 50, 26, 28, 46, 29, 30, 5, 50, 28, 10, 5, 8, 23, 35, 7, 20, 18 with a class-interval of 5.

(b) 2.15, 8.00, 3.75, 4.25, 7.00, 3.20, 5.00, 6.75, 4.30, 1.50, 7.80, 9.15, 7.25, 6.95, 8.75, 7.20, 1.85, 7.30, 8.75, 1.85, 6.00 with a class-interval of 2.50.

Chapter IV

MEASURES OF CENTRAL TENDENCY

Mean, Mode, Median and Quantiles

18. Most Frequency Distribution of quantitative data collected from different fields show some common characteristics—some uniformity in their mass behaviour, firstly, the individual items in the distribution take up different values and no single value can represent them and secondly, there is a marked tendency to cluster towards the middle or about a central value. (This tendency of massing in the central part of the distribution is called *central tendency*.)

If the different values in a distribution be spread over a scale there seems to exist a value at which the clustering of values appears to be greatest. This value at which the massing is maximum, is taken to represent the *measure of central tendency*.

19. Several are the measures of central tendency. All such measures are called *averages*. Sometimes averages are termed *measures of location*, because they locate the position of concentration of values. The average is a single value which typifies the whole group.

The **measures of central tendency** are the three types of averages : (1) Mean (2) Mode and (3) Median.

Again, **Mean** is of three kinds :

- (a) Arithmetic Mean
- (b) Geometric Mean
- (c) Harmonic Mean

Arithmetic Mean is commonly written as A, M or $\bar{A}$. The arithmetic mean $\bar{A}$ of a series of n values $x_1, x_2, \dots, x_n$ is
$$\bar{A} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\Sigma x}{n} \left(= \frac{\text{Sum of the values}}{\text{Number of values}} \right)$$

where the symbol Σ (pronounced as sigma) denotes summation. This is called *simple arithmetic mean*.

Evidently, $\Sigma x = n \cdot \bar{A}$.

Thus, the arithmetic mean of 5 numbers 8, 26, 40, 7, 19 is $A = \frac{8+26+40+7+19}{5} = \frac{100}{5} = 20$

There is another arithmetic mean called *weighted arithmetic mean*.

If $x_1, x_2, \dots, x_n$ be the different values whose frequencies are $f_1, f_2, \dots, f_n$, then the totality of the items is given by $f_1x_1 + f_2x_2 + \dots + f_nx_n$, and the weighted arithmetic mean is $A = \frac{f_1x_1 + f_2x_2 + \dots + f_nx_n}{f_1 + f_2 + \dots + f_n} = \frac{\sum fx}{\sum f}$.

Thus, if there are 5 boys of height 4' each, 8 boys of height 4'9" each and 7 boys of height 4'4" each; their weighted average is

$$A = \frac{4' \times 5 + 4'9'' \times 8 + 4'4'' \times 7}{5 + 8 + 7} = \frac{20' + 38' + 30'4''}{20} = \frac{88'4''}{20} = 4'5''.$$

✓ **20. Computation of Arithmetic Mean by deviation** (deviation of x from $A = x - A$)

✓ One property of arithmetic mean is that the algebraic sum of the deviations of the different items from the mean is zero.

The proof of this important property is very simple.

for, since $\frac{x_1 + x_2 + \dots + x_n}{n} = A$

We have, $x_1 + x_2 + \dots + x_n = nA = (A + A + \dots \text{to } n \text{ terms})$

By transposition, $(x_1 - A) + (x_2 - A) + \dots + (x_n - A) = 0$.

The above property may be written as

$\sum(x - A) = 0$ or $\sum d = 0$, where d denotes deviation.

For example, let 4, 5, 9, 10, 12 be 5 different items, then their arithmetic mean is $A = \frac{4+5+9+10+12}{5} = \frac{40}{5} = 8$.

The deviations of the items from A are

4-8, 5-8, 9-8, 10-8, 12-8, whose sum $= (-4) + (-3) + 1 + 2 + 4 = 0$.

To find simple A.M. by deviation

Let a be any arbitrary assigned mean, then

$$\begin{aligned} A &= \frac{\sum x}{n} = \frac{\sum (x-a+a)}{n} = \frac{\sum (x-a)}{n} + \frac{\sum a}{n} \\ &= \frac{\sum (x-a)}{n} + \frac{na}{n} = a + \frac{\sum (x-a)}{n} = a + \frac{\sum d}{n} \end{aligned}$$

Ex 1. Find the arithmetic mean of 5, 7, 10, 8, 12.

By deviation

$$A = a + \frac{\sum d}{n}$$

Let 9 be the assumed mean.

x	d	
5	$5-9=-4$	$\therefore \sum d = -3$
7	$7-9=-2$	$\sum d = -3$
10	$10-9=1$	$n = 5 = -\cdot 6$
8	$8-9=-1$	$\therefore A = a + \frac{\sum d}{n} = 9 + (-\cdot 6)$
12	$12-9=3$	$= 8\cdot 4$
	-3	

21. To find weighted A.M by deviation :

As before, let a be any arbitrary assigned mean, then

$$\begin{aligned} A &= \frac{\sum fx}{\sum f} = \frac{\sum f(x-a+a)}{\sum f} = \frac{\sum f(x-a)}{\sum f} + \frac{\sum fa}{\sum f} \\ &= \frac{\sum f(x-a)}{\sum f} + \frac{a\sum f}{\sum f} = a + \frac{\sum f(x-a)}{\sum f} = a + \frac{\sum fd}{n} \end{aligned}$$

(since $\sum f = \text{number of items} = n$)

$$\begin{aligned} \text{or } A &= a + \frac{\sum fx}{\sum f} - a = a + \frac{\sum fx - a\sum f}{\sum f} \\ &= a + \frac{\sum f(x-a)}{n} = a + \frac{\sum fd}{n} \end{aligned}$$

Ex. 2. Find the weighted arithmetic mean of the following ungrouped frequency distribution of 100 workers. classified according to weekly wages.

Ordinary Method

$$A = \frac{\sum fx}{n}$$

Weekly wages x	Number of workers Frequency f	Product fx	
Rs. 10	5	50	$\therefore A = \frac{\Sigma fx}{n} = \frac{2143}{100} = 21.43$
17	11	238	
20	20	400	
22	35	770	
25	19	475	
30	7	210	
	100	2143	

By deviation $A = a + \frac{\Sigma fd}{n}$

Let the assumed mean be 20

Weekly wages x	frequency	deviation d	fd	
10	5	-10	-50	$\therefore \frac{\Sigma fd}{n} = \frac{143}{100} = 1.43$
17	11	-3	-42	
20	20	0	0	
22	35	2	70	$\therefore A = a + \frac{\Sigma fd}{n} = 20 + 1.43 = 21.43$
25	19	5	95	
30	7	10	70	
	100		143	

Grouped Data—When the data are collected in groups and arranged with respective frequencies, the arrangement is called a *grouped frequency distribution*.

22. Arithmetic Mean from Grouped Data

Ex. 3. Compute the A. M. from the frequency distribution of 252 students in a Test Examination, classified according to marks.

Class-interval x	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89
frequency f	6	12	41	60	71	50	10	2

Long Method

$$A = \frac{\sum fx}{n}$$

The *midpoint* of each class interval = lower limit of the class + half the class interval.

Class-interval = 10	Midpoint x	Frequency f	Product fx
10-19	15	6	90
20-29	25	12	300
30-39	35	41	1435
40-49	45	60	2700
50-59	55	71	3905
60-69	65	50	3250
70-79	75	10	750
80-89	85	2	170
		252	12,600

$$A = \frac{\sum fx}{n} = \frac{12,600}{252} = 50$$

Short Method

$$A = a + \frac{\sum fd}{n} \times i$$

To shorten computation, we select ' a ' to be the midvalue of any class and take the deviations in terms of class-intervals as units.

This device is particularly advantageous when class-intervals are equal.

Class-interval	Midpoint	Frequency	(in class-interval)	
=10	x	f	d	fd
10-19	15	6	-4	-24
20-29	25	12	-3	-36
30-39	35	41	-2	-82
40-49	45	60	-1	-60
50-59	55	71	0	0
60-69	65	50	+1	50
70-79	75	10	+2	20
80-89	85	2	+3	6
		252		-202 76

Steps 1. The assumed mean is chosen as the midvalue of the 5th class. So that $a=55$.

2. The deviations are calculated in terms of class-interval units. The deviation of the next lower class = -1, that of next lower class = -2, that of next higher class = +1 etc.

3. Take the algebraic sum of the deviations. This is $= -202 + 76 = -126$

4. Divide this sum by the total frequency 252 and multiply the fraction by the class-interval i ($= 10$)

$$\text{The necessary correction} = \frac{-126}{252} \times 10 = -5.$$

$$\therefore A = 55 + (-5) = 50.$$

Note—The arithmetic mean is a calculated value and may not exist in the series of items.

23. Geometric Mean (G. M.)

The geometric mean G of a series of n values $x_1, x_2, \dots, x_n$ is

$$G = (x_1 \cdot x_2 \cdot \dots \cdot x_n)^{1/n}$$

Taking logarithm,

$$\log G = \frac{1}{n} \{ \log x_1 + \log x_2 + \dots + \log x_n \} = \frac{\Sigma \log x}{n}$$

Thus logarithm of the geometric mean of a number of values is the arithmetic mean of their logarithms.

Ex. 4. Find the geometric mean of 4, 8, 16, 128

$$G = (4 \cdot 8 \cdot 16 \cdot 128)^{\frac{1}{4}} = (8^4 \times 2^4)^{\frac{1}{4}} = 16.$$

24. Weighted Geometric Mean

If the values $x_1, x_2, \dots, x_n$ have weights $f_1, f_2, \dots, f_n$

$$\text{then } G = (x_1^{f_1} \cdot x_2^{f_2} \cdot \dots \cdot x_n^{f_n})^{\frac{1}{f_1 + f_2 + \dots + f_n}}$$

$$\therefore \log G = \frac{f_1 \log x_1 + f_2 \log x_2 + \dots + f_n \log x_n}{n} = \frac{\Sigma f \log x}{n}$$

where $n = f_1 + f_2 + \dots + f_n = \Sigma f = \text{total frequency.}$

Ex. 5. Find the G. M. of 18, 12, 48, 61. whose weights are 3, 5, 2, 8 respectively.

x	f	$\log x$	$f \cdot \log x$	
18	3	1.2553	3.7659	$\Sigma f \cdot \log x = 26.7507$
12	5	1.0792	5.3960	$\therefore \log G = \frac{26.7507}{18} = 1.4837$
48	2	1.6532	3.3064	$\therefore G = 30.41$
61	8	1.7853	14.2824	
18			26.7507	

25. Harmonic Mean (H. M.)

The Harmonic Mean H of a series of n values

$$x_1, x_2, \dots, x_n \text{ is } H = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} = \frac{\Sigma \frac{1}{x}}{n}$$

that is, reciprocal of the harmonic mean is equal to the arithmetic mean of the reciprocals of the values.

[If x be a value, then $\frac{1}{x}$ is called its reciprocal]

Ex. 6. Find the harmonic mean of 18, 12, 30.

$$H = \frac{1}{\frac{1}{18} + \frac{1}{12} + \frac{1}{30}} = \frac{10 + 15 + 6}{180 \times 3} = \frac{31}{540}$$

$$\therefore H = \frac{540}{31} = 17.4.$$

Note—When all the items are equal, the Arithmetic, Geometric and Harmonic means are equal.

As for example, the A.M, G.M, and H.M. of three numbers 8, 8, 8 are each 8.

26. The **Median** is defined as a value which divides a distribution arranged in order such that it lies between an equal number of items on either side of it.

The Median from ungrouped data.

To find the median, arrange the items in any order of magnitude. Then the middle value is the median.

Let n be the number of items.

If n is odd, the median is the value of $\frac{n+1}{2}$ th. item

If n is even, we say there are two middle items whose arithmetic mean is the median. In this case median is not a member of the series.

Ex. 7. Find the median of the following series of items.

(1) 40, 15, 39, 87, 52

(2) 30, 8, 25, 47, 35, 84

(1) Rearranging the data in the ascending order 15, 39, 40, 52, 87, the median = $\frac{5+1}{2}$ or 3rd item = 40.

(2) Rearranging, we have 8, 25, 30, 35, 47, 84

The number of items being 6, there are two middle items 30 and 35 whose arithmetic mean is the median.

$\therefore$ the required median = 32.5

Ex. 8. Compute the median from the following unclassified distribution of weekly wages of a group of workmen.

Wages in Rs	15	16	17	18	19	20	21	22	23	Total
Number of workmen	6	12	14	15	18	25	14	7	4	115

Here number of items = $n = 115$ (odd)

$\therefore$ the median is the $\frac{n+1}{2} = \frac{115+1}{2} = 56$ th item

(48th to 65th items are included under the frequency 18)
whose variate is 19.

Ex. 9. Find the median from the following unclassified data.

5, 50, 30, 8, 70, 15, 10, 5, 8, 20, 40, 32, 30, 8, 15, 60, 40, 28,
15, 20, 40, 40, 28, 30, 32, 42, 60, 10, 15, 32, 5, 20, 15, 42, 32, 40.

Form the following frequency table

(x)		(f)
5	///	3
8	///	3
10	//	2
15	////	5
20	///	3
28	//	2
30	///	3
32	////	4
40	////	5
42	//	2
50	/	1
60	//	2
70	/	1
		36

$$n=36$$

∴ There are two middle
items, the 18th and 19th.

The 18th item = 28.

19th item = 30.

$$\therefore \text{the median} = \frac{28+30}{2} = 29$$

Note—Here the value of the median does not coincide with any item.

27. Mode (M_2) is the value of the item at which the concentration is greatest. It is called the “thickest value” having the greatest massing of items.

Ex. 10. Find the mode of the ungrouped data

4, 3, 2, 5, 3, 4, 5, 2, 1, 7, 3, 2, 1

x	f
1	2
2	3
3	3
4	2
5	2
7	1
	13

There are two modes 3 and
2 each of frequency 3.

Ex. 11. Examine if the series

1, 3, 5, 6, 8, 2, 4 has any mode.

Every item has the same frequency 1. There is no modal value.

28. Mode from grouped data by interpolation formula.

The mode may be located by means of the following formula.

$$\text{Mode} = l + \frac{f_2 \times x}{f_1 + f_2}$$

where l = lower limit of the modal class (in which the mode lies)

f_1 = frequency of the next lower class

f_2 = " " " " higher "

x = size of the modal class.

Ex. 12. Compute the mode for 227 Apprentice Engineers on the basis of monthly allowance.

Monthly allowance (class-interval = 10)	Number of Apprentices
Rs 120-129.99	7
130-139.99	26
140-149.99	45
150-159.99	60
160-169.99	49
170-179.99	27
180-189.99	10
190-199.99	3
	227

The class 150—159.99 with frequency 60 is the largest group, that is, it is, the modal group whose mid-value = $150 + 5 = 155$. So the mid-value 155 appears to be the mode.

If we increase the class interval to 20, we have

Monthly allowance (class interval=20)	frequency
Rs. 120—139·99	33
140—159·99	105
160—179·99	76
180—199·99	<u>13</u>
	227

Here the modal group is 140—159·99 with midvalue $140+10=150$, which appears to be the mode.

Thus, the mode changes with different classifications.

Hence this method does not give a tangible result for mode. If the class-interval decreases, the computed mode approaches the true modal value.

Applying the formula

$$M_2 = l + \frac{f_2}{f_1 + f_2} \times x$$

For class interval 10

The mode lies in the class 150—159·99.

$$\therefore l=150, f_1=45, f_2=49, x=10.$$

$$\therefore M = 150 + \frac{49 \times 10}{45 + 49} = 150 + 5.21 = 155.21$$

For class interval 20

The mode lies in the class 140—159·99

$$\therefore l=140, f_1=33, f_2=76, x=20$$

$$\therefore M = 140 + \frac{76 \times 20}{33 + 76} = 140 + 13.94 = 153.94. \text{ These two}$$

modal values are not so widely different.

The interpolation formula gives an approximate mode.

Bimodal groups. Sometimes a distribution shows two modes—two cases of concentration. This may be due to various causes such as class limits, want of sufficient data etc.

In such cases we may find an approximate mode by altering class-limits.

29. Mode from Mean and Median.

$$\text{Mean} - \text{Mode} = 3 (\text{Mean} - \text{Median}).$$

Ex. 13. Find the mean and median from the following distribution and hence determine the mode.

1, 3, 5, 3, 2, 1, 8, 7, 2, 1

$$\text{Mean} = \frac{1+3+5+3+2+1+8+7+2+1}{10} = \frac{33}{10} = 3.3$$

Arranging the data in order of magnitude

1, 1, 1, 2, 2, 3, 3, 5, 7, 8, we have

$$\text{Median} = \frac{2+3}{2} = 2.5$$

$$\therefore \text{Mode} = \text{Mean} - 3 (\text{Mean} - \text{Median})$$

$$= 3.3 - 3 (3.3 - 2.5) = 3.3 - 3 \times .8 = 3.3 - 2.4 = .9$$

The above relation gives only approximate mode.

30. Relative advantages and disadvantages of Averages

A.M. (1) It admits of easy computation by summation and division only.

(2) It is easily intelligible and the commonest measure of average.

(3) It can be always found out and treated algebraically.

but (4) It is influenced by every member of the series and the presence of a very small or very large item gives a misleading mean.

G.M. (1) It is less influenced even by the presence of a high value.

(2) It can be found out if no member is zero or negative.

(It is zero if one member is zero, meaningless if one member is negative)

(3) It is most suitable for averaging ratios—such as in finding out percentage increase or decrease as in the formula for compound interest.

$$A = P(1+i)^n$$

(4) It can be treated algebraically.

H.M. (1) It is suitable for averaging prices and has limited use.

(2) It can be algebraically treated.

but (3) It is not a mean of wide application.

Median (1) It can be found out, even when data are not all known and the items are not numerically measurable.

(2) It is affected by the position of any extreme value, but not by its magnitude.

but (3) it can not be algebraically treated.

Mode (1) The mode gives an actual value—a value which occurs in the distribution.

(2) It is not affected by any extreme value.

but (3) it is not so easy of computation from unclassified and unarrayed distribution.

(4) It can not be treated algebraically.

(5) There may be even two or more modes. So in such a case, it fails to give a correct idea.

(6) The mode fails if importance is given to extreme values.

Generally, $A.M > G.M > H.M$

If all the items in the distribution are equal, then

$$A.M = G.M = H.M.$$

The arithmetic mean, median and mode are regarded as the most important measures of central tendency.

Graphical representation of Mean, Median and Mode

In a symmetric distribution the position of the mean, median and mode is the same.

But in an assymetrical or skew distribution the median lies between the mean and the mode and the distances between

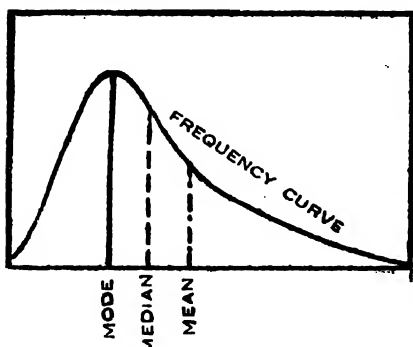


Fig. 1.

them are approximately given by $\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$.

The concepts of the above three measures may be graphically shown thus :—

The mean is the centre of gravity.

The median divides the frequency curve into two equal parts.

The mode lies just below the peak of the frequency curve.

31. Quantiles—Quartiles, Quintiles, Deciles

Quantiles are Partition values. A Quantile divides the total distribution in given proportions.

Quartiles—There are three quartiles which divide the total frequency into four equal parts.

Lower value		Upper value
Q_1	Q_2	Q_3
	M	

First Quartile (Q_1)—is that below which lies one-fourth of the total number of items and three-fourths above.

Third Quartile (Q_3)—is that above which lies one-fourth of the total number of items and three-fourths below.

Second Quartile (Q_2)—is identical with the Median, having one-half of the items below it and one-half above.

Quintile—There are four quintiles which divide the distribution into five equal parts.

Deciles (D) divide the total frequency into ten equal parts.

Percentiles—divide the total distribution into hundred equal parts.

If n be the number of items in a distribution

$Q_1 = \text{value of } \frac{n+1}{4} \text{ th. item and}$

and $Q_3 = \text{value of } 3\left(\frac{n+1}{4}\right) \text{ th. item beginning from}$
the lower value. Similarly, $D_1 = \text{value of } \frac{n+1}{10} \text{ th. item,}$
 $D_7 = \text{value of } \frac{7(n+1)}{10} \text{ th. item from the lower value.}$

For partition values the data are first put in ascending or descending order of magnitude.

Further worked out Examples

Ex. 1. Compute the arithmetic mean of the following.

Serial No.	1	2	3	4	5
Heights					
(in inches)	60	54	65	58	62

Serial No.	Height (x)	Deviation (x-a) (d)
1	60	0
2	54	-6
3	65	+5
4	58	-2
5	62	+2
$n=5$	$\Sigma x = 299$	$\Sigma d = -1$

$a = \text{assumed mean} = 60$

By ordinary method, $A. M. = \frac{\Sigma x}{n} = \frac{299}{5} = 59.8 \text{ inches}$

By short-cut method, $A. M. = a + \frac{\Sigma d}{n} = 60 + \frac{-1}{5} = 60 - .2$
 $= 59.8 \text{ inches.}$

Ex. 2. Compute the weighted arithmetic mean from the following frequency distribution.

Marks	0	15	30	40	60	75-85
No. of students	5	8	25	30	15	2

Marks (x)	Frequency (f)	f. x.	Deviation $d=x-a$	f. d.
0	5	0	-40	-200
15	8	120	-25	-200
30	25	750	-10	-250
40	30	1200	0	0
60	15	900	20	300
80	2	160	40	80
$n=85$		$\Sigma fx=3130$		$\Sigma fd=-270$

$a = \text{assumed mean} = 40$

By ordinary method, A.M. = $\frac{\Sigma fx}{n} = \frac{3130}{85} = 36.8$ marks.

By short-cut method, A.M. = $a + \frac{\Sigma fd}{n} = 40 + \frac{-270}{85}$
 $= 40 - 3.2 = 36.8$ marks.

Ex. 3. Find arithmetic mean from the following grouped frequency data.

Marks	frequency	Marks	frequency
0—10	2	50—50	20
10—20	10	60—70	14
20—30	25	70—80	8
30—40	40	80—90	2
40—50	60	90—100	1

Mid-value of class interval	frequency	deviation		
(x)	(f)	fx	$d=x-a$	fd
5	2	10	-35	-70
15	10	150	-25	-250
25	25	625	-15	-375
35	40	1400	-5	-200
45	60	2700	+5	+300
55	20	1100	+15	+300
65	14	910	+25	+350
75	8	600	+35	+280
85	2	170	+45	+90
95	1	95	+55	+55
Total	$n=182$	$\Sigma fx=7760$		$\Sigma fd=480$

$a = \text{assigned mean} = 40$

$$\frac{\sum fx}{n} = \frac{7760}{182} = 42.6 \text{ marks}$$

$$A.M = a + \frac{\sum fd}{n} = 40 + \frac{480}{182} = 40 + 2.6 = 42.6 \text{ marks.}$$

E.x 4. Find the mode, median and quartiles of the following distribution .

25, 1275, 748, 162, 967, 162

Mode is the value of the item which occurs most

$\therefore$ Mode = 162.

To find the median, arrange the data in any order of magnitude, say 25, 162, 162, 748, 967, 1275 (ascending order)

Number of items $n = 6$.

There are two middle items—whose values are 162 and 748

$\therefore$ Median = arithmetic average of 162, 748

$$= \frac{162 + 748}{2} = 455$$

1st. Quartile = $\left(\frac{n+1}{4}\right)$ th. item = $\frac{7}{4}$ th. item = 2nd. item nearly = 162.

3rd. Quartile = $\frac{3(n+1)}{4}$ th. item = $\frac{21}{4}$ th. item = 5th. item nearly = 967.

2nd. Quartile = $\frac{2(n+1)}{4}$ th. item = $\frac{14}{4}$ th. item = 3rd. or 4th item nearly (same as median).

E.x. 5. Calculate mode from the following classified distribution

Age	5-9	9-13	13-17	17-21	21-25
No. of boys	4	38	60	25	7

S—3

Age	mid value (x)	frequency (f)	cumulative frequency	$f. x.$
5-9	7	4	4	28
9-13	11	38	42	418
13-17	15	60	102	900
17-21	19	25	127	475
21-25	23	7	134	161
Total		$n=134$		$\Sigma fX=1982$

The highest frequency is 60

$\therefore$ modal group is 13-17

$$\text{Mode} = l + \frac{f_2}{f_1 + f_2} \times c \quad \text{or} \quad \text{mode} = l + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times c$$

where l = lower limit of modal group

f_0 = frequency of the modal group.

f_2 = frequency of the next group higher than the
modal group

f_1 = frequency of the next group lower than the
modal group

c = size of the class-interval

By the first formula

$$\therefore \text{Mode} = 13 + \frac{25}{38 + 25} \times 4 = 13 + \frac{100}{63} = 13 + 1.6 = 14.6$$

By the second formula

$$\text{Mode} = 13 + \frac{60 - 38}{2 \times 60 - 88 - 25} \times 4 = 13 + \frac{88}{57} = 13 + 1.5 = 14.5$$

Ex. 6. The weekly wages of 5 labourers are

40, 60, 36, 45, 25

Calculate the arithmetic mean, geometric mean and the harmonic mean.

$$\text{Arithmetic mean} = \frac{40 + 60 + 36 + 45 + 25}{5} = \frac{206}{5} = 41.2$$

$$\begin{aligned}
 \text{Geometric mean, } \log G &= \frac{\log 40 + \log 60 + \log 36 + \log 45 + \log 25}{5} \\
 &= \frac{1.6021 + 1.7782 + 1.5563 + 1.6532 + 1.3979}{5} \\
 &= \frac{7.9877}{5} = 1.598 = \log 39.63 \\
 \therefore G &= 39.63
 \end{aligned}$$

Harmonic mean

$$\begin{aligned}
 \frac{1}{H} &= \frac{\frac{1}{40} + \frac{1}{60} + \frac{1}{36} + \frac{1}{45} + \frac{1}{25}}{5} \\
 &= \frac{.025 + .017 + .028 + .022 + .04}{5} \\
 &= \frac{.132}{5} = \frac{132}{5000} \\
 \therefore H &= \frac{5000}{132} = 37.9
 \end{aligned}$$

Exercise 2

1. Define the three means and compare their advantages and disadvantages.
2. Distinguish between the simple and weighted arithmetic means.
3. Compare the relative advantages of mean, median and mode.
4. Find the arithmetic mean, geometric mean and the harmonic mean of the following unclassified data 4, 128, 16, 8.
5. If $M+x_1, M+x_2, \dots, M+x_n$ be n quantities whose simple arithmetic mean is M and $w_1, w_2, \dots, w_n$ be the weights of n quantities and W be their weighted arithmetic mean, show that

$$W - M = \frac{\sum w x}{\sum w}$$

6. Calculate simple and weighted average in the following distribution.

wages	3.50	5.25	2.75	8.50	0.29
weight	5	35	20	15	5

7. Calculate the arithmetic mean from the following 42 measurements of a line.

Length in cm.	860	859	858	857	856	855	854	853	852
Frequency	2	3	4	9	10	8	4	1	1

8. From the following distribution of monthly wages of a group of workmen, find arithmetic mean, mode and median.

Wages in rupees	20	21	22	23	24	25	26	27	28-29	total
Number of workmen	8	10	11	16	20	25	15	9	6	120

(C.U. B. Com. 1955)

9. Calculate arithmetic mean, mode and median from the following frequency distribution.

x	92	$91\frac{1}{2}$	84	75	$73\frac{1}{2}$	72	$70\frac{1}{2}$	75
f	4	12	13	9	8	7	12	18

10. The following table indicates the size and quantity of socks sold by a retail shop in January, 1962. Find the average size.

Size	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8
No. of socks	7	23	40	50	80	100	30

11. Find the arithmetic mean and mode from the following frequency distribution of population classified according to age and sex,

Age-group in years	20-29	30-39	40-49	50-59
Male	1500	1300	2000	1100
Female	1200	1100	1800	1000

12. The following table gives the results of certain examinations of three Universities in the year 1936. Which is the best University? Give reasons for your answer.

University Examination	Percentage results in the University		
	A	B	C
M.A.	80	75	70
M.Sc.	70	70	60
B.A.	65	80	70
B.Sc.	60	70	80
B.Com.	75	65	75

(C. U. M. A. 1937)

(Calculate median in each case with arbitrary weights.)

✓ 13. Find the arithmetic mean and the median of the following frequency distribution.

Weekly earning of the secretaries

Weekly earning	Number of Secretaries
Rs. 10·00—19·99	1363
20·00—29·99	2177
30·00—39·99	1948
40·00—49·99	579
50·00—59·99	214
60·00—69·99	69
70·00—79·99	17
80·00—89·99	5
90·00—99·99	4
100·00—109·99	1 (C. U.)

14. Find the mean, mode...for the following

Years under	10	20	30	40	50	60
No. of persons	15	32	51	78	97	109
(cumulative frequency)						

(P. C. S. 1952)

(Hints—arrange the figures in class—intervals of 10 as under
 0—10, 10—20, 20—30, 30—40 etc. with frequencies 15, 17, 19, 27, 19, 12 respectively)

Chapter V

Measures of Dispersion

32. We have already seen that masses of quantitative data when thrown into an organised form exhibit a central tendency. We have further obtained different measures of this central tendency—the averages—mean, mode and median.

But all the values of any distribution are not equally close or equally scattered from the average. Again, this scattering is not equally pronounced in all cases.

So average as a measure of central value is not enough to present a correct picture of the distribution.

We must, therefore, know by how much the different data are dispersed or spread about the average.

For example, the mean of 1, 9, 20 is 10 and that of 9, 10, 11 is also 10. But the values of the former group are far more scattered than those of the second group.

If the winter temperature of a place is as low as -20°F and the summer temperature as high as 116°F , then the mean annual temperature is 48°F , but the annual range is $116 - (-20)$ or 136°F . Whereas a place with a winter temperature of 36°F and summer temperature of 60°F will have a mean annual temperature of 48°F and an annual range of 24°F .

Thus although the mean annual temperature is the same at both places, the climate of the first is widely different from that of the second.

33. There are different measures of Dispersion.

1. *Range*
2. *Quartile Deviation or Semi-logarithm Range*
3. *Mean Deviation*
4. *Standard Deviation*

1. *Range*—The difference between the highest and lowest values in a series is called the range. The range between 96°F and -20°F is 116°F , but the range between 96°F and 20°F is 76°F .

It is the simplest measure of dispersion and its value depends on the two extreme values—the highest and the lowest. But it ignores all other intermediate values. Further, a single but inordinate value at the extremes of a distribution changes its magnitude materially. This is why it is a crude and erratic measurement.

Ex. 1. Find the range in the following cases

(i) 5, 50, 200, 1000

(ii) 2—5, 6—9, 10—13, 14—17. (grouped distribution)

(i) The range = $1000 - 5 = 995$.

(ii) The range = $17 - 2 = 15$ (difference between the higher limit of the highest class and lower limit of the lowest class).

34. Quartile Deviation (Q. D.)

If Q_1 and Q_3 are first and third quartiles, then

$$Q. D. = \frac{Q_3 - Q_1}{2}$$

It is a semi — interquartile range being measured by half the distance between Q_1 and Q_3 , i.e., between the first and third quartiles.

It is neither affected by extreme values nor by any of the values between the first and third quartiles.

It is, however, a better measure of dispersion than range.

35. Mean or Average Deviation (M. D)

It is the sum of the deviations of all the items from a central value—mean, mode or median, and taking their average. The deviations are all taken positive ignoring algebraic signs, for with proper signs the total deviation from the arithmetic mean is zero. Unless otherwise stated, by mean deviation we shall mean deviation from arithmetic mean.

The symbol $| \quad |$ is used to denote only positive value and read as modulus $d(\text{mod. } d)$.

Thus, $|-2| = 2$, $|2| = 2$, $|5-2| = |3| = 3$,
 $|2-5| = |-3| = 3$, etc.

Mean Deviation is obtained from the following formula

$$\text{M. D.} = \frac{\sum |d|}{n}$$

If the median is taken as the average, the sum of the deviations, taken all positive, is a minimum. Mathematically, it is unsound to ignore the negative signs and taking them all positive instead.

Ex. 2. Find the mean deviation of

4, 14, 39, 15, 18

The arithmetic mean $= \frac{\sum x}{n} = \frac{4+14+39+15+18}{5} = 18$

The deviations of the numbers from 18 are
 $|4-18|$, $|14-18|$, $|39-18|$, $|15-18|$, $|18-18|$
 or 14, 4, 21, 3, 0.

$$\therefore \text{M. D. from mean} = \frac{\sum |d|}{n} = \frac{42}{5} = 8.4$$

Again, arranging the data in ascending or descending order, the median = 15.

The deviations of the numbers from 15 are
 $|4-15|$, $|14-15|$, $|39-15|$, $|15-15|$, $|18-15|$
 or 11, 1, 24, 0, 3.

$$\therefore \text{M. D. from median} = \frac{\sum |d|}{n} = \frac{39}{5} = 7.8$$

36. If the data are given in a frequency distribution, the mean deviation is given by the expression

$$\text{M. D.} = \frac{\sum f |x-a|}{n} = \frac{\sum f |d|}{n}$$

Ex. 3. Calculate the mean deviation from the following frequency distribution.

x	4	8	13	20	23	25	30	Total
f	3	12	18	28	25	17	9	112

x	f	fx	Deviation from mean $ x-19.4 $	$f x-a $	Deviation from median $ x-20 $	$f x-a $
4	3	12	15.4	46.2	16	48
8	12	96	11.4	136.8	12	144
13	18	234	6.4	115.2	7	126
20	28	560	6	16.8	0	0
23	25	575	3.6	90.0	3	75
25	17	425	5.6	95.2	5	85
30	9	270	10.6	95.4	10	90
112	2172			595.6		568

$$\text{Arithmetic mean} = \frac{\sum fx}{\sum f} = \frac{2172}{112} = 19.4$$

$$\text{M. D. from mean} = \frac{\sum f|x-a|}{n} = \frac{595.6}{112} = 5.32$$

Again, median = the average value of 56th. and 57th. items, each of which is 20. $\therefore$ median = 20.

$$\text{M. D. from median} = \frac{\sum f|x-a|}{n} = \frac{568}{112} = 5.07$$

If the data are given in a frequency distribution, deviations are measured from the central value and multiplied by class-frequencies.

37. Standard Deviation (σ)

It is the most widely used measure of variation. It is the square root of the mean of the squared deviations from the arithmetic mean. By squared deviation we mean the sum of the squares of the deviations. The deviations are measured from the arithmetic mean. It is popularly known as root-mean-square deviation.

The standard deviation is usually denoted by the Greek letter σ (sigma). If A is the arithmetic mean of n numbers $x_1, x_2, \dots, x_n$, then

$$\sigma^2 = \frac{1}{n} \sum (x-a)^2$$

$$\begin{aligned}
&= \frac{1}{n} \{ (x_1 - A)^2 + (x_2 - A)^2 + \dots + (x_n - A)^2 \} \\
&= \frac{1}{n} \{ \Sigma x^2 - 2A \Sigma x + nA^2 \} \\
&= \frac{1}{n} \{ \Sigma x^2 - 2A \cdot n \cdot A + nA^2 \} \quad \left[A = \frac{\Sigma x}{n} \right] \\
&= \frac{1}{n} \{ \Sigma x^2 - nA^2 \} \\
&= \frac{\Sigma x^2}{n} - A^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n} \right)^2
\end{aligned}$$

Variance (σ^2) which is the square of the standard deviation is called *Variance* of the distribution.

38. Grouped Frequency

Let $f_1, f_2, \dots, f_n$ be the frequencies of the quantities $x_1, x_2, \dots, x_n$

$$\begin{aligned}
\text{then } \sigma^2 &= \frac{\Sigma f d^2}{n} \\
&= \frac{\Sigma f (x - A)^2}{n} \\
&= \frac{f_1(x_1 - A)^2 + f_2(x_2 - A)^2 + \dots + f_n(x_n - A)^2}{n} \\
&= \frac{\Sigma f x^2 - 2A \Sigma f x + nA^2}{n} \\
&= \frac{\Sigma f x^2 - 2A \cdot n \cdot A + nA^2}{n} \quad \left[A = \frac{\Sigma f x}{n} \right] \\
&= \frac{\Sigma f x^2}{n} - A^2 \\
&= \frac{\Sigma f x^2}{n} - \left(\frac{\Sigma f x}{n} \right)^2
\end{aligned}$$

Thus the standard deviation is given by

$$\sigma = \sqrt{\frac{\Sigma f d^2}{n}} = \sqrt{\frac{\Sigma f x^2}{n} - A^2} = \sqrt{\frac{\Sigma f x^2}{n} - \left(\frac{\Sigma f x}{n} \right)^2}$$

for ungrouped distribution.

* It can be shown that if 'a' be any arbitrary mean and $d_1, d_2, \dots, d_n$ be the deviations of $x_1, x_2, \dots, x_n$ from 'a', then

$$\sigma = \sqrt{\frac{\sum fd^2}{n}} = \sqrt{\frac{\sum fx^2}{n} - A^2} = \sqrt{\frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n}\right)^2}$$

for frequency distribution.

The standard deviation is affected by every value of the distribution. But it is less affected by change of sampling than any other measure of deviation. It is suitable for algebraic treatment.

39. Co-efficient of Variation developed by Pearson is the standard deviation expressed as a percentage of arithmetic mean. It is represented by capital letter V.

Let V = coefficient of variation

σ = standard deviation

A = arithmetic mean

$$\text{Then } V = \frac{\sigma}{A} \times 100$$

Ex. 4. Find the standard deviation and coefficient of variation for the following ungrouped distribution.

5, 8, 3, 15, 14

$$\sigma = \sqrt{\frac{\sum d^2}{n}}$$

or

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

Serial No	value x	Deviation from mean d	Square of deviation d^2
1	5	-4	16
2	8	-1	1
3	3	-6	36
4	15	6	36
5	14	5	25
			114

Serial No	x	x^2
1	5	25
2	8	64
3	3	9
4	15	225
5	14	196
		45
		519

The arithmetic mean

$$= \frac{5+8+3+15+14}{5} = 9.$$

$$\sigma = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{114}{5}}$$

$$= \sqrt{22.8}$$

$$= 4.78$$

$$V = \frac{\sigma}{A} \times 100 = \frac{4.78}{9} \times 100$$

$$= 531.1$$

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$= \sqrt{\frac{519}{5} - \left(\frac{45}{5}\right)^2}$$

$$= \sqrt{103.8 - 81}$$

$$= \sqrt{22.8} = 4.78$$

Ex 5. Calculate mean deviation and standard deviation from the following series

20, 85, 120, 60, 40

Item	x	deviation $ d = x - A$	d^2	deviation $d' = x - 60$	d'^2
1	20	45	2025	-40	1600
2	85	20	400	25	625
3	120	55	3025	60	3600
4	60	5	25	0	0
5	40	25	625	-20	400
$n=5$	$\Sigma x = 325$	$\Sigma d = 150$	$\Sigma d^2 = 6100$	$\Sigma d' = -25$	$\Sigma d'^2 = 6225$

By arithmetic mean

$$A = \frac{\Sigma x}{n} = \frac{325}{5} = 65.$$

$$\text{Mean deviation (M.D)} = \frac{\Sigma |d|}{n} = \frac{150}{5} = 30.$$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d^2}{n}}$$

$$= \sqrt{\frac{6100}{5}} = \sqrt{1220} = 34.9$$

By assumed mean

Let the assumed mean = 60

$$\begin{aligned}\text{Standard deviation } \sigma &= \sqrt{\frac{\sum d'^2}{n} - \left(\frac{\sum d'}{n}\right)^2} \\ &= \sqrt{\frac{6225}{5} - \left(\frac{-25}{5}\right)^2} \\ &= \sqrt{1245 - 25} = \sqrt{1220} = 34.9.\end{aligned}$$

Ex. 6. Calculate the standard deviation from the following frequency distribution.

x	2	3	4	5	6	7	8	9	Total
f	7	13	28	40	22	15	10	1	136

$$\sigma = \sqrt{\frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n}\right)^2}$$

x	f	fx	fx^2
2	7	14	28
3	13	39	117
4	28	112	448
5	40	200	1000
6	22	132	792
7	15	105	735
8	10	80	640
9	1	9	81
Total	136	691	3841

$$\sigma = \sqrt{\frac{3841}{136} - \left(\frac{691}{136}\right)^2} = \sqrt{28.24 - 25.81} = \sqrt{2.43} = 1.56$$

Ex. 7 Calculate the standard deviation for the following classified frequency distribution.

Height in inches x	40-44	45-49	50-54	55-59	60-64	Total
Number of students f	8	25	45	30	12	120

In calculating the standard deviation or variance the actual value of the arithmetic mean need not be known. Any arbitrary mean will serve the purpose.

$$\sigma = \sqrt{\frac{\sum fd^2}{n} - \left(\frac{\sum fd}{n}\right)^2}$$

Mid-point (inches) x	Frequency f	Deviation from arbitrary mean d	fd	fd^2
42	8	-2	-16	32
47	25	-1	-25	25
52	45	0	0	0
57	30	+1	30	30
62	12	+2	24	48
Total	120		13	135

Let the arbitrary mean or origin = 52 which is the midpoint of the third class-interval. Deviations are expressed in class-intervals as units.

$$\begin{aligned}\sigma &= \sqrt{\frac{135}{120} - \left(\frac{13}{120}\right)^2} = \sqrt{1.125 - .012} \\ &= \sqrt{1.113} = 1.055 \text{ in units of class-intervals} \\ &= 1.055 \times 5 = 5.275 \text{ inches.}\end{aligned}$$

40. Root-Mean-Square Deviation—relation with standard deviation.

Let $x_1, x_2, \dots, x_n$ be n quantities and 'a' be any quantity.

Let $d_1, d_2, \dots, d_n$ be their deviations from 'a' so that

$$x_1 - a = d_1, \quad x_2 - a = d_2, \quad \dots, \quad x_n - a = d_n.$$

Then $\delta = \sqrt{\frac{\sum d^2}{n}}$ is called the root-mean-square deviation of $x_1, x_2, \dots, x_n$ about 'a'.

Let A be the arithmetic mean of $x_1, x_2, \dots, x_n$, and $A - a = c$.

Then $\sigma^2 = \frac{\Sigma(x-A)^2}{n}$ and

$$\begin{aligned}
 s^2 &= \frac{\Sigma d^2}{n} = \frac{\Sigma(x-a)^2}{n} = \frac{\Sigma(x-A+A-a)^2}{n} = \frac{\Sigma(x-A+c)^2}{n} \\
 &= \frac{\Sigma(x-A)^2 + 2c \Sigma(x-A) + n.c^2}{n} \\
 &= \frac{\Sigma(x-A)^2}{n} + 0 + c^2 = \sigma^2 + c^2, \text{ since } \Sigma(x-A) = 0 \\
 \therefore s^2 &= \sigma^2 + c^2
 \end{aligned}$$

Exercise 3

1. What is dispersion and how is it measured?
2. Define mean deviation and standard deviation and explain how they are calculated.
3. What is coefficient of dispersion?
4. Calculate the mean deviation from the arithmetic mean for the following 150, 210, 180, 200
5. Calculate the mean deviation from the median of the following series of values 1, 2, 3, 4, 5, 6, 7, 8, 9
6. Calculate the deviation from the arithmetic mean and median for the following values.
68, 84, 96, 139, 24, 27, 13, 85
7. Find the mean deviation from the arithmetic mean and the median for the following frequency distribution.

Weights in lbs.	1	4	6	8	9
Frequency	10	15	20	25	10
8. Find the mean deviation and standard deviation of 100 estimates of the length of a rod 8 cm. long.

Length in cm.	7.8	7.9	8	8.1	8.2
Frequency	5	10	40	20	25
9. Calculate standard deviation and coefficient of deviation for the following distribution.

Yield per acre in mds.	0	4	5	6	8
Frequency	20	30	60	80	10

10. Calculate standard deviation and coefficient of variation from the following frequency distribution.

Number of persons in the family	1	2	3	4	5	6
Number of families	6	10	25	40	30	9

11. Calculate the average size and standard deviation from the following distribution.

Size of sample	0	1	2	3	4	5	6
Frequency	1	6	15	20	15	6	1

12.

Wages in Rs.	No. of workers	Wages in Rs.	No. of workers
20—21	4	25—26	40
21—22	8	26—27	30
22—23	14	27—28	16
23—24	20	28—29	6
24—25	28	29—30	2

Calculate mean deviation from the above frequency distribution.

13. Calculate standard deviation from the following table giving the age distribution of 542 members of the House of Commons.
(B. Com. Nag. 1942)

Age	20	30	40	50	60	70	80
No. of members	3	61	132	153	140	51	2

14. From the following figures, find the standard deviation and coefficient of variation.

Marks	No. of persons	Marks	No. of persons
0—10	5	40—50	30
10—20	10	50—60	20
20—30	20	60—70	10
30—40	40	70—80	4

(B. Com. Agra. 1942)

15. Find the mean deviation and standard deviation for the following

Years under	10	20	30	40	50	60
-------------	----	----	----	----	----	----

No. of persons	15	32	51	78	97	109
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(cumulative

frequency)

(P.C.S. 1952)

Chapter VI

Index Numbers

41. Definition. Index numbers are numbers used as devices for measuring changes in prices or quantities of commodities from time to time or from place to place. These commodities might be either produced, stocked, marketed sold or consumed. The index numbers may also relate to wages or any variable factor.

When comparison is made between different quantities or volumes of commodities, the index number is called *Index Number of quantities*; but when comparison is made between different prices, it is called *Index Number of prices*.

Each of the many commodities is subject to various forces and its quantity or price moves in its own way. The traditional purpose of index numbers is to find an average to measure the net resultant of their price or volume movements so as to discover their causes and effects and to take measures to control them.

42. Base year—For purpose of comparison it is necessary to fix some period of time as a base. The base may be any length of time—a month, a half year, a year etc. Usually the base is a year. The base year should be a normal year—i.e., a year of normal prices or production—which are neither too low nor too high. Sometimes the base is taken as the average of two or more years.

Index numbers are generally expressed as percentages of the base period which is set equal to 100.

Sometimes the basis of comparison *i.e.*, the base year is fixed, sometimes variable.

43. In the Fixed Base Method prices or quantities of any year are expressed as percentages of a fixed year. But in the **Chain Base Method** they are expressed as percentages of the preceding year.

The advantages of the chain index are that unnecessary commodities may be excluded, new ones may be included and also the weights may be changed according to their relative importance.

Ex. 1.

TABLE I

Construction of Index Numbers by Fixed Base Method and Chain Base Method.

Year	Price of rice in Rs per md.	Fixed Base Index	Chain Base Index
1958	10	100	100
1959	14	140	140
1960	12	120	85.7
1961	20	200	166.7

Fixed Base Method—The year 1958 is taken as base-year. It represents 100. The index numbers of different years are expressed as percentages of 1958 number.

$$\text{Index (1959)} = \frac{14}{10} \times 100 = 140$$

$$\text{,, (1960)} = \frac{12}{10} \times 100 = 120$$

$$\text{,, (1961)} = \frac{20}{10} \times 100 = 200.$$

Chain Base Method

$$\text{Index (1959)} = \frac{14}{10} \times 100 = 140 \quad (\text{with 1958 as base})$$

$$\text{Index (1960)} = \frac{12}{14} \times 100 = 85.7 \quad (\text{with 1959 as base})$$

$$,, \quad (1961) = \frac{20}{12} \times 100 = 166.7 \quad (\text{with 1960 as base})$$

Changing Base : (a) chain base index numbers from fixed base index numbers.

$$\text{Chain Base Index (1959)} = \frac{140}{100} \times 100 = 140$$

$$,, \quad (1960) = \frac{120}{140} \times 100 = 85.7$$

$$,, \quad (1961) = \frac{200}{120} \times 100 = 166.7$$

(b) fixed base index numbers from chain base index numbers.

$$\text{Fixed Base Index (1959)} = \frac{100}{100} \times 140 = 140$$

$$,, \quad (1960) = \frac{140}{100} \times 85.7 = 119.98 = 120 \quad (\text{app})$$

$$,, \quad (1961) = \frac{120}{100} \times 166.7 = 200.04 = 200 \quad (\text{app})$$

44. Methods of Construction

There are two principal methods of construction of index numbers.

(1) method of price aggregate

(2) method of price relative.

Aggregate may, however, be *simple or weighted*.

45. Simple Aggregate Method

Let the subscript 0 denote the base year and the subscript 1 denote the current year. Let p refer to price and q its quantity.

then p_0 = base year price,

q_0 = base year quantity,

p_1 = current year price,

q_1 = current year quantity,

Let p_0', p_0''denote the base year prices and p_1', p_1'' denote the current year prices of commodities

P_{01} = price index for the current year

$$\begin{aligned}\text{Then } P_{01} &= \frac{\text{total current year price}}{\text{total base year price}} \times 100 \\ &= \frac{p_1' + p_1'' + \dots}{p_0' + p_0'' + \dots} \times 100 = \frac{\sum p_1}{\sum p_0} \times 100\end{aligned}$$

The above method, however, ignores the importance of different commodities.

46. Weighted Aggregate Method—In order that each commodity may have its due influence on the index number, definite quantities of some selected group of commodities are taken. Each commodity price is multiplied by the respective quantity and the products are then totaled. The total for the current year is then expressed as a percentage of the total for the base year.

$$M = \frac{p_1' q' + p_1'' q'' + \dots}{p_0' q' + p_0'' q'' + \dots} \times 100 = \frac{\sum p_1 q}{\sum p_0 q} \times 100$$

The weights employed in any current year are the respective quantities of the selected group of commodities for a particular year. The weights need not be exact. According to Fisher they may be rounded to 1, 10, 100 or 1000. *The above index measures the changing value of a fixed quantity of goods.*

There are different systems of weighting.

Laspeyres Formula

The weights or multipliers used here are the base year quantities.

$$L = \frac{p_1' q_0 + p_1'' q_0'' + \dots}{p_0' q_0 + p_0'' q_0'' + \dots} \times 100 = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100.$$

Laspeyres method is easy to apply, for it requires only the base year quantities as weights.

Paasche Formula

The weights or multipliers used are the current year quantities

$$P = \frac{p_1' q_1' + p_1'' q_1'' + \dots}{p_0' q_1' + p_0'' q_1'' + \dots} \times 100 = \frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100.$$

This method requires a new set of weights for each year. The weights or quantities are changed with each successive year. Such changes are neither feasible nor desirable.

Marshall-Edgeworth Formula

The weights are the total or average quantities of the base and current years

$$E = \frac{(q_0' + q_1') p_1' + (q_0'' + q_1'') p_1'' + \dots}{(q_0' + q_1') p_0' + (q_0'' + q_1'') p_0'' + \dots} \times 100$$

$$= \frac{\sum (q_0 + q_1) p_1}{\sum (q_0 + q_1) p_0} \times 100$$

It is a sort of compromise between the Laspeyres and Paasche formula.

Prof. Fisher has adopted an ideal formula, simple and accurate. His ideal index is a geometric mean of the aggregate index numbers of Laspeyres and Paasche.

Fisher's Ideal Formula

$$I = \sqrt{\text{Laspeyres index} \times \text{Paasche index}}$$

$$= \sqrt{\left(\frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100 \right) \times \left(\frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100 \right)}$$

$$= 100 \times \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}}$$

Walsh has adopted a formula which is the arithmetic mean of the two aggregate types of Laspeyres and Paasche formulas.

Walsh formula

$$W = \frac{1}{2} \left(\frac{\sum p_1 q_0}{\sum p_0 q_0} + \frac{\sum p_1 q_1}{\sum p_0 q_1} \right) \times 100$$

Ex. 2.

TABLE 2

Construction of Index Numbers by Aggregate
Method for 1960

Commodity	1959		1960		$p_0 q_0$	$p_1 q_0$	$p_0 q_1$	$p_1 q_1$
	Price (p_0)	Quantity (q_0)	Price (p_1)	Quantity (q_1)				
A	8	12	10	8	96	120	64	80
B	4	18	7	15	72	126	60	105
C	9	8	8	10	72	64	90	80
	Σp_0		Σp_1		$\Sigma p_0 q_0$	$\Sigma p_1 q_0$	$\Sigma p_0 q_1$	$\Sigma p_1 q_1$
	=21		=25		=240	=310	=214	=265

Set 1959=100

$$\text{Simple Aggregate Price Index, } P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$\text{Index (1960)} = \frac{25}{21} \times 100 = 119$$

$$\text{Laspeyres formula, } L = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$\text{Index (1960)} = \frac{310}{240} \times 100 = 129.2$$

$$\text{Paasche formula } P = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$\text{Index (1960)} = \frac{265}{214} \times 100 = 123.8$$

$$\text{Marshall-Edgeworth formula, } E = \frac{\Sigma (q_0 + q_1) p_1}{\Sigma (q_0 + q_1) p_0} \times 100$$

$$\text{Index (1960)} = \frac{310 + 265}{240 + 214} \times 100 = 126.7$$

$$\text{Fisher's Ideal formula, } I = 100 \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} \times \frac{\sum p_1 q_1}{\sum p_0 q_1}}$$

$$\begin{aligned}\text{Index (1960)} &= 100 \sqrt{\frac{310}{240} \times \frac{265}{214}} \\ &= 100 \sqrt{1.5995} \\ &= 100 \times 1.26 = 126\end{aligned}$$

$$\text{Walsh formula, } W = \frac{1}{2} \left(\frac{\sum p_1 q_0}{\sum p_0 q_0} + \frac{\sum p_1 q_1}{\sum p_0 q_1} \right) \times 100$$

$$\text{Index (1960)} = \frac{1}{2} (129.2 + 123.8) = 126.5$$

47. Method of Price Relatives--If the commodity prices of any year are expressed as percentages of the base year prices, then they are called price relatives. So price relatives are but percentages and have no units. To put it in a different way, the ratios of current year prices of commodities to their respective base year prices multiplied by 100 are called price relatives.

The price relatives for each year are then averaged to give index numbers for that year.

The types of averages used are

1. Arithmetic Mean of relatives, simple and weighted.
2. Geometric Mean of relatives, simple and weighted.
3. Harmonic Mean of relatives, simple and weighted.
4. Mode.
5. Median.

The above distinction between six fundamental types of index number viz the price aggregate, the arithmetic, geometric, harmonic, mode and median are due to Prof. Fisher.

If, as before, p_0 and p_1 denote the base year and current year prices of a commodity, then $\frac{p_1}{p_0} \times 100$ is the current year price relative of the commodity.

Arithmetic Mean of Price Relatives of n quantities

$$P_{01} = \frac{\frac{p_1'}{p_0'} \times 100 + \frac{p_1''}{p_0''} \times 100 + \dots}{n} = \frac{\sum \frac{p_1}{p_0} \times 100}{n} = \frac{100}{n} \sum \frac{p_1}{p_0}$$

Weighted Arithmetic Mean of Relatives

Let w' , w'' , be the respective weights of the current year relatives, then

$$I = \frac{\frac{p_1'}{p_0'} \times 100 \times w' + \frac{p_1''}{p_0''} \times 100 \times w'' + \dots}{w' + w'' + \dots} = 100 \cdot \frac{\sum w \cdot \frac{p_1}{p_0}}{\sum w}$$

If the weightage be the base year gross values ($p_0 q_0$), this arithmetic mean of relatives become

$$100 \cdot \frac{\sum p_0 q_0 \cdot \frac{p_1}{p_0}}{\sum p_0 q_0} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100,$$

which is the same as Laspeyres Formula.

Geometric Mean of Relatives

$$G = \left\{ \left(\frac{p_1'}{p_0'} \times 100 \right) \times \left(\frac{p_1''}{p_0''} \times 100 \right) \times \dots \right\}^{\frac{1}{n}}$$

$$= 100 \left(\frac{p_1' \cdot p_1'' \cdot \dots}{p_0' \cdot p_0'' \cdot \dots} \right)^{\frac{1}{n}}$$

Taking logarithm,

$$\log G = \log 100 + \frac{1}{n} \left\{ \log \frac{p_1'}{p_0'} + \log \frac{p_1''}{p_0''} + \dots \right\}$$

$$= 2 + \frac{1}{n} \sum \log \frac{p_1}{p_0} = \frac{1}{n} \sum \log \frac{100 p_1}{p_0}$$

The antilog gives the index number.

Harmonic Mean of Relatives

$$H = \frac{1}{\frac{p_0'}{p_1'} \times 100 + \frac{p_0''}{p_1''} \times 100 + \dots} = \frac{1}{100n} \sum \frac{p_0}{p_1}$$

Evidently harmonic mean of relatives with current year prices $\times$ current year quantities i.e., current year values $p_1 q_1$ is given by

$$H = \frac{100 \cdot \Sigma p_1 q_1}{\Sigma \frac{p_0}{p_1} \times p_1 q_1} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100, \text{ which the same as Paasche}$$

Formula.

Ex. 3. From Table 2 find for 1960 the averages of relative prices.

Arithmetic Mean of Relatives

$$P_m (1960) = \frac{100}{n} \Sigma \frac{p_1}{p_0} = \frac{100}{3} \left(\frac{10}{8} + \frac{7}{4} + \frac{8}{9} \right) = \frac{100}{3} \cdot \frac{280}{72} = 129.6$$

Geometric Mean of Relatives

$$\begin{aligned} \log G &= 2 + \frac{1}{n} \Sigma \frac{p_1}{p_0} \\ &= 2 + \frac{1}{3} \{ \log \frac{10}{8} + \log \frac{7}{4} + \log \frac{8}{9} \} \\ &= 2 + \frac{1}{3} \{ \log 1.25 + \log 1.75 + \log .89 \} \\ &= 2 + \frac{1}{3} \{ .0969 + .2430 + \bar{1}.9494 \} \\ &= 2 + \frac{1}{3} (1.2893 - 1) \\ &= 2 + .0964 = \log 124.8 \end{aligned}$$

$$\therefore G = 124.8$$

Harmonic Mean of Relatives

$$H = \frac{3 \times 100}{\frac{8}{10} + \frac{4}{7} + \frac{9}{8}} = \frac{300}{\frac{880}{880}} = 120.2$$

Median of Relatives

The price relatives for 1960 are

$$\frac{10}{8} = 100 \times 125, \quad \frac{7}{4} \times 100 = 175 \quad \text{and} \quad \frac{8}{9} \times 100 = 88.9$$

When arranged in order of magnitude they are 88.9, 125, 175 and the median is 125. This is the index number for 1960.

Table showing the averages of relatives just obtained.

Year	Arithmetic mean of relatives	Geometric mean of relatives	Harmonic mean of relatives	Median of relatives
1959	100	100	100	100
1960	129.6	124.8	120.2	125

48. Principal Index Numbers of Production and Prices

In the construction of index numbers several difficulties are faced. These are

1. Ascertaining the purpose for the construction of index numbers.
2. Collection of prices prevailing in the market.
3. Selection of base.
4. Selection of representative commodities of the consumption pattern.
5. Selection of weightage.

Before collecting data for making index numbers, we must ascertain the specific purpose for which they are required. The usual purpose is to measure the general price levels. If we wish to prepare an index of cost of living of any particular class of people, we must use the retail prices of commodities in which they are interested. The retail price index requires only retail prices. Wholesale prices by hales, tons or grosses will be quite useless and misleading.

The commodities selected for index numbers must be *representative*. For this, the population from which the data are collected is divided into groups and subgroups. Each subgroup contains several items. For obtaining sufficiently accurate index numbers, a large number of items should be taken. A large sample is less effected even by abnormal price changes. The Consumer Price Index compiled by the United States Bureau of Statistics includes as many as 300 commodities and services.

For sources of prices we must look to periodical market bulletins from various sources or get price quotations from local shops or markets.

49. Index Numbers of Production and Prices

The consumer price index. This index formerly known as cost of Living Index is compiled by Government of India for different places in India. It relates to working class families. The retail prices of a selected pocket of goods or services are divided into 5 major groups (1) Food (2) Fuel and Light (3) Clothing (4) House rent and (5) Miscellaneous. Each of the major groups is divided into subgroups. Each subgroup contains several prices.

Separate indices are constructed for each group with weights. These group indices are again weighted. The weights are respectively the percentages of total expenditure on them. The index numbers are prepared monthly. From monthly index numbers annual indices are constructed.

50. Index Numbers of Agricultural Production

This index number is prepared by Govt. of India.

The crops are divided into 2 major groups and 6 subgroups with weights as follows :—

- | | |
|----------------------------|--------------------------------|
| 1. Food grains (66·9) | (a) Cereals (58·3) |
| | (b) Pulses (8·6) |
| 2. Non-Foodgrains (33·1) | (a) Oilseeds (9·9) |
| | (b) Fibres (4·5) |
| | (c) Plantation crops (3·6) |
| | (d) Miscellaneous (15·1) |

The index includes 28 crops. The year 1950 is taken as base. The weights for different subgroups are the percentages of total production in the base year. The index numbers are made by weighted arithmetic mean.

51. Index Numbers for Calcutta and Industrial area

The State Statistical Bureau of West Bengal prepares the index number for the city of Calcutta and Industrial

area. It covers as many as 487 items of commodities and services, divided into 5 major groups and 80 subgroups.

The major groups are (1) Food (71.28) (2) Fuel and Light (2.89) (3) Clothing (9.27) (4) House rent (6.69) and (5) Miscellaneous (9.87).

The figures in brackets indicate the weights.

The above index number is also prepared by Govt. of India.

52. Index Numbers of Industrial Production

There is a monthly official publication showing the general trend of industrial production in India. It is constructed by taking weighted arithmetic mean of production relatives for each item of a representative sample. To get the production relative of any year the ratio of the production in that year to production of the base period is taken and expressed as percentage. This method is independent of the units such as maunds, gallons, etc. taken for different items. The weights base year is 1951.

Ex. 4. From the following group index numbers and group weights, construct the cost of living index

Group (1)	Index No (2)	Weight (3)
Food	428	45
Fuel and Light	200	8
Clothing	240	15
House rent	125	20
Miscellaneous	170	12

$$\text{Reqd. index number} = \frac{\sum (2) \times (3)}{\sum (3)}$$

$$= \frac{428 \times 45 + 200 \times 8 + 240 \times 15 + 125 \times 20 + 170 \times 12}{45 + 8 + 15 + 20 + 12}$$

$$\begin{aligned}
 &= \frac{19260 + 1600 + 3600 + 2500 + 2040}{100} \\
 &= \frac{29000}{100} = 290
 \end{aligned}$$

Ex. 5. Construct the index numbers for the year 1960 on the basis of the year 1937 of the following :—

	Rice		Wheat		Barley	
year	price	quantity	price	quantity	price	quantity
1937	8	40	7	60	15	20
1960	20	75	12	85	18	32

Arrange the data as below :—

Commodity	Base year (1937)		Current Year (1960)		$p_0 q_0$	$p_1 q_1$	$p_0 q_1$	$p_1 q_0$
	Price (p_0)	Quantity (q_0)	Price (p_1)	Quantity (q_1)				
Rice	8	40	20	75	320	1500	600	800
Wheat	7	60	12	85	42	1020	595	720
Barley	15	20	18	32	300	576	480	360
Total					$\Sigma p_0 q_0 = 1040$	$\Sigma p_1 q_1 = 3096$	$\Sigma p_0 q_1 = 1675$	$\Sigma p_1 q_0 = 1880$

Index Number for 1960 by Fisher's Ideal Formula

$$\begin{aligned}
 &= 100 \times \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \\
 &= 100 \times \sqrt{\frac{1880}{1040} \times \frac{3096}{1675}} \\
 &= 100 \sqrt{1.81 \times 1.848} \\
 &= 100 \times 2.95 = 295.
 \end{aligned}$$

Exercise 4.

1. Explain what you mean by Index Numbers. What are their uses ?
2. Explain Fixed Base Method and Chain Base Method of construction of index numbers.
3. What are the two methods for construction of index numbers ? Explain the difference between them.

4. (a) What is the difference between Laspeyres Method and Paasche Method? How is a compromise between them achieved?

(b) What is Fisher's Ideal Formula?

5. What is a price relative? Give examples.

6. Name the principal Index Numbers of Production and Prices as computed by the Govt. of India and explain the construction of any one of them.

7. Find the simple index numbers to compare the annual changes in prices of rice at a given place with 1957 as base year.

Year	1957	1958	1959	1960	1961
Rice					
per md.	Rs.20	Rs.25	Rs.28	Rs.24	Rs.22

8. Construct index numbers by Fixed Base Method with 1948 as base from the following table showing the acreage under jute in the Indian Union.

year	1948	1949	1950	1951
acreage				
in '000	834	1158	1454	1851

9. From the above construct index numbers by Chain Base Method.

10. Find the price-relatives for the years 1960 and 1961 with 1959 as base from the following price of a commodity

year	1959	1960	1961
Price per ton in Rs.	325	390	420

11. Compute weighted arithmetic averages of relative prices from the following table.

Commodity	1959		1960	
	relative price	weight	relative price	weight
Cotton	100	740	90.8	740
Sugar	110	890	123.4	890
Rice	120	2040	135	2040

$$\left[\begin{aligned} \text{weighted arithmetic mean (1959)} &= \frac{\sum \text{price relative} \times \text{weight}}{\sum \text{weight}} \\ &= 113.5 \\ \text{,, ,, ,, (1960)} &= 123.3 \end{aligned} \right]$$

12. Construct Fisher's Ideal Index for the year 1961 on basis of 1948

Year	article A		article B		article C	
	price	quantity	price	quantity	price	quantity
1948	2	20	3	14	9	2
1961	4	15	6	10	8	5

13. Compute the index number of cost of living from the following data :—

Group	Index No.	Weights
Food	352	48
Fuel and Lighting	220	10
Clothing	230	8
Rent	160	12
Miscellaneous	190	15

(C.U.B. Com. 1955)

14. From the following table compute simple aggregate price index, Laspeyres index, Paasche index, Marshall-Edgeworth index and Fisher's ideal index.

Commodity	1959		1960	
	p_0	q_0	p_1	q_1
Rice	12	5	14	6
Wheat	10	6	12	4
Sugar	30	4	40	5

15. Given the following data, what index numbers would you use for purpose of comparison?

Year	Rice		Wheat		Jowar	
	price	quantity	price	quantity	price	quantity
1927	9.3	100	6.4	11	5.1	5
1934	4.5	90	3.7	10	2.7	3

(C.U.M.A. 1937)

16. Calculate suitable indices to indicate the rise in the demand liabilities of the scheduled banks in India, and in the note circulation in the country, taking the year 1939 as the basic year

At the end of	Crores of rupees	
	Demand Liabilities	Notes in Circulation
1939	139	169
1940	166	217
1941	213	268
1942	325	493
1943	500	760
1944	616	941
1945	672	1,142
1946	735	1,225

(G.C. 1948)

Chapter VII

Graphical Representation of Quantitative data

Graphic Method—We have already seen how statistical data can be presented by means of text and table. But graphs and charts can more easily bring out the chief characteristics of a statistical distribution. They present a clear and quick picture of the whole and are easier to grasp.

But charts and graphs have their limitations. They present only a limited amount of information. Then again the information is rather approximate than exact. For it is not possible to represent exact numerical figures on a graph or chart.

Yet graphical presentation is especially suitable when comparison is desired.

There are several forms of graphical presentation.

(1) one-dimensional : line charts, or line diagrams, or graphs

- (2) two—dimensional : bar-charts, circles.
- (3) three—dimensional : pictures.

53. Line diagrams.

For the purpose of presenting data by lines, we draw a pair of perpendicular straight lines XOX' and YOY' intersecting at O . The lines XOX' , YOY' are called the x -axis and y -axis and O is called the origin.

Let P be a point in the plane of the axes and PN , PM be drawn parallel to OX and OY .

Then PN is called the x - coordinate or abscissa of the point P and denoted by x , PM is called the y -coordinate or ordinate of P and denoted by y . The abscissa and ordinate are collectively called coordinates.

The system is called the Cartesian system of coordinates after the French philosopher De Cartes who developed it.

Clearly $OM=NP=x$, $ON=MP=y$. The point P is denoted by (x, y) , abscissa being always written first and ordinate next.

The axes divide the plane of the axes into 4 equal parts XOY , YOX' , $X'OY'$, $Y'OX$ which are respectively called the first, second, third and fourth quadrants.

All values of x are measured parallel to x -axis. Positive values of x are to the right of y -axis and negative values to the left of y -axis. Positive values of y are above the x -axis and negative values below it.

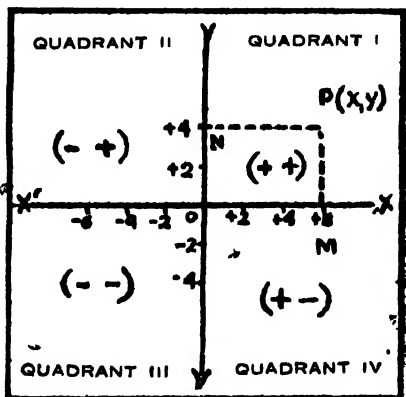


Fig. 2. Location of a Point with ref. to Rectangular Co-ordinates

The signs of x and y in the four quadrants are

Quadrant	XOY	YOX'	X'OY'	Y'OX
Signs of x and y	++	-+	--	+-

It is obvious that signs of the coordinates are opposite in opposite quadrants. Thus given any point P in the plane, we can determine its coordinates. Also conversely, if two numbers are given, we can locate the point in the plane.

The origin O has coordinates 0 and 0 and is denoted by (0,0).

For present purpose we shall usually require the first quadrant and sometimes the second and fourth.

The characteristics or variates must be measurable quantities. Generally data relating to time *i. e.*, time series are shown by line diagrams.

Periods such as years, months, weeks and days are plotted along the x -axis and other figures which vary in relation to time are shown along the y -axis.

54. Usually data are plotted at the middle of the time interval. But if data are plotted at the beginning or end of the time-interval there is merely a shifting of the curve to the left or to the right and the graph will present the same appearance.

Ex. 1. Plot the following data of the number of street accidents in a city A in the different months of 1960.

Month—	J	F	M	A	M	J	J	A	S	O	N	D
No. of accidents.	100	150	241	260	378	480	301	200	80	102	190	240

Scale

One small division along x -axis=1 month, one small division along y -axis=50 accidents.

• Coordinate Plotting

The months are plotted along the x -axis OX. and January at 1 which coincides with the origin. The figure 100 for January is shown on the y -axis according to

scale etc. The points are joined by broken lines. The monthly data are plotted at the middle of each month. If

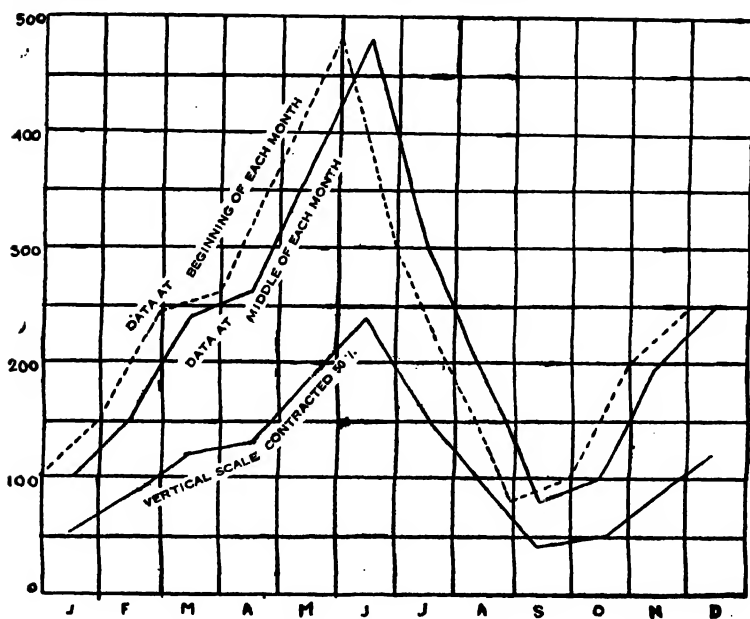


Fig. 3. Street Accidents, by months, during the year 1960

the monthly data are plotted at the beginning of each month, a horizontal shifting of the curve to the left is observed. The above graph shows the maximum incidence in June and the minimum in September (Fig. 3).

The graph gives broad impression. The figures can be plotted only approximately. Plotting of exact figures requires much larger space.

If the vertical scale is contracted the monthly variations are reduced as shown in fig 3. Likewise the vertical scale may be exaggerated giving an exaggerated impression.

Similar impressions are produced by exaggerating or contracting horizontal scale.

In the above case, months (values of x) are called independent variables, and the corresponding numbers of accidents (values of y) are dependent variables.

55. The graph as above should contain the origin 0. If the origin is not shown and only a part of the graph is exhibited, a wrong impression is produced.

Ex. 2. Show by line graph the following annual sales of Wash-Well Soap during the period 1945—1959.

Bring out, by suitable device, the relative variations in sales, cost and gross profit.

year	sales (Rs)	cost (Rs)	gross profit (Rs)
1945	30,000	17,000	13,000
1946	20,000	12,000	8,000
1947	28,000	15,000	13,000
1948	46,000	21,000	25,000
1949	50,000	24,000	26,000
1950	40,000	20,000	20,000
1951	60,000	28,000	32,000
1952	30,000	18,500	11,500
1953	48,000	23,000	25,000
1954	52,000	26,000	26,000
1955	60,000	31,000	29,000
1956	72,000	39,000	33,000
1957	68,000	40,000	28,000
1958	84,000	45,000	39,000
1959	96,000	50,000	46,000

Sales Graph

The years are laid off along the x -axis and sales along the y -axis. The scales used are the side of a small square = 2 years on the x -axis and the side of a small square = Rs 10,000 on the y -axis. Both the scales are arithmetic.

In plotting the ordinates, the origin 0 is included in the vertical scale so as to give correct impression. In plotting the abscissas, 1945 is plotted at 0, 1946 at '5 etc.

Thus the coordinates of the point representing the sales of Rs 30,000 for 1945 are 0 and 30,000; those for 1946 are .5 and 20,000 etc.

The points thus plotted represent the various annual values and these are joined by a series of straight lines.

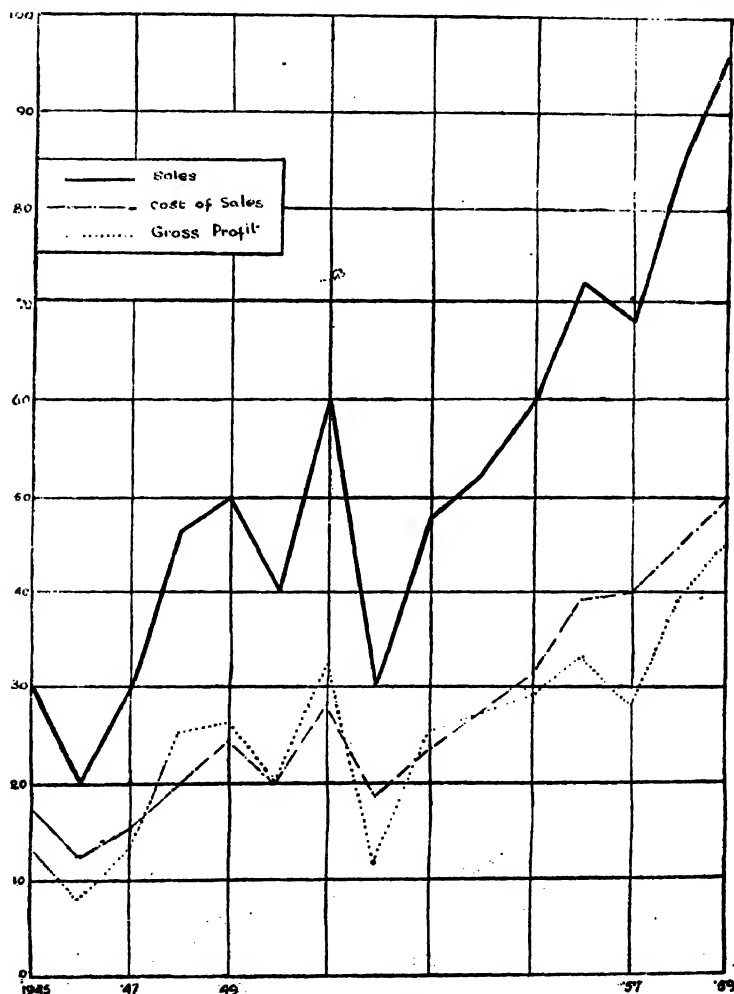


Fig. 4. Paralleling of three line graphs: Annual Sales, Cost & Gross Profits, 1945—1959.

The coordinate axes and the line graph are heavily ruled. The graph traces out annual variations in sales.

Effect of non-inclusion of Zero in the vertical scale.

Supposing that the *thick horizontal line* in fig. 4A which passes through the number 20,000 on the vertical scale is the *x*-axis, the impression conveyed by the graph is not in its true perspective.

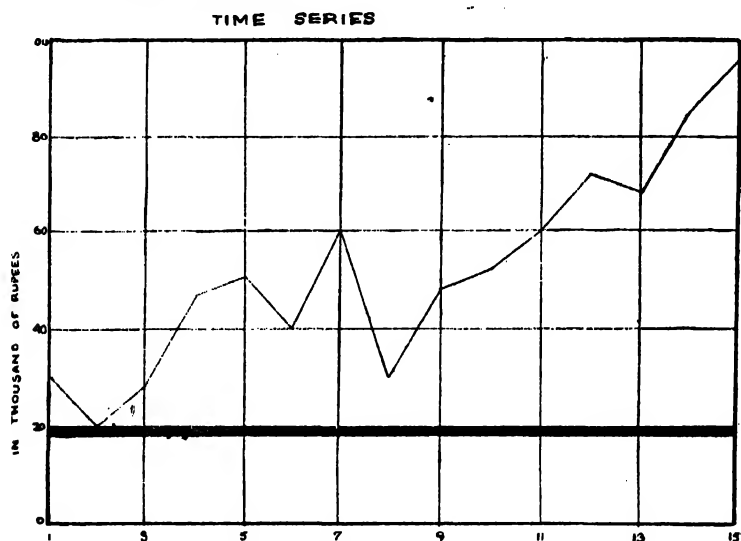


Fig. 4A. *Non-inclusion of zero in the Vertical Scale.*

The sale in 1959 appears to be about eight times that for 1945 while with the *x*-axis through the origin 0 it is just a little over thrice. The impression for 1946 is still more misleading as though the year is without any sales.

56. When several graphs are plotted on the same scale comparison of amplitudes and rhythms of change is made easier by the paralleling of graphs (Fig 4). If the three series are presented in the same chart in the arithmetic grid, a suitable scale to include the largest item (Rs 96,000) must be selected. Such a scale reduces importance of smaller items.

Ex. 3. The following table shows the wholesale price movements in India during the period 1939—1946.

	1939-40	40-41	41-42	42-43	43-44	44-45	45-46	46-47	47-48
Agricultural commodities	127.5	108.6	124.2	166.2	268.7	265.4	272.6	313.8	356.9
Manufactured articles	131.5	119.8	154.5	190.4	251.7	258.3	240.0	259.1	287.8

(Data from Reserve Bank of India, Report on Currency and finance, 1950-51, Page 147).

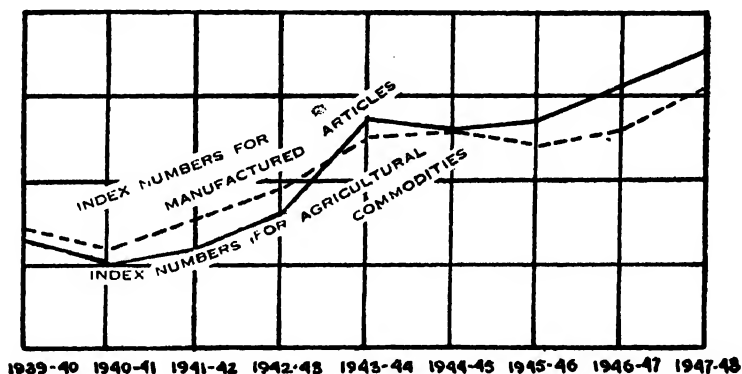


Fig 5. Price-movements of Agricultural Commodities and Manufactured Articles, base August, 1939=100.

57. Z-Chart—It is used to represent sales of a commodity. Three curves are used, one to represent (1) monthly figures as a *base curve* to ascertain figure of each month, another to represent (2) cumulative sale figures as a *diagonal curve* to ascertain the figures upto any month and the third to represent (3) the aggregate sale for one year upto any month as a *top curve*. These three curves form a letter Z.

The chart requires two vertical scales. But two scales on one chart are apt to give false impression.

Ex. 4. The sales of a certain commodity from a factory during 1959 and 1960 were as follows : (sales in '00 rupees).

	J	F	M	A	M	J	J	A	S	O	N	D
1959	—	16	10	11	14	10	13	9	13	8	12	12
1960	15	17	12	10	20	12	14	8	17	5	14	15

Represent the figures for 1960 by a Z-chart

Months	J	F	M	A	M	J	J	A	S	O	N	D
Sales (in '00 Rs)	15	17	12	10	20	12	14	8	17	5	14	15
Cumulative Sales (in '00 Rs)	15	32	44	54	74	86	100	108	125	130	144	159
One year's total including the month	143	144	146	145	151	153	154	153	157	154	156	159

Total upto Jan.=total sale from Feb. 1959 to

Jan. 1960=14300

Total upto Feb.=total sale from Mar.-1959 to

Feb. 1960=14400 etc.

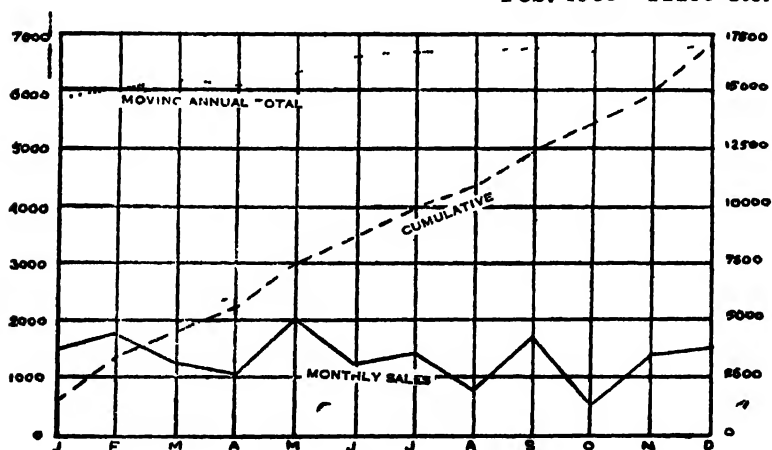


Fig. 6. *Factory Sales during 1960, by months. Cumulative and Moving Annual Totals.*

58. Range Chart. It may be used to show the range of commodity prices. Usually it is used to present daily or monthly high, low and closing prices of stocks and shares. The vertical distance between the high and low at any point of time measures the corresponding range. The portion between the high and low is coloured. The greater the range, the wider is the coloured band.

59. Net Balance Chart. It may be used to show the balance of trade, profit and loss.

Ex. 5. The income and expenditure of a family for 1960 were as follows.

	J	F	M	A	M	J	J	A	S	O	N	D
Income	500	700	400	600	550	640	800	720	580	610	525	900
Expenditure	720	580	530	420	510	600	710	690	660	500	680	560
Saving	...	120	...	180	40	40	90	30	...	110	...	340
Deficit	220	...	130	...	...	...	...	...	80	...	155	...

Show the monthly balance by graphs.

60. Component-part Chart. It is used to show the components of an aggregate when the data for a large number of years are known.

Ex. 6. The following table gives the population of a city X

In thousands	1952	1953	1954	1955	1956	1957	1958	1959	1960	1961
Under 20	200	250	280	300	350	400	460	515	530	500
20 & under 40	180	160	190	250	260	300	315	420	440	480
40 & under 60	80	80	95	100	115	117	160	190	180	200
60 & over	30	30	35	40	45	60	85	105	120	100

Use a component-part chart to show the distribution.

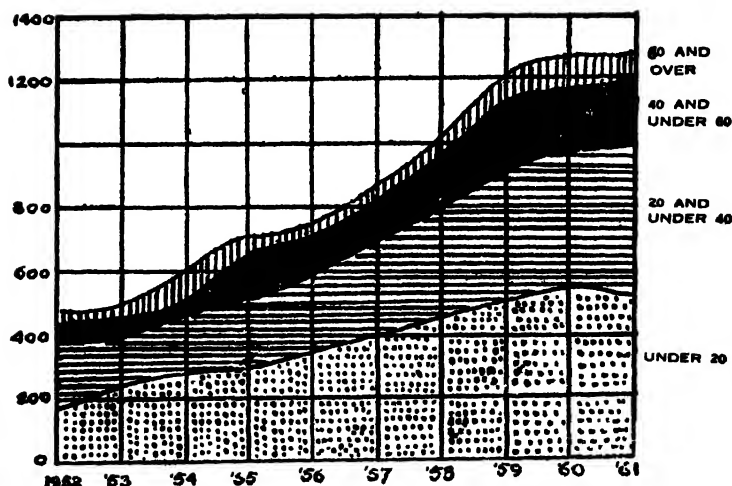


Fig. 7. Population Structure by age groups.

61. Bar Charts—These are widely used for graphical presentation. They consist of bars (rectangular areas of the same width) vertical or horizontal separated by spaces about one-half of the width of the bars. The bar charts are used for the comparison of magnitudes and relative values.

Horizontal bars are usually used to compare qualitative or geographical data, while vertical bars are used for quantitative data such as time series. A bar chart is used if only a very few years are to be shown.

The bar chart includes the zero in the vertical scale. A scale and guiding lines are found useful in reading the chart.

The vertical bar chart is also called *Column Chart*.

Ex. 7. Use bar chart to show the population of six leading states of India in 1951 from the following data.

State	Population in millions
U. P.	63.2
Madras	57.0
Bihar	40.2
Bombay	35.9
W. B.	24.8
M. P.	21.2

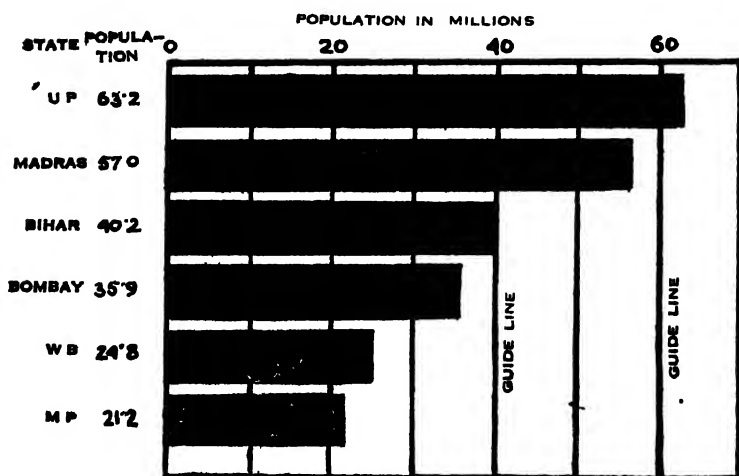


Fig. 8. Ranking of the Six Leading States of India according to Population in 1951.

The representation by horizontal bars is particularly advantageous, as corresponding figures may be shown against bars.

62. Compound bar Chart—It is the same as component bar chart. Different portions of the same bar are hatched or coloured differently so as to give effective visual impression.

63. Component Bar Chart—The component parts of any aggregate are nicely shown by bars. These bars may either be shown in absolute values or percentages of the total.

Ex. 8. The following is an account for the year ended 31st Dec. 1958 of a certain Degree College in West Bengal.

<i>Expenditure</i>	<i>Rs.</i>	<i>Income</i>	<i>Rs.</i>
To Salary of teaching staff	96 000	By Students' fee	90,000
„ Salary of others	30,000	„ State Grants	50,000
„ Rent	12,400		
„ Light & Fans	4,800	„ U.G.C. Grants	30,000
„ Miscellaneous	6 600		
„ Excess of income	20,200		
	Total 170,000		Total 170,000

Use component bar chart

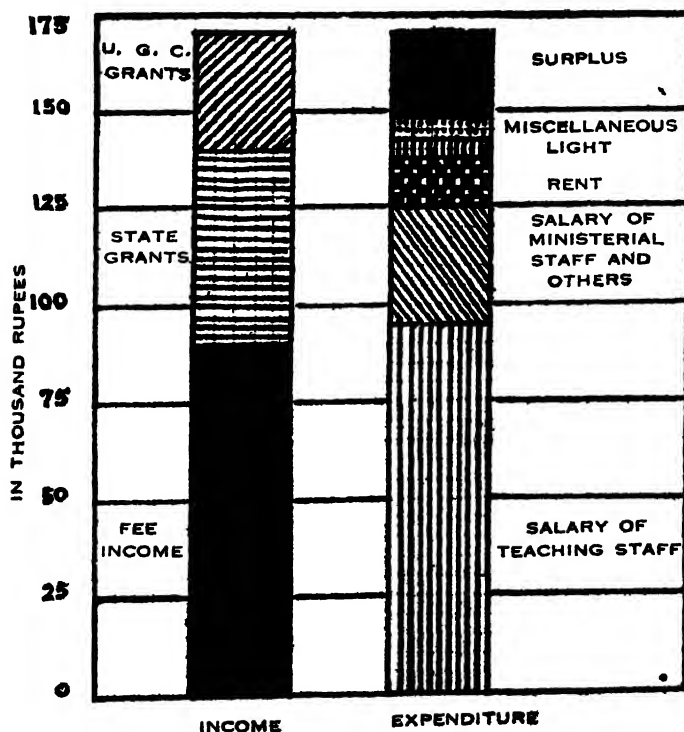


Fig. 9. *Income and Expenditure of an Institution in 1957-58.*

Ex. 9. Represent by bar graph the following data :—

Income of the Indian Union by industrial origin, 1948—1951 (in '000 crores)

	1948—49	1949—50	1950—51
Agriculture	42.5	44.9	48.9
Mining, manufacturing and hand trades	14.8	15.0	15.3
Commerce, transport and communications	16.0	16.6	16.9
Other sources	13.4	13.8	14.4
Total	86.7	90.3	95.5

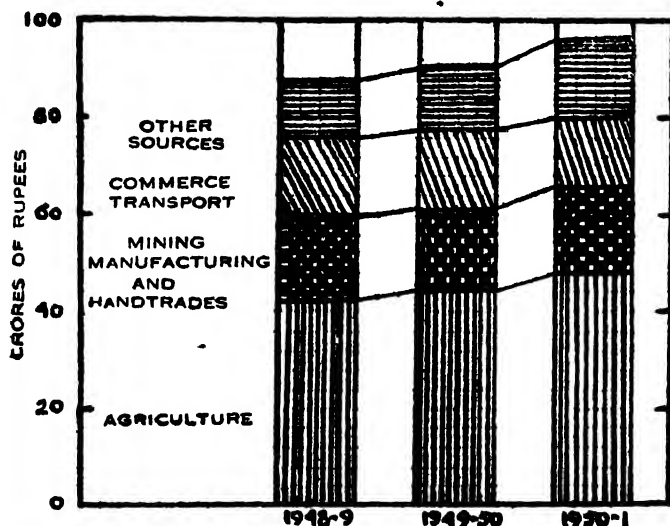


Fig 10. Changing National Income of the Indian Union, by Industrial Origin, 1948-1951.

64. Grouped Bar Chart.—Two or more sets of data for several years are compared by means of compound bar chart.

Ex. 10. Represent by bar chart the following distribution of population between towns and villages in India during the period 1921—51.

Census year (population in lakhs)	1921	1931	1941	1951 :
Villages	2199	2420	2710	2950
Towns	282	334	438	619

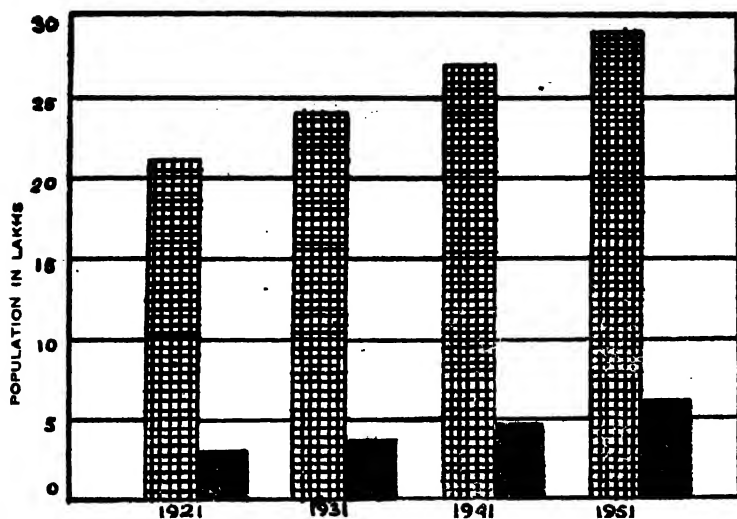


Fig. 11. *Trend of Rural and Urban Population in India, 1921—1951.*

65. The bar chart may also be paired. This is a **two-direction bar chart** which may be used to show decrease and increase.

Represent the following by a two-direction bar chart :

Ex. II. Heads of Revenue of the Union Govt. in crores of rupees.

	1953—54	1954—55
1. Customs	160·00	175·00
2. Union Excise Duties	78·00	76·31
3. Corporation Tax	38·40	38·35
4. Taxes on income	69·31	70·67
5. Railways	7·50	7·37
6. Posts and Telegraphs	2·02	1·50
7. Others	42·91	55·74
Total	398·14	424·94

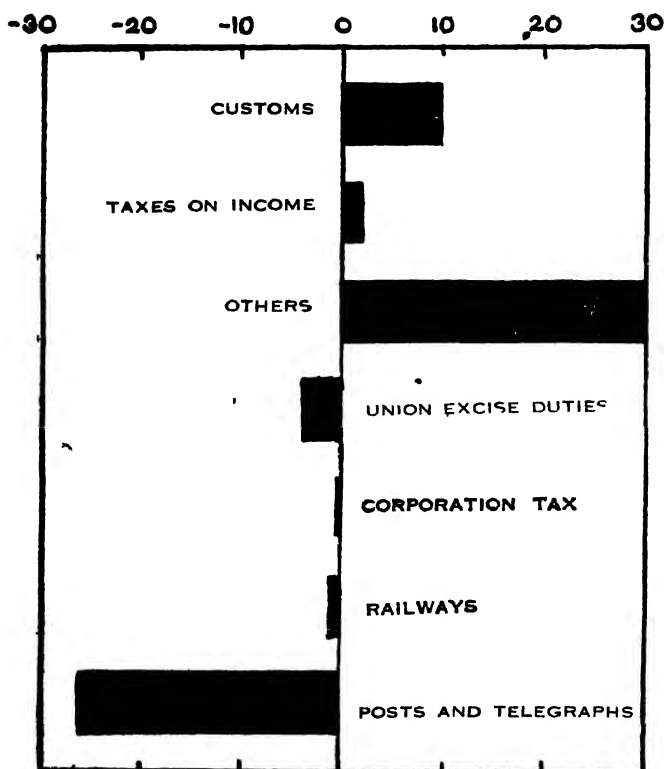


Fig. 12. Revenue Structure of the Union Govt. 1953-55 showing percentage increase and decrease.

66. Pie-Chart—Component-parts of a total may be shown in different ways—by means of a component-bar chart as in Fig. 9 or by means of a circle as in Fig 13.

Charts and circles are used when the parts are not many, otherwise curves are preferred.

The presentation by varying portions of a circle is called *Pie-Chart*. A pie-chart is two-dimensional as it represents data by sectors. As the area of a sector is proportional to the angle subtended by it at the centre, the data are proportional to the central angles.

The total angle at the centre is 360° . It represents the aggregate. The parts are expressed in terms of central angles.

Ex. 12. The following are the data of expenditure of a family in 1960.

Food.....Rs 3400
 Fuel and
 Light..... 200
 Clothing..... 960

House rent...Rs 780
 Education..... 1500
 Miscellaneous..... 1080

Represent the data by a pie-chart,

The total expenditure Rs 7920 is represented by 360° .

Food is represented by $\frac{360^\circ}{7920} \times 3400 = \frac{1^\circ}{22} \times 3400 = 154^\circ.5$

Fuel by $9^\circ.1$
 Clothing by $43^\circ.6$
 Rent by $35^\circ.5$

Education by $68^\circ.2$
 Miscellaneous by $49^\circ.1$
 Total 360°

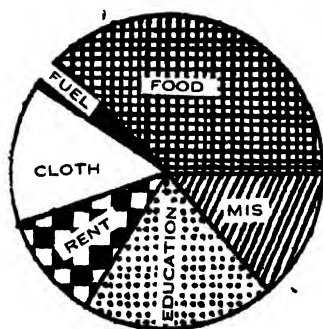


Fig. 13. Pie—Chart.

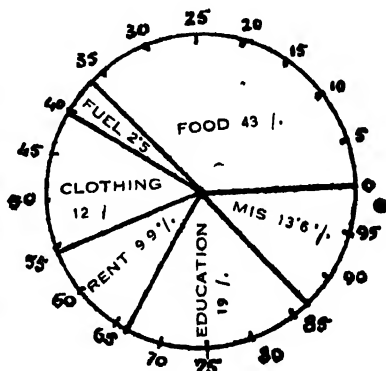


Fig. 13A. Percentage Chart.

The pie-chart can also be expressed as *percentages of the total*. For this purpose the entire circumference of the circle is divided into 100 equal parts.

The aggregate Rs 7920, in the above example is set equal to 100.

$$\text{Then Food} = \frac{3400}{7920} \times 100 = 43\%$$

$$\text{Fuel} = \frac{200}{7920} \times 100 = 2.5\%$$

$$\text{Clothing} = \frac{960}{7920} \times 100 = 12\%$$

$$\text{Rent} = \frac{780}{7920} \times 100 = 9.9\%$$

$$\text{Education} = \frac{1500}{7920} \times 100 = 19\%$$

$$\text{Miscellaneous} = \frac{1080}{7920} \times 100 = 13.6\%$$

It is seen that both the diagrams present the same appearance as it should be.

67. Logarithmic Chart or Ratio Chart.

So far we have used arithmetic scales (or arithmetic grids), both horizontally and vertically.

Arithmetic scales are used to represent absolute or exact amounts of change. For example, the figures 50, 150, 250, 350...when plotted on arithmetic scale show absolute changes of 100 ($150-50=100$, $250-150=100$, $350-250=100$...).

In the arithmetic scale equal changes are represented by equal lengths

Now the increase from 50 to 150 is 200%, that from 150 to 250 is $66\frac{2}{3}\%$ and that from 250 to 350 is 40%, i.e., the same length in arithmetic scale may represent different ratios or percentages of changes.

But sometimes *ratios* of changes and not the exact amounts of changes are desired. The ratios of changes may, however, be represented on arithmetic scales ; but the graphs of equal ratios appear on arithmetic scales as *curved lines*. It is very difficult to judge from curved lines whether they do represent a constant ratio of change or not. This difficulty may be avoided by plotting logarithms of constant ratios on an arithmetic grid.

Let us consider a series of values which maintain the same increase or decrease. Such values form an arithmetic progression (A. P).

The values 2, 4, 6, 8, 10, 12...form an A. P..... (A)

A series of values showing a constant ratio of increase or decrease is known to form a geometrical progression (G. P).

The values 2, 4, 8, 16, 32, 64...form a G. P. (B)

The graph of an A. P. is *always a straight line* plotted on an arithmetic grid. :

The graph of a G. P. is *always a curved line* plotted on an arithmetic grid, but logarithms of a G. P. is a straight

line plotted on such a grid because such logarithms form an A. P.

Ex 13 The numbers 2, 4, 8, 16, 32, 64. . . . form a G. P. but the logarithms of the numbers are

$\log 2, \log 4, \log 8, \log 16, \log 32, \log 64. . . .$ which, by the property of logarithms, may be written as

$\log 2, 2 \log 2, 3 \log 2, 4 \log 2, 5 \log 2, 6 \log 2. . .$ (C)
These logarithms are clearly in A. P., the common difference being $\log 2$ or 0.301

Let the above data correspond to 6 years, 1955-60.

The three series (A), (B) and (C) are represented in Fig.14 by three lines, the A. P. by a straight line (A), the G. P. by a curve (B) and the A. P. of logarithms by a straight line (C).

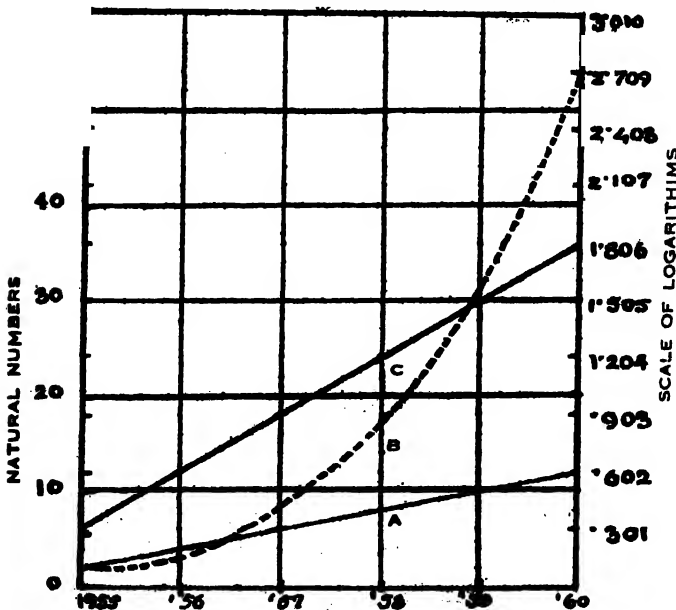


Fig. 14 Arithmetic Progression Curve A, Geometric Progression Curve B and Logarithms of Geometric Progression Curve C plotted on arithmetic scale. Vertical scales different.

That the ratio of change is constant is not discernable from the curve (B), but the fact is apparent from graph (C):

Both the graphs (B) and (C) are shown in Fig. 14 in the same grid with different vertical scales.

Hence in order that a particular graph may represent a constant ratio of change when plotted on an arithmetic grid, it is necessary that the *logarithms of the geometric progression be plotted* instead of the numbers themselves. The graph will appear as a straight line.

68. But the above method of finding out logarithms is not very convenient. We may use a different scale and a different grid by which a constant ratio of change can be made to appear as a straight line. Such a scale or

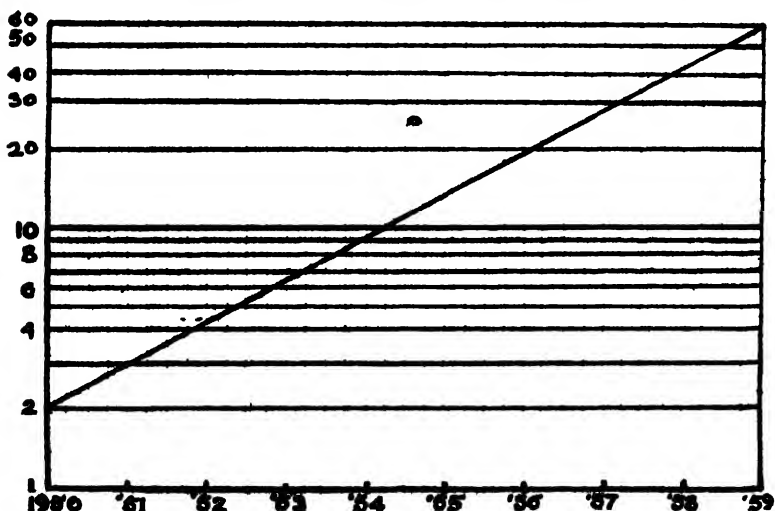


Fig. 14A Geometric Progression Curve B plotted on Semi—logarithmic scale.

grid is called *logarithm scale or grid*. More accurately it is called *Semi-logarithm scale or grid* because it is scaled arithmetically along the x -axis and logarithmically on the y -axis. In this grid the horizontal axis is marked off in years (or months) as usual, but the vertical axis is marked off logarithmically so that equal lengths along it represent equal ratios of change.

Fig. 14A represents the geometrical progression of the previous illustration in the semi-logarithmic scale. The curve is straightened out.

From the vertical scale in Fig 14A, we find that the distance from 2 to 4 is the same as that from 4 to 8 or from 10 to 20. In each case the increase is 100% or the ratio of increase is 1 : 1, *Equal ratios of increase have been represented by equal lengths on the vertical scale.*

69. Some advantages of Semi—logarithmic scale

1. Equal ratios of changes are represented by equal vertical distances.

2. The ratio of change is given by the *slope* of the line.

3. Exaggeration or contraction of vertical scale does not change the nature of the graph but only causes an *upward* or *downward shifting*. This facilitates comparison between two curves by placing them close to each other.

4. Data with very great range, which can not be ordinarily plotted on arithmetic scale, can be easily plotted on logarithmic scale.

It is to be noted that the vertical in semi-logarithmic grid does not contain zero and further negative values can not be represented in such a chart.

70. Histogram or Column Diagram.

Classified frequency distribution may be presented graphically.

Ex. 14. Consider the following table showing frequency distribution of 323 workmen classified according to weekly wages. Class-interval=Rs2

Weekly wages (in Rs.)	Number of workmen	Cumulated downward (more than lower limit)	Cumulated upward (less than upper limit)
10 and under 12	8	323	8
12 " " 14	15	315	23
14 " " 16	20	300	43
16 " " 18	60	280	103
18 " " 20	90	220	193
20 " " 22	80	130	273
22 " " 24	40	50	313
24 " " 26	10	10	323

71. Ogive

A frequency distribution may be 'cumulated downward or upward. The cumulated frequency curve is called *Ogive*. A cumulated downward curve is also called 'less than ogive'. For at any stage, we can know the

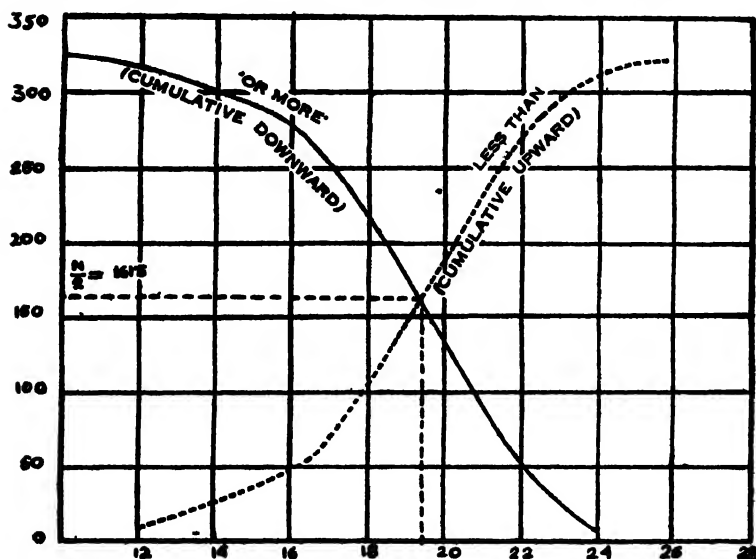


Fig. 15 Ogive—Cumulative Frequency Curve showing distribution of wages of 323 workmen.

number of workmen receiving certain wages or less. The cumulated upward curve is called "or more than ogive" for similar reason (Fig. 15).

72. The two ogives determine the *median* which is the abscissa of their point of intersection. The median can also be determined from one ogive. If n be the cumulative frequency, then through $\frac{n}{2}$ on the y -axis, draw a straight line parallel to the x -axis. The abscissa of the point where it meets the ogive is the median.

The *mode* is most elusive. It is, however, the abscissa of the maximum ordinate in an ideal frequency curve.

73. How to plot class intervals

The class-intervals are plotted along the x -axis and the corresponding frequencies along the y -axis. The horizontal scale does not begin with zero. Two extra intervals, one at the beginning and the other at the end,

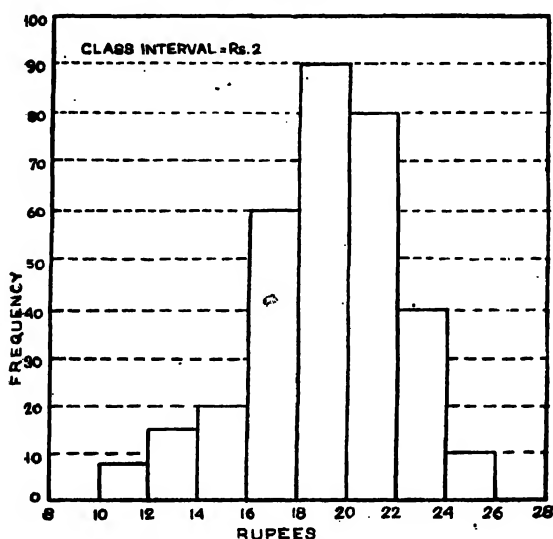


Fig. 16 *Histogram or Column Diagram.*
Distribution of 323 workmen classified according to weekly wages, Class-interval = Rs 2.

are taken. Rectangles are drawn with class-intervals as base and corresponding class-frequencies as heights. The figure formed by all such rectangles is called a *Histogram* Fig. 16. The *geometrical interpretation* of the diagram is that the total area of the histogram represents the total frequency, the area of each rectangle representing the corresponding frequency.

74. Frequency Polygon.

Take the midpoints of the class-intervals as abscissas and the corresponding class-frequencies as ordinates. The points so obtained are joined by broken lines. The ends of the polygon thus drawn are joined to the midpoints of two

extra: class-intervals whose frequencies are each taken zero,

CLASS INTERVAL = Rs 2

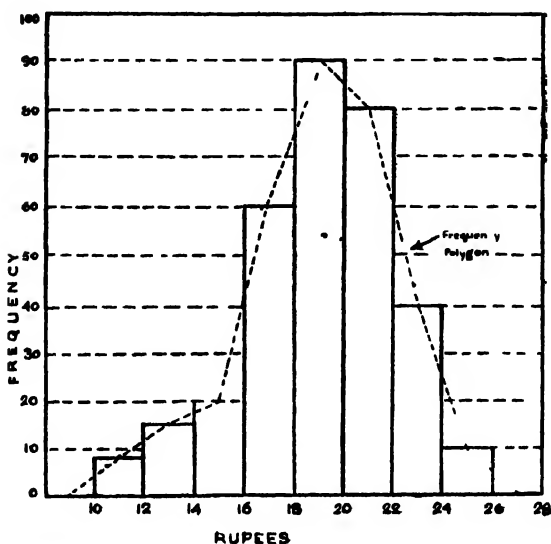


Fig. 16A Frequency Polygon Class—interval=Rs 2

This closed polygon is called *frequency polygon* whose area represents the total frequency Fig. 16A.

75. Frequency Curve

As the class-interval is gradually decreased and the number of observations increased, the histogram and polygon appear more and more regular. There is, however, a limit beyond which irregularities reappear. The smoothed line so obtained is called *frequency curve*. The area enclosed by the frequency curve and the x -axis represents the total frequency.

76. Pictographs.

These are picture charts, mere modification of bar charts to attract attention. The pictures are self-explanatory. They can not, however, show minute details and are only approximate. Interesting pictures are now-a-days used.

The population of three cities A, B, and C may be shown by the following pictograph (Fig. 17).

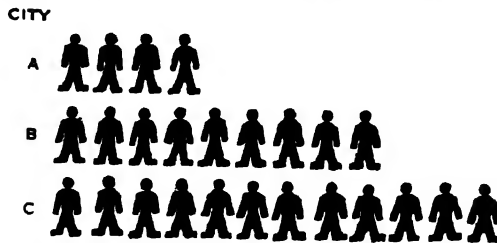


Fig. 17 Pictograph showing comparative population in three cities A, B & C.

Each complete symbol represents 10,00,000 persons.

Similarly, comparative number of cattle and quantity of crop production of the different countries may be shown by pictograph

(Fig 17A & B)

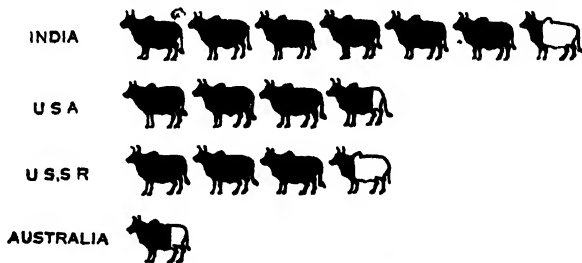


Fig. 17A. Pictograph showing comparative number of cattle, by countries.

Each complete symbol represents 20 million animals.

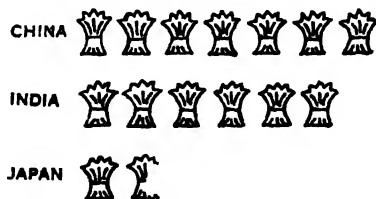


Fig. 17B Pictograph showing comparative rice—production by countries. Each complete symbol represents 20 million tons.

The pictures give only rough estimate and overall impression.

Ex. 15. The following table gives the figures of advertisement cost and sales of a Tea House.

Year	1950	1952	1954	1956	1958	1960
Advt. cost (Rs)	800	1,000	1,500	1,800	1,950	2,100
Sales (Rs)	10,000	12,000	17,000	18,500	22,000	24,000

Draw the graph and

(a) Read off the advertisement cost for the year in which the sales amounted to Rs. 15,000.

(b) Read off the sales for the year in which the advertisement cost was Rs. 1,600.

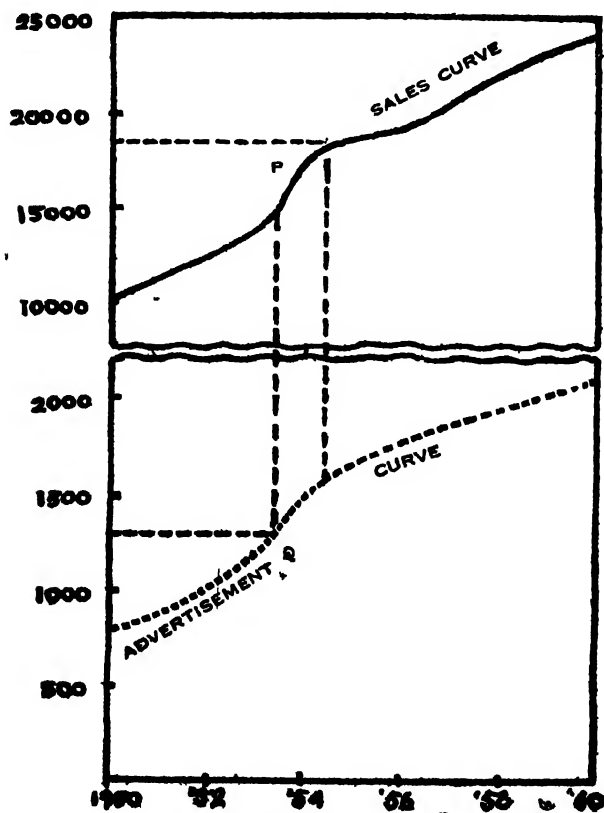


Fig. 18

Two vertical scales are used for drawing the two graphs.

(a) Let P be the point on the sales curve corresponding to the ordinate 15000. This ordinate cuts the advertisement curve at Q(below P) whose ordinate is 1250. So the advt. cost was Rs. 1250.

(b) Similarly, sales were Rs. 17,500.

77. Interpolation. ✓

When the values of a continuous variable are plotted, any intermediate value can be found approximately from its graph as in Fig. 18. The method is called graphical interpolation. Even if the variable is not continuous, such interpolation gives a fairly good result.

Exercise 5

1. Name the different kinds of graphical presentation and indicate their uses.

2. Distinguish between a logarithmic scale graph and arithmetic scale graph. What are the advantages of the former over the latter ?

3. What is a pie-chart and what are its uses ?

4. Distinguish between a histogram, a frequency polygon and a frequency curve.

5. Arrange the following observations as a frequency distribution of class-interval 10 and construct (a) a histogram and (b) a frequency polygon.

68, 30, 50, 70, 18, 10, 28, 35, 20, 40, 38, 69, 25, 50, 40
60, 50, 29, 62, 10, 35, 16, 38, 41, 64, 6, 18, 30, 46, 25
40, 58, 38, 47, 20, 40, 25, 48, 60, 10, 42, 39, 56, 48, 62

6. Calculate by means of a graph the profit on the sale of an article for Rs 22, when the rate of profit is 10% on cost.*

7. A retailer sells his goods at 10% above cost. Draw a graph from which can be read the selling prices. Read from

the graph, the selling prices corresponding to the cost price Rs 2.50 and Rs 5.69.

8. The table gives the number N of passengers in thousands, carried each year in British aircraft on journeys between England and abroad.

Year	1928	1929	1930	1931	1932	1933	1934	1935	1936	1937
N	24.8	26.2	22.0	21.9	41.6	53.5	58.1	70.0	72.0	78.0

Illustrate these data graphically. (L. C. C)

9. Represent on the graph paper the following data relating to a hotel.

No. of guests	200	260	320	360
Average Profits (Rs) —	1000	100	400	600

What should be the minimum number of guests to guarantee no loss? What will be the probable profit when the number of guests is 350? (I. C. W. A)

10. (a) Draw the graph of the price of rice from the following data and read off the price in 1954.

Year	1949	1951	1953	1955	1957	1959
Price (Rs)	19	19	20	22	26	29

(b) Find graphically the weight of a man at the age of 20 from the following observations.

Age in years	15	17	19	21	22	23	24	25	26
Weight in lbs	50	54	59	64	70	80	82	86	95

11. The following were the different heads of income of a Railway Company during 1949 and 1950.

	1949	1950
	crores (Rs)	crores (Rs)
Coaching	26	31
Goods	39	38.75
Other	1	1.25

(G.C. 1951)

Draw bar-charts.

12. Exhibit the following in suitable graphs or bar charts for representing the comparative position of production

and consumption of cement as estimated for the next five years :—

Year	Estimated Production (000 tons)	Estimated Consumption (000 tons)
1947-48	2,978	3,519
1948-49	3,582	3,852
1949-50	4,113	3,829
1950-51	4,527	3,972
1951-52	5,100	4,143

(G.C. 1947)

13. Make a suitable chart to represent the following data :—

Sale of petrolium

Territory.....	A	B	C	D	E
Quantity '000, 000 gallons...	25	31	48	14	2

Road mileage

'000 miles	7.8	10.2	16.3	3.2	2.5
------------------	-----	------	------	-----	-----

14. Draw two pie-charts for the following data relating to imports into and exports from Br. India during 1943-44.

	UK	USA	Egypt	Iran	Australia	Ceylon
Imports (Rs) crores	29.80	18.40	11.19	27.56	4.92	3.51
Exports (Rs) crores	60.19	40.29	—	—	13.31	14.36

15. Represent the following by component bar chart

Expenses of a-family in 1950 and 1960 in rupees .

	1950	1960		1950	1960
Food	1200	1500	Education	1000	1600
Clothing	800	1300	Fuel	280	680
Rent	600	900	Miscellaneous	1000	1500

16. Draw a cumulative frequency graph of the following distribution of monthly wages of a group of workmen and hence or otherwise find (a) the mode, and (b) the median.

Wages in rupees	20	21	22	23	24	25	26	27	28-29	Total
Number of workmen	8	10	11	16	20	25	15	9	6	120

17. Draw a frequency polygon for the following distribution of 860 boys, classified according to weight.

Weight in lbs	Numbers of boys
10—85	50
85—90	70
90—95	120
95—100	380
105—110	160
110—115	80
	860

Construct the cumulative frequency curves and hence determine the mode.

18. Represent the following in a semi-logarithmic scale

Year	Value
1952	64
1953	96
1954	144
1955	216
1956	324
1957	486
1958	729

19. The monthly production for 1961 of Hind Motor cars in India are given below :—

January 175 cars, February 185 cars, March 160 cars, April 110 cars, May 120 cars, June 80 cars, July 85 cars, August 88 cars. September 130 cars, October 110 cars, November 155 cars and December 170 cars.

• Represent the production figures by (a) Line chart and (b) Bar graph.

(C.U. B.Com.I, 1962)

Principal Sources of Main Statistical Data

The census of India published every ten years by the Government of India is a primary source containing a mine of information about semi-economic interests. The Statistical Abstract of India (Annual) is another important publication.

The following is a list of some important publications of the Government of India.

a. Population Statistics

1. The Census of India (Decennial)
2. The Statistical Abstract of India (Annual)

b. Agricultural Statistics

1. Agricultural Statistics of India (Annual)
2. Agricultural Situation in India (Monthly)
3. Estimates of Area and Yield of principal crops in India (Annual)
4. Bulletin of Agricultural prices.
5. Reports of the Ministry of Agriculture

c. Trade Statistics

1. Indian Trade Journal (Weekly)
2. Annual Statement on Sea-borne Trade of India (Annual)
3. Reports of Tariff Commission.
4. Review of the Trade of India (Annual)
5. Monthly Survey of Business Conditions

d. Industrial Statistics

1. Monthly Statistical Bulletin, Indian Textile Industry (Cotton, Woollen and Rayon)
2. Indian Coal Statistics (Annual)
3. Journal of Industry and Trade (Monthly)

e. Finance

1. Joint Stock Companies of India (Monthly)
2. Insurance year Book
3. Reserve Bank of India Bulletin (Monthly)
4. Report of Currency and Finance (Annual)

f. Communications

1. Reports on the working of Railways
2. Annual Report of Posts and Telegraph Department
3. Reports on Road-Rail Transport Problems.

APPENDIX

How to use a table of random numbers.

Reproduced below is a portion of the table of five columns random numbers constructed by the United States Interstate Commerce Commission.

The digits in each column as well as in each row are in random order. So it is possible to select the random numbers at any point in the table and in any direction.

Random Numbers

line	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
11	08337	89989	24260	08618	66798	25889	52860	57375
12	76829	47229	19706	30094	69430	92399	98749	22081
13	39708	30641	21267	56501	95182	72442	21445	17276
14	89836	55817	56747	75195	06818	83043	47403	58266
15	25903	61370	66081	54076	67442	52964	23823	02718
16	71345	03422	01015	68025	19703	77313	04555	83425
17	61454	92263	14647	08473	34124	10740	40839	05620
18	80376	08909	30470	40200	46558	61742	11643	92121
19	45144	54373	05505	90074	24783	86299	20900	15144
20	12191	88527	58852	51175	11534	87218	04876	85584

To obtain a random sample with the help of random numbers the total population of n units is numbered serially from 1 to n . Let us suppose that the total population consists of 600 units out of which a random sample of 5 is desired. We number the 600 units from 1 to 600 in any suitable manner. Next we start at any point in the table and in any direction. As the serial numbers consist of 3 digits at most, we shall use three columns out of five column random numbers from any group. We propose to choose columns instead of rows. Let us start at the first three columns of the group (5) from the 13th line.

The numbers so selected at random are

951 068 674 197 341 465 247 115

Now number 951 is not in the list. It is ignored.

The number 068 or 68 is in the list. It is included.

In this way excluding numbers higher than 600 and including all others in succession, we select the following 5 numbers

68 197 ~~341~~ 465 247 115

The units in the population having the above serial numbers form the desired random sample.

PART III

Exercise 2 Pages 35—3

- 6 4'06, 4 815' 7. 856'26. 8 24'08, 25, 24.
9. 74'95, 75, 73½. 10. $AM=7$, Mode = $7\frac{1}{2}$, Median = 7.
11. 39'7, 44 6, 40 0, 44'8. 12. B . 13. 29 4, 25 14. 30, 35.

Exercise 3 Pages 47—49

4. 20. 5. $2\frac{2}{3}$. 6. 34 25, 34.
7. 2, 2 8. 6'095, '112. 9. 1'87, '38'2.
10. 1'2, 30 9 11. 3, 1'2. 12 1'6.
13 11'6. 14 15'6, 39 2 15. 13'3, 15'49.

Exercise 4 Pages 61—64

7. 125, 140, 120, 110. 8. 138'8, 174'3, 233'9.
9 138'8, 125'6, 134'2. 10. 120, 129'2.
12 166. 13. 276'4.
14. 127, 125'8, 126'7, 126'3. 15. 49 2. (Fisher's)

Exercise 5 Pages 89—92

6. Rs. 2. 7. Rs. 2'75, Rs. 6'19. 9. 245.
10 (a) 21. (b) 61. 16. (a) 25. (b) 24.

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	12 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	59 13	17 21 26	30 34 38
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	48 12	16 20 24	28 32 36
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	47 11	15 18 22	26 29 33
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	37 10	14 18 21	25 28 32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	36 10	13 16 19	23 26 29
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	37 10	13 16 19	22 25 29
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	36 9	12 15 19	22 25 28
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	35 8	10 13 15	20 23 26
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	35 8	10 12 15	17 20 22
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	25 7	9 12 14	17 19 21
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	24 7	8 11 13	16 18 20
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	24 6	8 11 13	15 17 19
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	24 6	8 10 12	14 16 18
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	24 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	24 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	23 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	23 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	23 5	6 8 9	11 13 15
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	23 5	6 8 9	11 13 15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	13 4	6 7 9	10 12 14
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	13 4	6 7 9	10 12 14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	13 4	6 7 8	10 12 14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13 4	5 7 8	9 11 13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13 4	5 6 8	9 11 13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13 4	5 6 8	9 11 13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12 4	5 6 7	8 10 12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12 4	5 6 7	8 10 12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12 3	5 6 7	8 10 12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	12 3	5 6 7	8 10 12
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	12 3	4 5 7	8 10 12
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	12 3	4 5 6	8 10 12
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	12 3	4 5 6	8 10 12
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	12 3	4 5 6	8 10 12
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	12 3	4 5 6	8 10 12
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	12 3	4 5 6	8 10 12
45	6532	6542	6551	6561	6571	6580	6590	6600	6609	6618	12 3	4 5 6	8 10 12
46	6627	6637	6646	6656	6665	6675	6684	6693	6703	6712	12 3	4 5 6	8 10 12
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	12 3	4 5 6	8 10 12
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	12 3	4 5 6	8 10 12
49	6902	6911	6920	6929	6937	6946	6955	6964	6973	6982	12 3	4 5 6	8 10 12

	0	1	2	3	4	5	6	7	8	9	123	456	789
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	123	345	678
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	123	345	678
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	122	345	677
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	122	345	667
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	122	345	667
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	122	345	567
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	122	345	567
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	122	345	567
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	112	344	567
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	112	344	567
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	112	344	566
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	112	344	566
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	112	344	566
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	112	334	556
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	112	334	556
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	112	334	556
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	112	334	556
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	112	334	556
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	112	334	456
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	112	234	456
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	112	234	456
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	112	234	455
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	112	234	455
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	112	234	455
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	112	234	455
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	112	233	455
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	112	233	455
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	112	233	445
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	112	233	445
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	112	233	445
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	112	233	445
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	112	233	445
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	112	233	445
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	112	233	445
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	112	233	445
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	112	233	445
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	112	233	445
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	011	223	344
88	9445	9450	9455	9460	9465	9470	9475	9480	9485	9490	011	223	344
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	011	223	344
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	011	223	344
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	011	223	344
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	011	223	344
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	011	223	344
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	011	223	344
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	011	223	344
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	011	223	344
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	011	223	344
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	011	223	344
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	011	223	334

